Hankel determinants for the Fibonacci word  
and Padé approximation

by

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1. Introduction. The aim of this paper is to give a concrete and interesting example of the Padé approximation theory as well as to develop the general theory so as to find a quantitative relation between the Hankel determinant and the Padé pair. Our example is the formal power series related to the Fibonacci word.

The Fibonacci word $\varepsilon(a,b)$ on an alphabet $\{a,b\}$ is the infinite sequence
\begin{equation}
\varepsilon(a,b) = \hat{\varepsilon}_0 \hat{\varepsilon}_1 \ldots \hat{\varepsilon}_n \ldots := ababaababaab\ldots \quad (\hat{\varepsilon}_n \in \{a,b\}),
\end{equation}
which is the fixed point of the substitution
\begin{equation}
\sigma: \quad a \rightarrow ab, \quad b \rightarrow a.
\end{equation}

The Hankel determinants for an infinite word (or sequence) $\varphi = \varphi_0 \varphi_1 \ldots (\varphi_n \in \mathbb{K})$ over a field $\mathbb{K}$ are
\begin{equation}
H_{n,m}(\varphi) := \det(\varphi_{n+i+j})_{0 \leq i,j \leq m-1} \quad (n = 0,1,\ldots; \; m = 1,2,\ldots). \tag{3}
\end{equation}

It is known [2] that the Hankel determinants play an important role in the theory of Padé approximation for the formal Laurent series
\begin{equation}
\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-k+h}. \tag{4}
\end{equation}

Let $\mathbb{K}((z^{-1}))$ be the set of formal Laurent series $\varphi$ as above of $z$ with coefficients in $\mathbb{K}$ and $h \in \mathbb{Z}$ providing a nonarchimedean norm $\|\varphi\| := \exp(-k_0+h)$ with $k_0 = \inf\{k : \varphi_k \neq 0\}$. Let $\varphi$ be as above with $h = -1$. We say that a
pair \((P,Q)\in \mathbb{K}[z]^2\) of polynomials of \(z\) over \(\mathbb{K}\) is a Padé pair of order \(m\) for \(\varphi\) if
\[
\|Q \varphi - P\| < \exp(-m), \quad Q \neq 0, \quad \deg Q \leq m.
\]
A Padé pair \((P,Q)\) of order \(m\) for \(\varphi\) always exists and the rational function \(P/Q \in \mathbb{K}(z)\) is uniquely determined for each \(m = 0, 1, \ldots\). The element \(P/Q \in \mathbb{K}(z)\) with \(P,Q\) satisfying (5) is called the \(m\)th diagonal Padé approximation for \(\varphi\). A number \(m\) is called a normal index if (5) implies \(\deg Q = m\). Note that \(P/Q\) is irreducible if \(m\) is a normal index, although it can be reducible for a general \(m\). A normal Padé pair \((P,Q)\), i.e., \(\deg Q\) is a normal index, is said to be normalized if the leading coefficient of \(Q\) is equal to 1. It is a classical result that \(m\) is a normal index for \(\varphi\) if and only if the Hankel determinant \(\det(\varphi_{i+j})_{0 \leq i,j \leq m-1}\) is nonzero. Note that 0 is always a normal index and the determinant for the empty matrix is considered to be 1, so that the above statement remains valid for \(m = 0\).

We succeed in obtaining a quantitative relation between the Hankel determinant and the normalized Padé pair. Namely,
\[
\det(\varphi_{i+j})_{0 \leq i,j \leq m-1} = (-1)^{\lfloor m/2 \rfloor} \prod_{z: Q(z) = 0} P(z)
\]
for any normal index \(m\) with the normalized Padé pair \((P,Q)\), where \(\prod_{z: Q(z) = 0}\) indicates a product taken over all zeros \(z\) of \(Q\) with their multiplicity (Theorem 6).

We are specially interested in the Padé approximation theory applied to the Fibonacci words \(\varepsilon := \varepsilon(1,0)\) and \(\overline{\varepsilon} := \varepsilon(0,1)\), where 0, 1 are considered as elements in the field \(\mathbb{Q}\), since we have the following remark.

**Remark 1.** Let \(M\) be a matrix of size \(m \times m\) with entries consisting of two independent variables \(a\) and \(b\). Then \(\det M = (a - b)^{m-1}(pa + (-1)^{m-1}qb)\), where \(p\) and \(q\) are integers defined by
\[
p = \det M|_{a=1, b=0}, \quad q = \det M|_{a=0, b=1}.
\]

**Proof.** Subtracting the first column vector from all the other column vectors of \(M\), we see that \(\det M\) is divisible by \((a - b)^{m-1}\) as a polynomial in \(\mathbb{Z}[a, b]\). Hence, \(\det M = (a - b)^{m-1}(xa + yb)\) for integers \(x,y\). Setting \((a,b) = (1,0), (0,1)\), we get the assertion.

In Section 2, we study the structure of the Fibonacci word, in particular, its repetition property. The notion of singular words introduced in Z.-X. Wen and Z.-Y. Wen [5] plays an important role.

In Section 3, we give the value of the Hankel determinants \(H_{n,m}(\varepsilon)\) and \(H_{n,m}((\overline{\varepsilon})\) for the Fibonacci words in some closed forms. It is a rare case where the Hankel determinants are determined completely. Another such case is for the Thue–Morse sequence \(\varphi\) consisting of 0 and 1, where the Hankel
determinants \( H_{m,n}(\varphi) \) modulo 2 are obtained, and the function \( H_{m,n}(\varphi) \) of \((m, n)\) is proved to be 2-dimensionally automatic (see [1]).

In Section 4, we consider the self-similar property of the values \( H_{n,m}(\varepsilon) \) and \( H_{n,m}(\bar{\varepsilon}) \) for the Fibonacci words. The quarter plane \( \{(n, m) : n \geq 0, m \geq 1\} \) is tiled by 3 kinds of tiles with the values \( H_{n,m}(\varepsilon) \) and \( H_{n,m}(\bar{\varepsilon}) \) on it with various scales.

In Section 5, we develop a general theory of Padé approximation. We also obtain the admissible continued fraction expansion of \( \varphi_{\varepsilon} \) and \( \varphi_{\bar{\varepsilon}} \), the formal Laurent series (4) with \( h = -1 \) for the sequences \( \varepsilon \) and \( \bar{\varepsilon} \), and determine all the convergents \( p_k/q_k \) of the continued fractions. It is known in general that the set of the convergents \( p_k/q_k \) for \( \varphi \) is the set of diagonal Padé approximations and the set of degrees of \( q_k \)'s in \( z \) coincides with the set of normal indices for \( \varphi \).

2. Structure of the Fibonacci word. In what follows, \( \sigma \) denotes the substitution defined by (2), and

\[
\hat{\varepsilon} = \hat{\varepsilon}_0 \hat{\varepsilon}_1 \ldots \hat{\varepsilon}_n \ldots \quad (\hat{\varepsilon}_n \in \{a, b\})
\]

is the (infinite) Fibonacci word (1). A finite word over \{a, b\} is sometimes considered to be an element of the free group generated by a and b with inverses \( a^{-1} \) and \( b^{-1} \). For \( n = 0, 1, \ldots \), we define the \( n \)th Fibonacci word \( F_n \) and the \( n \)th singular word \( W_n \) as follows:

\[
F_n := \sigma^n(a) = \sigma^{n+1}(b), \quad W_n := \beta_n F_n \alpha^{-1}_n,
\]

where we put

\[
\alpha_n = \beta_m = \begin{cases} a & \text{(n even, \( m \) odd)}, \\ b & \text{(n odd, \( m \) even)}, \end{cases}
\]

and we define \( W_{-2} \) to be the empty word and \( W_{-1} := a \) for convenience.

Let \((f_n; n \in \mathbb{Z})\) be the Fibonacci sequence:

\[
f_{n+2} = f_{n+1} + f_n \quad (n \in \mathbb{Z}), \quad f_{-1} = f_0 = 1.
\]

Then \( |F_n| = |W_n| = f_n \) \((n \geq 0)\), where \( |\xi| \) denotes the length of a finite word \( \xi \).

For a finite word \( \xi = \xi_0 \xi_1 \ldots \xi_{n-1} \) and a finite or infinite word \( \eta = \eta_0 \eta_1 \ldots \) over an alphabet, we denote

\[
\xi \prec_k \eta
\]

if \( \xi = \eta_k \eta_{k+1} \ldots \eta_{k+n-1} \). We simply write

\[
\xi \prec \eta
\]

and say that \( \xi \) is a subword of \( \eta \) if \( \xi \prec_k \eta \) for some \( k \). For a finite word \( \xi = \xi_0 \xi_1 \ldots \xi_{n-1} \) and \( i \) with \( 0 \leq i < n \), we denote the \( i \)th cyclic permutation
of \( \xi \) by \( C_i(\xi) := \xi_1\xi_{i+1} \cdots \xi_{n-1}\xi_n \xi_1 \cdots \xi_i \). We also define \( C_{i'}(\xi) := C_{i'}(\xi) \) with \( i' := i - n[i/n] \) for any \( i \in \mathbb{Z} \).

In this section, we study the structure of the Fibonacci word \( \hat{\varepsilon} \) and discuss the repetition property. The following two lemmas were obtained by Z.-X. Wen and Z.-Y. Wen [5] and we omit the proofs.

**Lemma 1.** We have the following statements:

1. \( \hat{\varepsilon} = F_nF_{n-1}F_nF_{n+1}F_{n+2} \cdots \) \( (n \geq 1) \),
2. \( F_n = F_{n-1}F_n - F_{n-2}F_{n-1} \beta_n^{-1} \alpha_n^{-1} \beta_n \alpha_n \) \( (n \geq 2) \),
3. \( F_nF_n < \hat{\varepsilon} \) \( (n \geq 3) \),
4. \( \hat{\varepsilon} = W_{-1}W_0W_1W_2W_3 \cdots \),
5. \( W_n = W_{n-2}W_{n-3}W_{n-2} \) \( (n \geq 1) \),
6. \( W_n \) is a palindrome, that is, \( W_n \) stays invariant under reading the letters from the end \( (n \geq -2) \),
7. \( C_i(F_n) < \hat{\varepsilon} \) \( (n \geq 0, 0 \leq i < f_n) \),
8. \( C_i(F_n) \neq C_j(F_n) \) for any \( i \neq j \), moreover, they are different already before their last places \( (n \geq 1, 0 \leq i < f_n) \),
9. \( W_n \neq C_i(F_n) \) \( (n \geq 0, 0 \leq i < f_n) \),
10. \( \xi < \hat{\varepsilon} \) and \( |\xi| = f_n \) imply that either \( \xi = C_i(F_n) \) for some \( i \) with \( 0 \leq i < f_n \) or \( \xi = W_n \) \( (n \geq 0) \).

**Lemma 2.** For any \( k \geq -1 \), we have the decomposition of \( \hat{\varepsilon} \) as follows:

\[
\hat{\varepsilon} = (W_{-1}W_0 \cdots W_{k-1})W_k\gamma_0W_k\gamma_1 \cdots W_k\gamma_n \cdots ,
\]

where all the occurrences of \( W_k \) in \( \hat{\varepsilon} \) are picked up and \( \gamma_n \) is either \( W_{k+1} \) or \( W_{k-1} \) corresponding to \( \hat{\varepsilon}_n \) is a or b, respectively. That is, any two different occurrences of \( W_k \) do not overlap and are separated by \( W_{k+1} \) or \( W_{k-1} \).

We introduce another method to discuss the repetition property of \( \hat{\varepsilon} \). Let \( \mathbb{N} \) be the set of nonnegative integers. For \( n \in \mathbb{N} \), let

\[
n = \sum_{i=0}^{\infty} \tau_i(n)f_i ,
\]

\[
\tau_i(n) \in \{0,1\} \quad \text{and} \quad \tau_i(n)\tau_{i+1}(n) = 0 \quad (i \in \mathbb{N})
\]

be the regular expression of \( n \) in the Fibonacci base due to Zeckendorf. For \( m, n \in \mathbb{N} \) and a positive integer \( k \), we define

\[
m \equiv_k n
\]

if \( \tau_i(m) = \tau_i(n) \) for all \( i < k \).

**Lemma 3.** We have \( \hat{\varepsilon}_n = a \) if and only if \( \tau_0(n) = 0 \).

**Proof.** We use induction on \( n \). The lemma holds for \( n = 0, 1, 2 \). Assume that it holds for any \( n \in \mathbb{N} \) with \( n < f_k \) for some \( k \geq 2 \). Take any \( n \in \mathbb{N} \) with \( f_k \leq n < f_{k+1} \). Then, since \( 0 \leq n - f_k < f_{k-1} \), we have
which gives the regular expression if \( \tau_{k-1}(n - f_k) = 0 \). If \( \tau_{k-1}(n - f_k) = 1 \), then we have the regular expression \( n = \sum_{i=0}^{k-2} \tau_i(n - f_k)f_i + f_{k+1} \). In any case, we have \( \tau_0(n) = \tau_0(n - f_k) \). On the other hand, since \( \tilde{\varepsilon} \) starts with \( F_kF_{k-1} \) by Lemma 1, we have \( \tilde{\varepsilon}_n = \tilde{\varepsilon}_{n-f_k} \). Hence, \( \tilde{\varepsilon}_n = a \) if and only if \( \tau_0(n) = 0 \) by the induction hypothesis. Thus, we have the assertion for any \( n < f_{k+1} \), and by induction, we complete the proof. ■

**Lemma 4.** Let \( n = \sum_{i=0}^{\infty} n_i f_i \) with \( n_i \in \{0, 1\} \) \( (i \in \mathbb{N}) \). Assume that \( n_i n_{i+1} = 0 \) for \( 0 \leq i < k \). Then \( n_i = \tau_i(n) \) for \( 0 \leq i < k \).

**Proof.** If there exists \( i \in \mathbb{N} \) such that \( n_i n_{i+1} = 1 \), let \( i_0 \) be the maximum such \( i \). Take the maximum \( j \) such that \( n_i+1 = n_i+3 = n_i+5 = \ldots = n_j = 1 \). Then, replacing \( f_{i_0} + f_{i_0+1} + f_{i_0+3} + f_{i_0+5} + \ldots + f_j \) by \( f_{j+1} \), we have a new expression of \( n \):

\[
n = \sum_{i=0}^{\infty} n'_i f_i := \sum_{i=0}^{i_0-1} n_i f_i + f_{j+1} + \sum_{i=j+3}^{\infty} n_i f_i.
\]

This new expression is unchanged at the indices less than \( k \), and is either regular or has a smaller maximum index \( i \) with \( n'_i n'_{i+1} = 1 \). By continuing this procedure, we finally get the regular expression of \( n \), which does not differ from the original expression at the indices less than \( k \). Thus, \( n_i = \tau_i(n) \) for any \( 0 \leq i < k \). ■

**Lemma 5.** For any \( n \in \mathbb{N} \) and \( k \geq 0 \), \( \tau_0(n + f_k) \neq \tau_0(n) \) if and only if either \( n \equiv_{k+2} f_{k+1} - 2 \) or \( n \equiv_{k+2} f_{k+1} - 1 \). Moreover,

\[
\tilde{\varepsilon}_{n+f_k} - \tilde{\varepsilon}_n = \begin{cases} (-1)^{k+1}(a - b) & (n \equiv_{k+2} f_{k+1} - 2), \\ (-1)^k(a - b) & (n \equiv_{k+2} f_{k+1} - 1), \end{cases}
\]

where \( a \) and \( b \) are considered as independent variables.

**Proof.** If \( k = 0 \), we can verify the statement by a direct calculation.

Assume that \( k \geq 1 \) and \( \tau_k(n) = 0 \). Then

\[
n + f_k = \sum_{i=0}^{k-1} \tau_i(n)f_i + f_k + \sum_{i=k+1}^{\infty} \tau_i(n)f_i.
\]

By Lemma 4, we have \( \tau_0(n + f_k) = \tau_0(n) \) if \( k \geq 2 \) or if \( k = 1 \) and \( \tau_0(n) = 0 \). In the case where \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 0 \), since

\[
n + f_k = 1 + 2 + \sum_{i=3}^{\infty} \tau_i(n)f_i = f_2 + \sum_{i=3}^{\infty} \tau_i(n)f_i,
\]

we have \( \tau_0(n + f_k) = 0 \) by Lemma 4. On the other hand, in the case where \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 1 \), since
we have \( \tau_0(n + f_k) = 1 \) by Lemma 4.

Thus, in the case where \( k \geq 1 \) and \( \tau_k(n) = 0 \), \( \tau_0(n + f_k) \neq \tau_0(n) \) if and only if \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 0 \), or equivalently, if and only if \( n \equiv_{k+2} f_{k+1} - 2 \) with \( k = 1 \). Note that \( n \equiv_{k+1} f_{k+1} - 1 \) with \( k = 1 \) contradicts \( \tau_k(n) = 0 \).

Now assume that \( k \geq 1 \) and \( \tau_k(n) = 1 \). Take the minimum \( j \geq 0 \) such that \( \tau_k(n) = \tau_{k-2}(n) = \tau_{k-4}(n) = \ldots = \tau_j(n) = 1 \). Then since \( 2f_i = f_{i+1} + f_{i-2} \) for any \( i \in \mathbb{N} \), we have

\[
(14) \quad n + f_k = \sum_{i=0}^{j-3} \tau_i(n)f_i + f_{j-2} + f_{j+1} + f_{j+3} + f_{j+5} + \ldots + f_{k+1} + \sum_{i=k+2}^{\infty} \tau_i(n)f_i,
\]

where the first term on the right-hand side vanishes if \( j = 0, 1, 2 \). Hence by Lemma 4, \( \tau_0(n + f_k) = \tau_0(n) \) if \( j \geq 4 \).

In the case where \( j = 3 \), \( \tau_0(n + f_k) = \tau_0(n) \) holds if \( \tau_0(n) = 0 \) by (14) and Lemma 4. If \( \tau_0(n) = 1 \), then by (14) and Lemma 4, \( \tau_0(n + f_k) = 0 \). Thus, for \( j = 3 \), \( \tau_0(n + f_k) \neq \tau_0(n) \) if and only if \( \tau_0(n) = 1 \).

If \( j = 2 \), then by the assumption on \( j \), we have \( \tau_0(n) = 0 \). On the other hand, since \( f_0 = 1 \), by (14) and Lemma 4, we have \( \tau_0(n + f_k) = 1 \). Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).

If \( j = 1 \), then \( \tau_0(n) = 0 \) since \( \tau_1(n) = 1 \) by the assumption on \( j \). On the other hand, since \( f_{-1} = 1 \), we have \( \tau_0(n + f_k) = 1 \) by (14) and Lemma 4. Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).

If \( j = 0 \), then by the assumption on \( j \), \( \tau_0(n) = 1 \). On the other hand, since \( f_{-2} = 0 \), we have \( \tau_0(n + f_k) = 0 \) by (14) and Lemma 4. Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).

By combining all the results as above, we get the first part.

The second part follows from Lemma 3 and the fact that for any \( k \geq 0 \),

\[
f_{k+1} - 1 = f_k + f_{k-2} + \ldots + f_i
\]

with \( i = 0 \) if \( k \) is even and \( i = 1 \) if \( k \) is odd. Hence,

\[
\tau_0(f_{k+1} - 1) = \tau_0(f_{k+1} - 2) = \begin{cases} a & (k \text{ odd}, \text{ } h \text{ even}), \\ b & (k \text{ even}, \text{ } h \text{ odd}). \end{cases}
\]

**Lemma 6.** For any \( k \geq 0 \), \( W_k \prec_n \hat{e} \) if and only if \( n \equiv_{k+2} f_{k+1} - 1 \).

**Proof.** By Lemma 2, the smallest \( n \in \mathbb{N} \) such that \( W_k \prec_n \hat{e} \) is

\[
f_{-1} + f_0 + f_1 + \ldots + f_{k-1} = f_{k+1} - 1,
\]

but
which is the smallest $n \in \mathbb{N}$ such that $n \equiv k+2 \ f_{k+1} - 1$. Let $n_0 := f_{k+1} - 1$. Then the regular expression of $n_0$ is
\[
    n_0 = f_k + f_{k-2} + f_{k-4} + \ldots + f_d,
\]
where $d = (1 - (-1)^k)/2$. The next $n$ with $n \equiv k+2 \ n_0$ is clearly
\[
    n = f_{k+2} + f_k + f_{k-2} + \ldots + f_d,
\]
which is, by Lemma 2, the next $n$ such that $W_k \prec_n \hat{\epsilon}$ since $f_k + f_{k+1} = f_{k+2}$.

For $i = 1, 2, \ldots$, let
\[
    n_i = n_0 + \sum_{j=0}^{\infty} \tau_j(i) f_{k+2+j}.
\]
Then it is easy to see that $n_i$ is the $i$th $n$ after $n_0$ such that $n \equiv k+2 \ f_{k+1} - 1$. We prove by induction on $i$ that $n_i$ is the $i$th $n$ after $n_0$ such that $W_k \prec_n \hat{\epsilon}$. Assume that it is so for $i$. Then by Lemma 4, $W_k \gamma_i W_k \prec_{n_i} \hat{\epsilon}$. Hence, the next $n$ after $n_i$ such that $W_k \prec_n \hat{\epsilon}$ is $n_i + f_k + |\gamma_i|$. Thus, we have
\[
    n_i + f_k + |\gamma_i| = n_i + f_k + f_{k+1} |\hat{\epsilon}_{i=a} + f_{k-1} |\hat{\epsilon}_{i=b}
\]
\[
    = n_i + f_{k+2} |\tau_0(i)=0 + f_{k+1} |\tau_0(i)=1 = n_{i+1},
\]
which completes the proof. ■

Lemma 7. Let $k \geq 0$ and $n, i \in \mathbb{N}$ satisfy $n \equiv k+1 \ i$.

1. If $0 \leq i < f_k$, then $\tau_0(n+j) = \tau_0(i+j)$ for any $j = 0, 1, \ldots, f_{k+2} - i - 3$.

2. If $f_k \leq i < f_{k+1}$, then $\tau_0(n+j) = \tau_0(i+j)$ for any $j = 0, 1, \ldots, f_{k+3} - i - 3$.

Proof. (1) We prove the lemma by induction on $k$. The assertion holds for $k = 0$. Let $k \geq 1$ and assume that the assertion is valid for $k - 1$. For $j = 0, 1, \ldots, f_k - i$, we have $n + j \equiv k \ i + j$ and hence, $\tau_0(n+j) = \tau_0(i+j)$. Let $j_0 = f_k - i$. Then, since $n + j_0 \equiv k \ i + j_0 \equiv k \ 0$, we have $\tau_0(n+j_0 + j) = \tau_0(i+j_0 + j)$ for any $j = 0, 1, \ldots, f_{k+1} - 3$ by the induction hypothesis. Thus, $\tau_0(n+j) = \tau_0(i+j)$ for any $j = 0, 1, \ldots, f_{k+2} - i - 3$. This proves (1).

(2) In this case, $\tau_{k+1}(n) = 0$. Hence, $n \equiv k+2 \ i$. Therefore, we can apply (1) with $k + 1$ for $k$. Thus, we get (2). ■

Let $n, m, i \in \mathbb{N}$ with $m \geq 2$ and $0 < i < m$. We call $n$ an $(m, i)$-shift invariant place in $\hat{\epsilon}$ if
\[
    \hat{\epsilon}_{n+i} \hat{\epsilon}_{n+1+i} \ldots \hat{\epsilon}_{n+m-1+i} = \hat{\epsilon}_{n+i} \hat{\epsilon}_{n+i+1} \ldots \hat{\epsilon}_{n+i+m-1}.
\]
We call $n$ an $m$-repetitive place in $\hat{\epsilon}$ if there exist $i, j \in \mathbb{N}$ with $i > 0$ and $i + j < m$ such that $n + j$ is an $(m, i)$-shift invariant place in $\hat{\epsilon}$. Let $R_m$ be the set of $m$-repetitive places in $\hat{\epsilon}$. 

\[\]
Lemma 8. (1) Let \( n \equiv_{k+1} 0 \) for some \( k \geq 1 \). Then \( n \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place in \( \hat{\mathcal{E}} \).

(2) Let \( n \equiv_{k+1} f_k \) for some \( k \geq 2 \). Then \( n \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place in \( \hat{\mathcal{E}} \).

Proof. (1) Since the least \( i \geq n \) such that either \( i \equiv_{k+2} f_{k+1} - 1 \) or \( i \equiv_{k+2} f_{k+1} - 2 \) is not less than \( n + f_{k+1} - 2 \), by Lemma 5, we have
\[
\hat{\mathcal{E}}_n \hat{\mathcal{E}}_{n+1} \cdots \hat{\mathcal{E}}_{n+f_{k+1}-3} = \hat{\mathcal{E}}_n f_k \hat{\mathcal{E}}_{n+f_k+1} \cdots \hat{\mathcal{E}}_{n+f_k+f_{k+1}-3}.
\]

(2) Since the minimum \( i \geq n \) such that either \( i \equiv_{k+1} f_k - 1 \) or \( i \equiv_{k+1} f_k - 2 \) is \( n + f_{k+1} - 2 \), by Lemma 5, we have
\[
\hat{\mathcal{E}}_n \hat{\mathcal{E}}_{n+1} \cdots \hat{\mathcal{E}}_{n+f_k-3} = \hat{\mathcal{E}}_n f_k \hat{\mathcal{E}}_{n+f_k+1} \cdots \hat{\mathcal{E}}_{n+f_k+f_{k+1}-3}.
\]

Theorem 1. The pair \((n,m)\) of nonnegative integers satisfies \( n \in \mathcal{R}_m \) if one of the following two conditions holds:

(1) \( f_k + 1 \leq m \leq f_{k+1} - 2 \), \( n-i \equiv_{k+1} 0 \) and \( i \leq n \) for some \( k \geq 1 \) and \( i \in \mathbb{Z} \) with \( f_k + 1 \leq m + i \leq f_{k+1} - 2 \).

(2) \( f_{k-1} + 1 \leq m \leq f_{k+1} - 2 \), \( i \leq n \) and \( n-i \equiv_{k+1} f_k \) for some \( k \geq 2 \) and \( i \in \mathbb{Z} \) with \( f_{k-1} + 1 \leq m + i \leq f_{k+1} - 2 \).

Remark 2. The “if and only if” statement actually holds in Theorem 1 in place of “if” since we will prove later that \( H_{n,m} \neq 0 \) if none of the conditions (1) and (2) hold.

Proof (of Theorem 1). Assume (1) and \( i \geq 0 \). By Lemma 8(1), \( n-i \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place. Then \( n \) is an \((m, f_k)\)-shift invariant place since \( i + m \leq f_{k+1} - 2 \). Thus, \( n \in \mathcal{R}_m \) as \( f_k < m \).

Assume (1) and \( i < 0 \). Then, since \( n-i \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place and \( m \leq f_{k+1} - 2 \), it is an \((m, f_k)\)-shift invariant place. Moreover, since \( f_k - i < m \), \( n \) is an \( m \)-repetitive place.

Assume (2) and \( i \geq 0 \). Then, \( n-i \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place by Lemma 8(2). Then, \( n \) is an \((m, f_{k-1})\)-shift invariant place since \( i + m \leq f_{k+1} - 2 \). Thus, \( n \) is an \( m \)-repetitive place as \( f_{k-1} < m \).

Assume (2) and \( i < 0 \). Then, since \( n-i \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place and \( m \leq f_{k+1} - 2 \), it is an \((m, f_{k-1})\)-shift invariant place. Then \( n \) is an \( m \)-repetitive place, since \( f_{k-1} - i < m \). Thus, \( n \in \mathcal{R}_m \).

Corollary 1. The place 0 is \( m \)-repetitive for an \( m \geq 2 \) if \( m \notin \bigcup_{k=1}^{\infty} \{f_k - 1, f_k\} \).

Remark 3. The “if and only if” statement actually holds in Corollary 1 in place of “if” since we will prove later that \( H_{0,m} \neq 0 \) if \( m \in \bigcup_{k=1}^{\infty} \{f_k - 1, f_k\} \).

Proof (of Corollary 1). Let \( i = 0 \) in (1) of Theorem 1. Then 0 is \( m \)-repetitive if \( f_k + 1 \leq m \leq f_{k+1} - 2 \) for some \( k \geq 1 \).
Corollary 2. Let $k \geq 2$. The place $n$ is $f_k$-repetitive if
\[ W_k \prec \hat{x}_{n+1} \hat{x}_{n+2} \cdots \hat{x}_{n+2f_k-3}. \]

Proof. By (2) of Theorem 1, for any $k \geq 2$, $n$ is an $f_k$-repetitive place if
\[ n-i \equiv_{k+1} f_k \] for some $i$ with $i \leq n$ and $-f_{k-2} + 1 \leq i \leq f_{k-1} - 2$. Since the
condition $n-i \equiv_{k+1} f_k$ is equivalent to $n-i \equiv_{k+2} f_k$ and there is no carry
in addition of $-i$ to both sides of $n \equiv_{k+2} f_k + i$, the condition $n-i \equiv_{k+1} f_k$
is equivalent to $n \equiv_{k+2} f_k + i$. Hence, the place $n$ is $f_k$-repetitive if $n \equiv_{k+2} j$
for some $j$ with $f_{k-1} + 1 \leq j \leq f_{k+1} - 2$. By Lemma 6, this condition is
equivalent to $W_k$ starting at one of the places in \{ $n + 1, n + 2, \ldots, f_k - 2$ \},
which completes the proof. \bull

3. Hankel determinants. The aim of this section is to find the value
of the Hankel determinants
\[
H_{n,m} := H_{n,m}(\varepsilon) = \det(\varepsilon_{n+i+j})_{0 \leq i,j \leq m-1},
\]
\[
\bar{H}_{n,m} := H_{n,m}(\bar{\varepsilon}) = \det(\bar{\varepsilon}_{n+i+j})_{0 \leq i,j \leq m-1}
\]
\[ (n = 0, 1, \ldots; m = 1, 2, \ldots) \]
for the Fibonacci word $\varepsilon(a, b)$ at $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$:
\[ \varepsilon := \varepsilon(1, 0) = 10110101101101 \ldots, \]
\[ \bar{\varepsilon} := \varepsilon(0, 1) = 01001010010010 \ldots. \]

It is clear that $H_{n,m}(\varepsilon(a, b)) = 0$ if $n$ is the $m$-repetitive place in $\varepsilon(a, b)$,
where $a, b$ are considered to be two independent variables, and that, in general,
$H_{n,m}(\varepsilon(a, b))$ becomes a polynomial in $a$ and $b$ as stated in Remark 1.

In the following lemmas, theorems and corollary, we give parallel state-
ments for $\varepsilon$ and $\bar{\varepsilon}$, while we give the proofs only for $\varepsilon$ since those for $\bar{\varepsilon}$ are
similar. The only difference is the starting point, Lemma 5, where $a - b$ on
the right-hand side is 1 for $\varepsilon$ and $-1$ for $\bar{\varepsilon}$.

We use the following notation: for every subset $S$ of \{ $0, 1, 2, 3, 4, 5$ \},
$\chi(k : S)$ is the function on $k \in \mathbb{Z}$ such that
\[ \chi(k : S) = \begin{cases} 1 & \text{otherwise.} \\
-1 & \text{if } k \equiv s \pmod{6} \text{ for some } s \in S, \end{cases} \]

The following corollary follows from Theorem 1.

Corollary 3. $H_{n,m} = 0$ if one of the conditions (1), (2) in Theorem 1
is satisfied. The same statement holds for $\bar{H}_{n,m}$.

Lemma 9. For any $k \geq 2$, we have
\[
H_{0,f_k} = \chi(k : 2, 3)(H_{0,f_{k-1}} - (-1)^{f_{k-1}} H_{f_{k-1}, f_{k-1}}),
\]
\[
\bar{H}_{0,f_k} = \chi(k : 1, 3, 4, 5)(\bar{H}_{0,f_{k-1}} - (-1)^{f_{k-1}} \bar{H}_{f_{k-1}, f_{k-1}}). \]
Proof. The matrix \((\varepsilon_{i+j})_{0 \leq i,j < f_k}\) is decomposed into three parts:

\[
(\varepsilon_{i+j})_{0 \leq i,j < f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},
\]

where

\[
A = (\varepsilon_{i+j})_{0 \leq i < f_{k-2}, 0 \leq j < f_k},
\]

\[
A' = (\varepsilon_{f_{k-2}+i+j})_{0 \leq i < f_{k-3}, 0 \leq j < f_k},
\]

\[
B = (\varepsilon_{f_{k-1}+i+j})_{0 \leq i < f_{k-2}, 0 \leq j < f_k}.
\]

By Lemma 5, the following two subwords of \(\varepsilon\):

\[
\varepsilon_0\varepsilon_1 \ldots \varepsilon_{f_{k-2}+f_{k-2}} \quad \text{and} \quad \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1} \ldots \varepsilon_{f_{k-1}+f_{k-2}+f_{k-2}}
\]

differ only at two places, namely, \(\varepsilon_{f_{k-2}} \neq \varepsilon_{f_{k-1}+f_{k-2}}\) and \(\varepsilon_{f_{k-1}} \neq \varepsilon_{f_{k-1}+f_{k-1}}\). Thus, we get

\[
B - A = \begin{pmatrix} (-1)^k & (-1)^{k-1} \\ (-1)^k & (-1)^{k-1} \\ \vdots & \vdots \\ (-1)^k & (-1)^{k-1} & \cdots & 0 \end{pmatrix}.
\]

Let \(A_0, A_1, \ldots, A_{f_{k-1}}\) be the columns of the matrix \(\begin{pmatrix} A \\ A' \end{pmatrix}\) in order from the left. Since

\[
(A_0A_1 \ldots A_{f_{k-2}-2}) = (\varepsilon_{i+j})_{0 \leq i < f_{k-1}, 0 \leq j < f_{k-2}-1},
\]

\[
(A_{f_{k-1}}A_{f_{k-1}+1} \ldots A_{f_{k-2}}) = (\varepsilon_{f_{k-1}+i+j})_{0 \leq i < f_{k-1}, 0 \leq j < f_{k-2}-1}
\]

and

\[
\varepsilon_0\varepsilon_1 \ldots \varepsilon_{f_{k-2}+f_{k-1}-3} = \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1} \ldots \varepsilon_{f_{k-1}+f_{k-2}+f_{k-1}-3}
\]

by Lemma 5, we get

\[
(A_0A_1 \ldots A_{f_{k-2}-2}) = (A_{f_{k-1}}A_{f_{k-1}+1} \ldots A_{f_{k-2}}).
\]

Thus, from (15) and (16) we obtain

\[
H_{0,f_k} = \det \begin{pmatrix} A_0 & \ldots & A_{f_{k-1}-1} & A_{f_{k-1}} & \ldots & A_{f_{k-2}} & A_{f_{k-1}} \\ 0 & \ldots & (A_0)_{f_{k-1}-1} & (A_{f_{k-1}})_{f_{k-1}} & \ldots & (A_{f_{k-2}})_{f_{k-1}} & (A_{f_{k-1}})_{f_{k-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & (A_0)_{f_{k-1}-1} & (A_{f_{k-1}})_{f_{k-1}} & \ldots & (A_{f_{k-2}})_{f_{k-1}} & (A_{f_{k-1}})_{f_{k-1}} \end{pmatrix}
\]

\[= \det \begin{pmatrix} A_0 & \ldots & A_{f_{k-1}-1} & 0 & \ldots & 0 & A_{f_{k-1}} \\ 0 & \ldots & (A_0)_{f_{k-1}-1} & (A_{f_{k-1}})_{f_{k-1}} & \ldots & (A_{f_{k-2}})_{f_{k-1}} & (A_{f_{k-1}})_{f_{k-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & (A_0)_{f_{k-1}-1} & (A_{f_{k-1}})_{f_{k-1}} & \ldots & (A_{f_{k-2}})_{f_{k-1}} & (A_{f_{k-1}})_{f_{k-1}} \end{pmatrix}.
\]
\[ (-1)^{(k-1)k-2} (-1)^{[k-2]/2} \det(A_0A_1 \ldots A_{k-1-1}) + (-1)^kk^{-2} (-1)^{[k-2]/2} + k^{-1} \det(A_{k-1}A_0A_1 \ldots A_{k-1-2}) . \]

Since
\[ \varepsilon_0 \varepsilon_1 \ldots \varepsilon_2 k_{k-1-3} = \varepsilon_k \varepsilon_{k+1} \ldots \varepsilon_{k+2 k_{k-1-3}} \]
by Lemma 5, we get
\[ \det(A_{k-1}A_0A_1 \ldots A_{k-1-2}) = \det(\varepsilon_{k-1+i+j})_{0 \leq i,j < k_{k-1-1}} = H_{k-1,k_{k-1}-1} . \]
Thus we get
\[ H_{0,k_k} = (-1)^{(k-1)k-2} (-1)^{[k-2]/2} H_{0,k_{k-1}} + (-1)^{k_{k-2}} (-1)^{[k-2]/2} + k^{-1} H_{k_{k-1},k_{k-1}-1} = \chi(k:2,3)(H_{0,k_{k-1}} - (-1)^{k_{k-1}} H_{k_{k-1},k_{k-1}-1}) , \]
where we have used the fact that
\[ (-1)^{(k-1)k-2} (-1)^{[k-2]/2} = \chi(k:2,3) . \]

**Lemma 10.** For \( k \geq 2 \), we have
\[ H_{k_{k+1}-1,k_k} = \chi(k:1,3,4,5) H_{k_{k+1}-1,k_{k-1}-1} , \]
\[ \overline{H}_{k_{k+1}-1,k_k} = \chi(k:2,3) \overline{H}_{k_{k+1}-1,k_{k-1}-1} . \]

**Proof.** Just as in the proof of Lemma 9, we decompose the matrix \( (\varepsilon_{k_{k+1}-1+i+j})_{0 \leq i,j < k_k} \) into three parts:
\[ (\varepsilon_{k_{k+1}-1+i+j})_{0 \leq i,j < k_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix} , \]
where
\[ A = (\varepsilon_{k_{k+1}-1+i+j})_{0 \leq i < k_{k-2}, 0 \leq j < k_k} , \]
\[ A' = (\varepsilon_{k_{k+1}-1+k_{k-2}-1+i+j})_{0 \leq i < k_{k-3}, 0 \leq j < k_k} , \]
\[ B = (\varepsilon_{k_{k+1}-1+k_{k-1}-1+i+j})_{0 \leq i < k_{k-2}, 0 \leq j < k_k} . \]
By Lemma 5, the following two subwords of \( \varepsilon \):
\[ \varepsilon_{k_{k+1}-1} \varepsilon_{k_{k+1}} \ldots \varepsilon_{k_{k+1}+k_{k-2}+k_{k-3}} \] and
\[ \varepsilon_{k_{k+1}-1+k_{k-1}} \varepsilon_{k_{k+1}+k_{k-1}-1} \ldots \varepsilon_{k_{k+1}+k_{k-1}+k_{k-2}+k_{k-3}} \]
differ only at two places. Namely, \( \varepsilon_{k_{k+1}+k_{k-2}} \neq \varepsilon_{k_{k+1}+k_{k-1}+k_{k-2}} \) and \( \varepsilon_{k_{k+1}+k_{k-1}} \neq \varepsilon_{k_{k+1}+k_{k-1}+k_{k-1}} \). Therefore, we get
\[ B - A = \begin{pmatrix} 0 & (-1)^k & (-1)^{k-1} \\ (-1)^k & \ddots & (-1)^{k-1} \\ (-1)^k & (-1)^{k-1} & 0 \end{pmatrix} . \]
Thus, we have
\begin{equation}
\det(\varepsilon_{f_{k+1}+1+i+j})_{0 \leq i, j < f_k} = \det\begin{pmatrix}
A_0 & A_1 & \ldots & A_{f_k-2} & A_{f_k-1} \\
0 & \ldots & \ldots & \ldots & \ldots \\
(-1)^k & (-1)^{k-1} & \ldots & 0
\end{pmatrix}
= (-1)^{k-2}(-1)^{(f_k-2)/2}\det(A_0A_1\ldots A_{f_k-1})
= \chi(k : 1, 3, 4, 5)H_{f_{k+1}-1, f_{k-1}}.
\end{equation}

**Lemma 11.** For any \( k \geq 2 \), we have
\[ H_{f_{k+1}-1, f_{k-1}} = \chi(k : 2, 5)H_{0, f_{k-1}}, \]
\[ \overline{H}_{f_{k+1}-1, f_{k-1}} = \chi(k : 2, 5)\overline{H}_{0, f_{k-1}}. \]

**Proof.** Since, by Lemma 5,
\[ \varepsilon_{f_{k+1}}\varepsilon_{f_{k+1}}\ldots\varepsilon_{f_{k+1}+f_k-1-2} = \varepsilon_{f_{k+1}+f_{k-1-1}}\varepsilon_{f_{k+1}+f_{k-1}}\ldots\varepsilon_{f_{k+1}+2f_k-1-2}, \]
we get
\[ (\varepsilon_{f_{k+1}+1+i+j})_{0 \leq i, j < f_k} = \begin{pmatrix}
0 & 0 & 1 \\
1 & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
0 & 1 & 0
\end{pmatrix}(\varepsilon_{f_{k+1}+1+i+j})_{0 \leq i, j < f_k-1}. \]

Also, by Lemma 5,
\[ (\varepsilon_{f_{k+1}+i+j})_{0 \leq i, j < f_k} = (\varepsilon_{i+j})_{0 \leq i, j < f_k}. \]

Thus we obtain
\[ H_{f_{k+1}-1, f_{k-1}} = \det(\varepsilon_{f_{k+1}+1+i+j})_{0 \leq i, j < f_k-1} = (-1)^{f_k-1} \det(\varepsilon_{f_{k+1}+1+i+j})_{0 \leq i, j < f_k-1} = \chi(k : 2, 5)H_{0, f_{k-1}}. \]

**Lemma 12.** For any \( k \geq 3 \), we have
\[ H_{0, f_k} = \chi(k : 2, 3)H_{0, f_{k-1}} + \chi(k : 2, 4)H_{0, f_{k-2}}, \]
\[ \overline{H}_{0, f_k} = \chi(k : 1, 3, 4, 5)\overline{H}_{0, f_{k-1}} + \chi(k : 0, 1, 2, 3)\overline{H}_{0, f_{k-2}}. \]

**Proof.** Clear from Lemmas 9–11.

**Lemma 13.** For any \( k \geq 0 \), we have
\[ H_{0, f_k} = \chi(k : 2)_{f_{k-1}}, \]
\[ \overline{H}_{0, f_k} = \chi(k : 1, 2, 4)_{f_{k-2}}. \]
Proof. We have

\[ H_{0,f_0} = 1, \quad H_{0,f_1} = 1, \quad H_{0,f_2} = -2, \]
\[ \overline{H}_{0,f_0} = 0, \quad \overline{H}_{0,f_1} = -1, \quad \overline{H}_{0,f_2} = -1. \]

Thus, the assertion holds for \( k = 0, 1, 2 \). For \( k \geq 3 \), we can prove it by induction on \( k \) using Lemma 12.

Lemma 14. For any \( k \geq 1 \), we have

\[ H_{0,f_k-1} = \chi(k : 0, 4)f_{k-2}, \]
\[ \overline{H}_{0,f_k-1} = \chi(k : 2, 3, 4, 5)f_{k-3}. \]

Proof. Since the matrix \((\varepsilon_{i+j})_{0 \leq i, j < f_k - 1}\) is obtained from \((\varepsilon_{i+j})_{0 \leq i, j < f_k}\) by removing the last row and the last column, for any \( k \geq 2 \) we have by (17),

\[ H_{0,f_k-1} = \det \begin{pmatrix} A_0 & A_1 & \ldots & A_{f_k-1} & 0 & \ldots & 0 & 0 & (-1)^k & (-1)^{k-1} \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (-1)^k & (-1)^{k-1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \end{pmatrix} \]

\[ = (-1)^k(f_{k-2})((-1)^{(f_{k-2}-1)/2}) \det(A_0A_1\ldots A_{f_k-1-1}) \]

\[ = (-1)^k(f_{k-2})((-1)^{(f_{k-2}-1)/2})H_{0,f_k-1}. \]

Hence, in view of Lemma 13, we obtain the formula for \( H_{0,f_k-1} \).

Theorem 2. For any \( m, k \geq 1 \) with \( f_{k-1} < m \leq f_k \) and \( n \in \mathbb{N} \) with \( n \equiv k+1 \mod 0 \), we have

\[ H_{n,m} = \begin{cases} \chi(k : 2)f_{k-1} & \text{if } m = f_k, \\ \chi(k : 0, 4)f_{k-2} & \text{if } m = f_k - 1, \\ 0 & \text{otherwise}, \end{cases} \]

\[ \overline{H}_{n,m} = \begin{cases} \chi(k : 1, 2, 4)f_{k-2} & \text{if } m = f_k, \\ \chi(k : 2, 3, 4, 5)f_{k-3} & \text{if } m = f_k - 1, \\ 0 & \text{otherwise}. \end{cases} \]

Proof. By Lemmas 3 and 7, the matrix for \( H_{n,m} \) coincides with that for \( H_{0,m} \) so that \( H_{n,m} = H_{0,m} \). Thus, the first two cases follow from Lemmas 13 and 14. For the last case, by Corollary 1, there exist two identical rows in the matrix \((\varepsilon_{i+j})_{0 \leq i, j < m}\), so that \( H_{0,m} = 0 \).
THEOREM 3. For any \( k, n, i \in \mathbb{N} \) with \( n \equiv k+1 i \) and \( 0 \leq i \leq f_{k+1} - 1 \), we have

\[
H_{n, f_k} = \begin{cases} 
\chi(k : 2) \chi(k : 1, 4)^i f_{k-1} \\
\quad \text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \leq i < f_{k-1} \\
\quad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \leq i < f_k, \\
\chi(k : 1, 2, 4)^i f_{k-2} \\
\quad \text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\
\quad \text{or } i = f_{k+1} - 1, \\
0 \\
\quad \text{otherwise,}
\end{cases}
\]

\[
\Pi_{n, f_k} = \begin{cases} 
\chi(k : 1, 2, 4)^i \chi(k : 1, 4)^i f_{k-2} \\
\quad \text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \leq i < f_{k-1} \\
\quad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \leq i < f_k, \\
\chi(k : 2)^i f_{k-3} \\
\quad \text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\
\quad \text{or } i = f_{k+1} - 1, \\
0 \\
\quad \text{otherwise.}
\end{cases}
\]

Proof. The assertion holds for \( k = 0 \). Let \( k \geq 1 \).

Assume that either \( \tau_{k+1}(n) = 0 \) and \( 0 \leq i < f_{k-1} \) or \( \tau_{k+1}(n) = 1 \) and \( 0 \leq i < f_k \). Then by Lemmas 3 and 7 we have

\[
\varepsilon_{i+j} = \varepsilon_{n+j} \quad (j = 0, 1, \ldots, f_k - i - 1),
\]

\[
\varepsilon_{i+j-f_k} = \varepsilon_{n+j} \quad (j = f_k - i, f_k, \ldots, 2f_k - 2),
\]

\[
\varepsilon_j = \varepsilon_{j+f_k} \quad (j = 0, 1, \ldots, f_k - 1).
\]

Hence, the columns of the matrix \((\varepsilon_{n+j})_{0 \leq h, j \leq f_k}\) coincide with those of \((\varepsilon_{h+j})_{0 \leq h, j \leq f_k}\). The \( j \)th column of the former is the \((i+j) \text{ (mod } f_k)\)th column of the latter for \( j = 0, \ldots, f_k - 1 \). Therefore, we get

\[
H_{n, f_k} = (-1)^{i(f_k-i)} H_{0, f_k},
\]

which leads to the first case of our theorem by Theorem 2.

Assume that \( i = f_{k+1} - 1 \). Then \( H_{n, f_k} = H_{f_{k+1}-1, f_k} \) by Lemmas 3 and 7. Thus, by Lemmas 10–12 we get

\[
H_{n, f_k} = \chi(k : 1, 2, 4)^i f_{k-2}.
\]

Assume that \( \tau_{k+1}(n) = 0 \) and \( i = f_{k-1} \). Then, since \( n \equiv k+2 i \), we have

\[
H_{n, f_k} = H_{f_{k+1}-1, f_k} \text{ by Lemmas 3 and 7. By Lemma 1,}
\]

\[
\xi := \varepsilon_{f_{k-1}} \varepsilon_{f_{k-1}+1} \cdots \varepsilon_{f_{k-1}+2f_k-2} <_1 W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2},
\]

\[
\eta := \varepsilon_{f_{k+1}-1} \varepsilon_{f_{k+1}} \cdots \varepsilon_{f_{k+1}+2f_k-3} <_{f_k} W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}.
\]

Since the last letter of \( \eta \) comes one letter before the last letter of the palindrome word \( W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2} \), it follows that \( \xi \) is the mirror image of \( \eta \), so that
\((\varepsilon_{f_k-1+i+j})_{0 \leq i,j < f_k}\)

\[
\begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
1 & 0
\end{pmatrix} \quad \begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
1 & 0
\end{pmatrix}
\]

Thus, we obtain \(H_{f_k-1, f_k} = H_{f_k+1-1, f_k}\) and

\[
H_{n, f_k} = \chi(k : 1, 2, 4)f_{k-2}.
\]

Assume that \(n\) does not belong to the above two cases. Then, since \(\tau_{k+1}(n) = 1\) implies \(i < f_k\), we have the following condition:

\[
\tau_{k+1}(n) = 0 \quad \text{and} \quad f_k - 1 < i \leq f_k - 1 - 2.
\]

This condition is nonempty only if \(k \geq 2\), which we assume. Then the condition (2) of Theorem 1 is satisfied with \(f_k\) (resp. \(i - f_k\)) in place of \(m\) (resp. \(i\)). Thus, by Corollary 3, \(H_{n, f_k} = 0\).

**Lemma 15.** For any \(k, n, i \in \mathbb{N}\) with \(k \geq 1\) and \(n \equiv_{k+1} i\), assume that either \(\tau_{k+1}(n) = 0\) and \(0 \leq i < f_k - 1\) or \(\tau_{k+1}(n) = 1\) and \(0 \leq i < f_k\). Then

\[
H_{n, f_k} = \begin{cases}
\chi(k : 1, 2, 4)f_{k-2} & (i = 0), \\
\chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^iH_{i+f_k, f_k-1} & (0 < i \leq f_k-2), \\
\chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^iH_{i+f_k, f_k-1} & (f_k-2 < i \leq f_k-1), \\
\chi(k : 2, 3, 4, 5)^iH_{i+f_k, f_k-1} & (f_k-1 < i < f_k).
\end{cases}
\]

\[
\overline{H}_{n, f_k} = \begin{cases}
\chi(k : 2, 3, 4, 5)^iH_{i+f_k, f_k-1} & (i = 0), \\
\chi(k : 1, 3, 4, 5)^iH_{i+f_k, f_k-1} & (0 < i \leq f_k-2), \\
\chi(k : 1, 3, 4, 5)^iH_{i+f_k, f_k-1} & (f_k-2 < i \leq f_k-1), \\
\chi(k : 2, 3, 4, 5)^iH_{i+f_k, f_k-1} & (f_k-1 < i < f_k).
\end{cases}
\]

**Proof.** If \(i = 0\), then the statement follows from Theorem 2. Let

\[
A_i = (\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_k-1-1}),
\]

\[
A_i' = (\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_k-2-1}),
\]

\[
B_j = (\varepsilon_{j+f_k-1}, \varepsilon_{j+f_k-1+1}, \ldots, \varepsilon_{j+f_k-1}) \quad (j = 0, 1, \ldots).
\]

Then, by the same argument as in the proof of Theorem 3, we obtain

\[
H_{n, f_k-1} = \det \begin{pmatrix}
A_i & \ldots & A_{i+f_k-1-1} & A_{i-f_k-1} & \ldots & A_{i-1} \\
B_i' & \ldots & B_{i+f_k-1-1}' & B_{i-f_k-1}' & \ldots & B_{i-1}'
\end{pmatrix}
\]

\[
= (-1)^{(i-1)(f_k-1)} \det \begin{pmatrix}
A_0 & \ldots & A_{i-2} & A_i & \ldots & A_{f_k-1} \\
B_0' & \ldots & B_{i-2}' & B_i' & \ldots & B_{f_k-1}'
\end{pmatrix}.
\]
Therefore, if $f_{k-2} < i \leq f_{k-1}$, then by the same argument as for (17), we obtain

\[
(-1)^{(i-1)(f_{k-2}-i)} H_{n,f_{k-1}}
\]

\[
= \det \begin{pmatrix}
A_0 \ldots A_{i-2} A_1 \ldots A_{f_{k-1}-1} & 0 & \ldots & 0 & A_{f_{k-1}-1} \\
0 & \ddots & \vdots & \vdots & \vdots \\
-1^{k-1} & \vdots & \ddots & \vdots & \vdots \\
0 & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.
\]

Since by Lemma 5,

\[
A_{f_{k-1}} - A_{f_{k-2}} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
(-1)^{k}
\end{pmatrix},
\]

we get

\[
(-1)^{(i-1)(f_{k-2}-i)} H_{n,f_{k-1}}
\]

\[
= \det \begin{pmatrix}
A'_0 \ldots A'_{i-2} A'_1 \ldots A'_{f_{k-1}-1} & 0 & \ldots & 0 & 0 \\
0 & \ddots & \vdots & \vdots & \vdots \\
-1^{k-1} & \vdots & \ddots & \vdots & \vdots \\
0 & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

\[
= (-1)^{k} H_{i+f_{k},f_{k-1}-1}.
\]

Thus we obtain

\[
H_{n,f_{k-1}} = \chi(k : 2,3) \chi(k : 1,2,4,5)^i H_{i+f_{k},f_{k-1}-1}.
\]

Assume that $f_{k-1} < i < f_k$. Then as above we have

\[
(-1)^{(i-1)(f_{k-1}-i)} H_{n,f_{k-1}}
\]

\[
= \det \begin{pmatrix}
A_0 \ldots A_{f_{k-1}-1} & 0 & \ldots & 0 & A_{f_{k-1}-1} \\
0 & \ddots & \vdots & \vdots & \vdots \\
0 & \ddots & \vdots & \vdots & \vdots \\
0 & \ddots & \vdots & \ddots & \vdots \\
(-1)^{k} & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]

\[
= (-1)^{k(i-f_{k-1}-1)+(k-1)(f_{k-1}-i)+(f_{k-2}-1)/2} \det(A_0 \ldots A_{f_{k-1}-1}).
\]
Hence, by Lemma 13,
\[ H_{n, f_k - 1} = \chi(k : 0, 3, 4)\chi(k : 1, 4)^t H_{0, f_k - 1} = \chi(k : 0, 4)\chi(k : 1, 4)^t f_{k-2}. \]

Assume that \(0 < i < f_k - 2\). Then, since \(A_{i-1 + f_k - 1} = A_{i-1}\), by the same arguments as above we get
\[
(-1)^{(i-1)(f_k-2)} H_{n, f_k - 1} = \begin{vmatrix}
    A_0' \ldots A_{i-2}' A_i' \ldots A'_{f_k - 1 - 1} & 0 & \ldots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \ldots & 0
\end{vmatrix}
\]
\[= (-1)^k \begin{vmatrix}
    0 \\
    0 \\
    \vdots \\
    0
\end{vmatrix}
\]
\[= (-1)^k f_{k-2} (-1)^{\lfloor f_k - 2/2 \rfloor} \det(A_0' \ldots A_{i-2}' A_i' \ldots A'_{f_k - 1 - 1})
\]
\[+ (-1)^k (i-1)+(k-1)(f_k-2-i)(-1)^{i-1+\lfloor (f_k-2-1)/2 \rfloor}
\times \det(A_0 \ldots A_{i-2} A_i \ldots A_{f_k - 1 - 1} A_{i-1}).
\]

Since
\[\det(A_0 \ldots A_{i-2} A_i \ldots A_{f_k - 1 - 1} A_{i-1}) = (-1)^{i-1} f_{k-1} H_{0, f_k - 1},\]
by Lemma 13 we obtain
\[(21) \quad H_{n, f_k - 1} = \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^t H_{i+f_k, f_k - 1 - 1}
\]
\[+ \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^t f_{k-2}.\]

Note that (21) holds also for \(i = f_k - 2\) since in this case,
\[H_{n, f_k - 1} = (-1)^{k(f_k-2-i-1)}(-1)^{f_k-2-i+\lfloor (f_k-2-1)/2 \rfloor}
\times \det(A_0 \ldots A_{f_k-2} A_{f_k-2} \ldots A_{f_k-1} A_{f_k-1})
\]
and
\[A_{f_k-1} = A_{f_k-1} + i(0, \ldots, 0, (-1)^k). \]

**Lemma 16.** For any \(k, n, i \in \mathbb{N}\) with \(k \geq 1\) and \(n \equiv_{k+1} i\), assume that either \(\tau_{k+1}(n) = 0\) and \(0 \leq i < f_k - 1\) or \(\tau_{k+1}(n) = 1\) and \(0 \leq i < f_k\). Then
\[H_{n, f_k - 1} = \begin{cases}
    \chi(k : 0, 4)^t f_{k-2} & (i = 0), \\
    \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^t f_{k-3} & (0 < i < f_k - 1), \\
    \chi(k : 0, 4)\chi(k : 1, 4)^t f_{k-2} & (f_k - 1 < i < f_k),
\end{cases}
\]
\[\overline{H}_{n, f_k - 1} = \begin{cases}
    \chi(k : 2, 3, 4, 5)^t f_{k-3} & (i = 0), \\
    \chi(k : 0, 1)\chi(k : 1, 4)^t f_{k-4} & (0 < i < f_k - 1), \\
    \chi(k : 2, 3, 4, 5)\chi(k : 1, 4)^t f_{k-3} & (f_k - 1 < i < f_k).
\end{cases}
\]
Proof. The first and third cases have already been proved in Lemma 15. Consider the second case where \( 0 < i \leq f_{k-1} \). We divide it into two subcases, and use induction on \( k \).

**Case 1:** \( i = 1 \). If \( k = 1 \), then
\[
H_{n,f_{k-1}} = H_{n,1} = \varepsilon_n = 0
\]
since \( n \equiv 2 \) and \( \tau_0(n) = 1 \). On the other hand, \( f_{k-3} = f_{k-2} = 0 \), and hence, we get the statement. Assume that \( k \geq 2 \) and the assertion holds for \( k - 1 \). Then, by Lemma 15 and the induction hypothesis, we get
\[
H_{n,f_{k-1}} = \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i + f_k, f_{k-1} - 1}
\]
\[
+ \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2}
\]
\[
= \chi(k : 1, 3, 4, 5)H_{1 + f_k, f_{k-1} - 1} + \chi(k : 2, 3, 4, 5)f_{k-2}
\]
\[
= \chi(k : 1, 3, 4, 5)\chi(k - 1 : 2, 3, 4, 5)f_{k-4} + \chi(k : 2, 3, 4, 5)f_{k-2}
\]
\[
= \chi(k : 0, 1)f_{k-4} + \chi(k : 2, 3, 4, 5)f_{k-2}
\]
\[
= \chi(k : 2, 3, 4, 5)f_{k-3},
\]
which is the desired statement.

**Case 2:** \( i \geq 2 \). If \( f_{k-2} < i \leq f_{k-1} \), then it follows from the third case and then the fourth case of Lemma 15 that
\[
H_{n,f_{k-1}} = \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i + f_k, f_{k-1} - 1}
\]
\[
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i \chi(k - 1 : 0, 4)\chi(k - 1 : 1, 4)^i f_{k-3}
\]
Assume that \( i \leq f_{k-2} \) and the statement holds for \( k - 1 \). Then by Lemma 15, we get
\[
H_{n,f_{k-1}} = \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i + f_k, f_{k-1} - 1}
\]
\[
+ \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2}
\]
\[
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i \chi(k - 1 : 1, 2, 3, 5)\chi(k - 1 : 1, 4)^i f_{k-4}
\]
\[
+ \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2}
\]
\[
= \chi(k : 0, 4)\chi(k : 1, 4)^i f_{k-2} + \chi(k : 2, 3, 4, 5)\chi(k : 1, 4)^i f_{k-2}
\]
\[
= \chi(k : 2, 3, 4, 5)f_{k-3}.
\]

**Lemma 17.** For any \( k, n \in \mathbb{N} \) with \( k \geq 2 \) and \( \tau_{k+1}(n) = 0 \), we have
\[
H_{n,f_{k-1}} = \begin{cases} 
\chi(k : 2, 3, 4, 5)f_{k-3} & (n \equiv_{k+1} f_{k-1}), \\
\chi(k : 0, 4)f_{k-2} & (n \equiv_{k+1} f_{k-1} + 1), 
\end{cases}
\]
\[
\bar{H}_{n,f_{k-1}} = \begin{cases} 
\chi(k : 0, 4)f_{k-4} & (n \equiv_{k+1} f_{k-1}), \\
\chi(k : 2, 3, 4, 5)f_{k-3} & (n \equiv_{k+1} f_{k-1} + 1). 
\end{cases}
\]
Proof. Assume that \( n \equiv k + 1 \). Then since \( \tau_{k+1}(n) = 0 \), we have \( n \equiv k + 2 \). Therefore, by Lemmas 3 and 7, we get

\[
H_{n, f_k - 1} = \det \begin{pmatrix}
A_{f_k - 1} & \ldots & A_{f_k - 1} & A_{f_k - 1} & \ldots & A_{f_k + 1 - 2} \\
B'_{f_k - 1} & \ldots & B'_{f_k - 1} & B'_{f_k - 1} & \ldots & B'_{f_k + 1 - 2}
\end{pmatrix},
\]

where we use the notation (20). By Lemma 5, the following two subwords of \( \varepsilon \):

\[
\varepsilon_{n+1} \varepsilon_{n+f_k - 2+k-3} \quad \text{and} \quad \varepsilon_{n+f_k - 1} \varepsilon_{n+f_k - 1+1} \varepsilon_{n+f_k - 1+f_k - 2+k-3}
\]
differ only at two places, namely, at the \((f_k - 2 - f_k - 1)\)th and the \((f_k - 1 - f_k - 1)\)th places. Hence, we have

\[
H_{n, f_k - 1} = \det \begin{pmatrix}
A_{f_k - 1} & \ldots & A_{f_k - 1} & A_{f_k - 1} & \ldots & A_{f_k + 1 - 2} \\
B'_{f_k - 1} & \ldots & B'_{f_k - 1} & B'_{f_k - 1} & \ldots & B'_{f_k + 1 - 2}
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
A_{f_k - 1} & \ldots & \ldots & A_{f_k - 1} & A_{f_k - 1} & \ldots & A_{f_k + 1 - 2} \\
0 & \ldots & \ldots & 0 & \ldots & \ldots & 0 \\
(-1)^k & \ldots & \ldots & (-1)^k & \ldots & \ldots & (-1)^k
\end{pmatrix}.
\]

By adding the first \( f_k - 2 \) columns and subtracting the last \( f_k - 2 \) columns to and from the column beginning by \( A_{f_k - 1} \), we get the column

\[
t(A_{f_k - 1} 0 \ldots 0) + t((-1)^{k-1} 0 \ldots 0(-1)^k 0 \ldots 0),
\]

where \((-1)^k\) is at the \((f_k - 2 - 1)\)th place. Since, by Lemma 5,

\[
(A_{f_k - 1} \ldots A_{f_k - 2}) - (A_{2f_k - 1} \ldots A_{f_k - 2})
\]

\[
= \begin{pmatrix}
0 & \ldots & (-1)^{k-1} & \ldots & (-1)^k \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(-1)^{k-1} & \ldots & \ldots & 0 \\
(-1)^k & \ldots & \ldots & \ldots
\end{pmatrix},
\]

we get

\[
H_{n, f_k - 1} = (-1)^{k(f_k - 2 - 1)}(-1)^{f_k - 1(f_k - 2 - 1) + (f_k - 2 - 1)/2}
\]

\[
\times \left\{ \det(A_{f_k - 1} A_{f_k} \ldots A_{f_k + 1 - 2}) + (-1)^{k-1} \det(A_{f_k} \ldots A_{f_k + 1 - 2}) + (-1)^{k+f_k - 2} \det(A_{f_k} \ldots A_{f_k + 1 - 2}) \right\}.
\]
\[ A''_j := t(\varepsilon_{j+1} \cdots \varepsilon_{j+f_k-1}-1), \]
\[ A'''_j = t(\varepsilon_j \cdots \varepsilon_{j+f_k-2} \varepsilon_{j+f_k-1} \cdots \varepsilon_{j+f_k-1}). \]

Here, we have
\[
\begin{align*}
\det(A_{f_k} A_{f_k} \cdots A_{f_{k+1}-2}) &= H_{f_{k-1}, f_{k-1}}, \\
\det(A'''_{f_k} A'''_{f_{k+1}-2}) &= H_{f_{k+1}, f_{k-1}-1},
\end{align*}
\]
and by Lemma 5,
\[
\det(A'''_{f_k} \cdots A'''_{f_{k+1}-2})
= \begin{pmatrix}
 0 & (1)^{k-1} & (1)^k \\
 A''''_{f_k} \cdots A''''_{f_k+f_k-2} & \cdots & \cdots \\
 & \cdots & \cdots \\
 & \cdots & \cdots \\
 & & \cdots \\
 C_{f_k+1-1} & (1)^k & 0 \\
\end{pmatrix},
\]
where we put
\[ C_j = (\varepsilon_j \varepsilon_{j+1} \cdots \varepsilon_{j+f_k-1}). \]

Since \( C_{f_k+f_k-2+j} = C_{f_k+j} \) for \( j = 0, 1, \ldots, f_k-3 - 2 \) by Lemma 5, we have
\[
\det(A'''_{f_k} \cdots A'''_{f_{k+1}-2})
= (-1)^{(k-1)(f_k-3) + f_k-3 - 1 + (f_k-3-1)/2} \det \begin{pmatrix}
  C_{f_k} \\
  \vdots \\
  C_{f_k+f_k-2} \\
  C_{f_{k+1}-1} \\
\end{pmatrix}.
\]
Moreover it follows from Lemma 5 that
\[
\det \begin{pmatrix} C_{f_{k}} & & \\ \vdots & & \\ C_{f_{k}+f_{k-2}-2} & & \\ C_{f_{k+1}-1} & & \\ & \vdots & \\ & & C_{f_{k+1}+f_{k-2}-2} & \\ & & & C_{f_{k+1}-1} \end{pmatrix} = \det \begin{pmatrix} C_{f_{k+1}} & & \\ \vdots & & \\ C_{f_{k+1}+f_{k-2}-2} & & \\ C_{f_{k+1}-1} & & \\ & \vdots & \\ & & C_{f_{k+1}+f_{k-2}-2} & \\ & & & C_{f_{k+1}-1} \end{pmatrix} = (-1)^{f_{k-2}-1}H_{f_{k+1}-1,f_{k-2}},
\]
which implies
\[
\det(A'' \ldots A''_{f_{k+1}-2}) = \chi(k : 0, 3, 5)H_{f_{k+1}-1,f_{k-2}}.
\]
Thus by (22), (23), Theorem 3 and Lemma 16, we obtain
\[
H_{n,f_k-1} = \chi(k : 4)H_{f_{k-1}, f_{k-1}} + \chi(k : 0, 2)H_{f_{k+1}, f_{k+1}-1} + \chi(k : 1, 3, 4)H_{f_{k+1}-1, f_{k-2}} = \chi(k : 2, 3, 4, 5)f_{k-3} + \chi(k : 2, 3, 4, 5)f_{k-4} + \chi(k : 0, 1)f_{k-4} = \chi(k : 2, 3, 4, 5)f_{k-3},
\]
which is the first case of our lemma.

To prove the second case, assume that \(n \equiv_{k+1} f_{k-1} + 1\). Then as above we get
\[
H_{n,f_k-1} = \det \begin{pmatrix} A_{f_{k-1}+1} \ldots A_{f_{k-1}}A_{f_{k}} \ldots A_{f_{k+1}-1} \\ B'_{f_{k-1}+1} \ldots B'_{f_{k-1}}B'_{f_{k}} \ldots B'_{f_{k+1}-1} \end{pmatrix} = \det \begin{pmatrix} A_{f_{k-1}+1} & \ldots & \ldots & A_{f_{k-1}} & A_{f_{k}} & \ldots & A_{f_{k+1}-1} \\ 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ (-1)^k & \ldots & \ldots & (-1)^k \\ (-1)^{k-1} & \ldots & \ldots & (-1)^{k-1} \end{pmatrix} = (-1)^{(k-1)(f_{k-2}-1)}(-1)^{(f_{k-2}-1)f_{k-1}+(f_{k-2}-1)/2} \times \det(A_{f_{k}} \ldots A_{f_{k+1}-1}).
\]
Therefore, by Theorem 3 we get
\[
H_{n,f_k-1} = \chi(k : 0, 3, 4)\chi(k - 1 : 2)f_{k-2} = \chi(k : 0, 4)f_{k-2}. \quad \blacksquare
\]

**Theorem 4.** For any \(k, n, i \in \mathbb{N}\) with \(k \geq 1\), \(n \equiv_{k+1} i\) and \(0 \leq i < f_{k+1}\), we have
Therefore by Theorem 1, 
\[i \text{ and } f \text{ independent of } \tau \text{ for } i \text{ and } n\]

If those used in the proof of Lemma 15, we get, with the notation (20),

Therefore, by Theorems 3 and 4,

**Proof.** The first four cases follow from Lemmas 16 and 17. Note that for \(i = f_{k-1}\), the assertions in these lemmas coincide, so that \(H_{n,fk-1}\) is independent of \(\tau_{k+1}(n)\). Consider the last case, where \(\tau_{k+1}(n) = 0 \text{ and } f_{k-1} + 2 \leq i \leq f_{k+1} - 1\). We may assume that \(k \geq 2\). Then, with \(m = f_k - 1\) and \(i - f_k\) in place of \(i\) there, the condition (2) of Theorem 1 is satisfied. Therefore by Theorem 1, \(n \in R_m\), which implies that \(H_{n,fk-1} = 0\). ■

**Lemma 18.** For any \(n,m \in \mathbb{N}\) such that \(f_{k-2} + 1 \leq m \leq f_k - 2\), \(i \leq n\) and \(n - i \equiv k+1 \mod 0\) for some \(i,k \in \mathbb{Z}\) with \(k \geq 2\) and \(m + i = f_k\), we have

\[H_{n,m} = \chi(k : 2)\chi(k : 3, 4, 5)(-1)^{i/2}f_{k-3},\]

\[\overline{H}_{n,m} = \chi(k : 1, 4)\chi(k : 0, 1, 2)(-1)^{i/2}f_{k-3}.\]

**Proof.** First, we consider the case \(i < f_k - 2\). By arguments similar to those used in the proof of Lemma 15, we get, with the notation (20),

\[H_{n,m} = \det \begin{pmatrix} A_i A_{i+1} & \ldots & A_{f_k-1} & 0 & \ldots & 0 & A_{f_k-1} \\ 0 & \ldots & 0 & (-1)^k & (-1)^{k-1} & \ldots & 0 \end{pmatrix}.\]

Therefore, by Theorems 3 and 4,

\[H_{n,m} = (-1)^k(f_{k-2}-i+1) + [(f_{k-2}-i+1)/2]H_{i,f_k-1} - (-1)^{(k-1)}(f_{k-2}-i) + [(f_{k-2}-i)/2]H_{i,f_k-1}.
\]

If \(i = f_k - 2\), then the statement follows from Theorem 3.
Finally, we consider the case $f_{k-2} < i < f_{k-1}$. Then, setting
\begin{equation}
A'_j = \epsilon_j \epsilon_{j+1} \ldots \epsilon_{j+r-1},
\end{equation}
by Theorem 3 we obtain
\[H_{n,m} = \det(A'_{i+1}^{-1} A'_{i+2}^{-1} \ldots A'_{i+m-1}^{-1})
\]
\[
= \det\begin{pmatrix}
A'_{i+1}^{-1} & A'_{i+2}^{-1} & \ldots & A'_{i+2}^{-1} & A'_{i+3}^{-1} & \ldots & A'_{i+m-1}^{-1} \\
0 & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
\end{pmatrix}
\]
\[
= (-1)^{k-1}(f_{k-2}-i)(-1)(f_{k-2}-i)f_{k-2}+(f_{k-2}-i)/2) H_{f_{k-2},f_{k-1}} 
\]
\[
= \chi(k : 2) \chi(k : 3, 4, 5)^i(-1)^{[i/2]} f_{k-1}. \]

\textbf{Lemma 19.} For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \leq m \leq f_{k-2}$, $i \leq n$, $n - i \equiv k f_{k-1}$ for some $i, k \in \mathbb{Z}$ with $k \geq 2$ and $m + i = f_k$, we have
\[H_{n,m} = \chi(k : 1, 2, 4) \chi(k : 0, 1, 2)^i(-1)^{[i/2]} f_{k-2}, \]
\[\bar{H}_{n,m} = \chi(k : 2) \chi(k : 3, 4, 5)^i(-1)^{[i/2]} f_{k-3}. \]

\textbf{Proof.} By the same arguments and in the same notations as in the second part of the proof of Lemma 18, we obtain
\[H_{n,m} = \det(A'_{i+1}^{-1} A'_{i+2}^{-1} \ldots A'_{i+m-1}^{-1})
\]
\[
= \det\begin{pmatrix}
A'_{i+1}^{-1} & A'_{i+2}^{-1} & \ldots & A'_{i+2}^{-1} & A'_{i+3}^{-1} & \ldots & A'_{i+m-1}^{-1} \\
0 & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
(\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) & (\ldots) \\
\end{pmatrix}
\]
\[
= (-1)^{k-2}(f_{k-2}-i)(-1)(f_{k-2}-i)f_{k-2}+(f_{k-2}-i)/2) H_{f_{k-2},f_{k-1}} 
\]
\[
= \chi(k : 1, 2, 4) \chi(k : 0, 1, 2)^i(-1)^{[i/2]} f_{k-2}. \]

\textbf{Lemma 20.} For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \leq m \leq f_{k-2}$, $i \leq n$ and $n - i \equiv k f_{k-1}$ for some $i, k \in \mathbb{Z}$ with $k \geq 2$ and $m + i = f_k - 1$, we have
\[H_{n,m} = \chi(k : 0, 4) \chi(k : 3, 4, 5)^i(-1)^{[i/2]} f_{k-2}, \]
\[\bar{H}_{n,m} = \chi(k : 2, 3, 4, 5) \chi(k : 0, 1, 2)^i(-1)^{[i/2]} f_{k-3}. \]
Proof. The proof is similar to the first part of the proof of Lemma 18. With the notation in (20), we get

\[
H_{n,m} = \det \begin{pmatrix}
A_i A_{i+1} & \cdots & A_{f_k-1+i-1} & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & (\chi k) & (\chi k) & \cdots & (\chi k) \\
(\chi k) & (\chi k) & \cdots & (\chi k) & (\chi k) & \cdots & (\chi k) \\
\end{pmatrix}
\]

Hence, by Theorem 3

\[
H_{n,m} = \chi(k : 0, 4)\chi(k : 3, 4, 5)i(\chi k) f_k - 2. \quad \Box
\]

Lemma 21. For any \(n, m \in \mathbb{N}\) such that \(f_k + 1 \leq m \leq f_k - 2\), \(i \leq n\) and \(n - i \equiv k f_k - 1\) for some \(i, k \in \mathbb{Z}\) with \(k \geq 2\) and \(m + i = f_k - 1\), we have

\[
H_{n,m} = \chi(k : 2, 3, 4, 5)\chi(k : 0, 1, 2)i(\chi k) f_k - 3,
\]

\[
\bar{H}_{n,m} = \chi(k : 0, 4)\chi(k : 3, 4, 5)i(\chi k) f_k - 4.
\]

Proof. Since \(i = f_k - 1 - m\), we get \(1 \leq i \leq f_k - 1 - 2\). If \(i = f_k - 1\), then \(m = f_k - 1\) and \(n \equiv k f_k - 1\). Therefore, by Theorem 3, we get

\[
H_{n,m} = \chi(k : 0, 4) f_k - 3,
\]

which coincides with the required identity since

\[
\chi(k : 0, 1, 2)f_k - 1 = \chi(k : \{0, 1, 2\} \cap \{0, 3\}) = \chi(k : 0),
\]

\[
(\chi k) f_k - 2 = \chi(k : 0, 4).
\]

If \(i = f_k - 2\), then \(m = f_k - 1 - 1\) and \(n \equiv 0\). Therefore, by Theorem 4, we get

\[
H_{n,m} = \chi(k : 1 : 2, 4) f_k - 3,
\]

which coincides with the required statement since

\[
\chi(k : 0, 1, 2)f_k - 2 = \chi(k : \{0, 1, 2\} \cap \{1, 2, 4, 5\}) = \chi(k : 1, 2),
\]

\[
(\chi k) f_k - 2 = \chi(k : 3, 4).
\]

If \(f_k + 1 \leq i \leq f_k - 1 - 2\), then \(m + i \equiv 0\) with \(i' := i - f_k - 2\). Then, since \(m + i' = f_k - 1\) and \(f_k - 2 + 1 \leq m \leq f_k - 1 - 2\), applying Lemma 20, we obtain

\[
H_{n,m} = \chi(k : 1 : 0, 4) f_k - 3
\]

\[
= \chi(k : 1, 5) \chi(k : 0, 4, 5)i(\chi k : \{0, 4, 5\} \cap \{1, 2, 4, 5\})(\chi k) f_k - 3
\]

\[
= \chi(k : 1, 4) \chi(k : 0, 4, 5)i(\chi k : \{0, 4, 5\} \cap \{1, 2, 4, 5\})(\chi k) f_k - 3
\]

\[
= \chi(k : 2, 3, 4, 5) \chi(0, 1, 2)i(\chi k) f_k - 3.
\]
Now, we consider the case $1 \leq i \leq f_{k-2} - 2$. Then, with the notations in (24) and in (20), we get
\[
H_{n,m} = \det(A_{f_{k-1}+i}^{f_{k-1}} \cdots A_{f_{k-1}}^{f_{k-1}}A_{f_k}^{f_{k-1}} \cdots A_{f_{k+1}-1}^{f_{k+1}-1}) \\
= \det \begin{pmatrix} A_{f_{k-1}+i} & A_{f_{k-1}+i+1} & \cdots & A_{f_{k-2}} & A_{f_{k-1}+1}^{f_{k-1}} & \cdots & 0 \\ 0 & (1)^k & (1)^{k-1} & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (1)^k & (1)^{k-1} & \cdots & \cdots & 0 & \cdots & \cdots \\ \end{pmatrix}.
\]
Therefore, by arguments similar to those used in the first part of the proof of Lemma 17, we get
\[
H_{n,m} = (-1)^{(f_{k-2}-1-i)}(-1)^{f_{k-1}}(f_{k-2}-1-i)+[f_{k-2}-1-i)/2] \\
\times \{\det(A_{f_{k-1}}^{f_{k-1}}A_{f_{k}}^{f_{k}} \cdots A_{f_{k+1}-1}^{f_{k+1}-1}) + (-1)^{k-1} \det(A_{f_{k}}^{f_{k}} \cdots A_{f_{k+1}-1}^{f_{k+1}-1}) \\
+ (-1)^{k+f_{k-2}-1-i} \det(A_{f_{k}}^{f_{k}} \cdots A_{f_{k+1}-1}^{f_{k+1}-1})\},
\]
where we use the same notations as in the proof of Lemma 17 except for $A_j''s$ which are defined by
\[
A_j'' = t(\varepsilon_j \cdots \varepsilon_{j+f_{k-2}-i-2} \varepsilon_j+f_{k-2}-i \cdots \varepsilon_j+f_{k-1}-1).
\]
Then, following the arguments there, we get
\[
H_{n,m} = \chi(k:4)\chi(k:0, 1, 2)^i(-1)^{[i/2]}\{H_{f_{k-1},f_{k-1}} \\
+ (-1)^{k-1}H_{f_{k+1},f_{k-1}} + (-1)^{k+f_{k-2}-1-i}E\}
\]
with
\[
E := \det(A_{f_{k}}^{f_{k}} \cdots A_{f_{k+1}-1}^{f_{k+1}-1}) \\
= \det(A_{f_{k}}^{f_{k}} \cdots A_{f_{k+1}+f_{k-2}-i-2}^{f_{k+1}+f_{k-2}-i-2}A_{f_{k+1}+f_{k-2}-i}^{f_{k+1}+f_{k-2}-i} \cdots A_{f_{k+1}-1}^{f_{k+1}-1}) \\
= (-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} \det(A_{f_{k}+f_{k-2}-i}^{f_{k}+f_{k-2}-i} \cdots A_{f_{k+1}+f_{k-2}-i}^{f_{k+1}+f_{k-2}-i-2}) \\
= (-1)^{(f_{k-2}-2-i-1)(f_{k-3}+i)}H_{f_{k-2}-i, f_{k-1}-1},
\]
where we have used Lemma 5. Therefore, by Theorems 3 and 4, we have
\[
H_{n,m} = \chi(k:4)\chi(k:0, 1, 2)^i(-1)^{[i/2]}\{\chi(k-1:1, 2, 4)f_{k-3} \\
+ (-1)^{k-1}\chi(k-1:2, 3, 4, 5)f_{k-4} \\
+ (-1)^{k+f_{k-2}-1-i}(-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} \\
\times \chi(k-1:1, 2, 3, 5)\chi(k-1:1, 4)f_{k-4} \\
= \chi(k:2, 3, 4, 5)\chi(k:0, 1, 2)^i(-1)^{[i/2]}f_{k-3}. \quad \blacksquare
\]
4. Tiling for $H_{n,m}$ and $\overline{H}_{n,m}$. In this section, we collect the values of $H_{n,m}$ and $\overline{H}_{n,m}$ obtained in the last section and arrange them in the quarter plane $\Omega := \{0, 1, \ldots\} \times \{1, 2, \ldots\}$. We will tile $\Omega$ by the following tiles on which the values $H_{n,m}$ are written in. That is, $U_1 := V_1 := \{(1, -1)\}$, and for $k \geq 2$,

$$U_k := \{(i, j) \in \mathbb{Z}^2 : 0 \leq i + j \leq f_{k-1} - 1, \ -f_{k-1} \leq j \leq -1\},$$

$$V_k := \{(i, j) \in \mathbb{Z}^2 : 0 \leq i + j \leq f_{k-2} - 1, \ -f_{k-2} \leq j \leq -1\},$$

with the written-in values $u_k : U_k \rightarrow \mathbb{Z}$ and $v_k : V_k \rightarrow \mathbb{Z}$ given by $u_1(1, -1) := 0$, $v_1(1, -1) := 1$, and for $k \geq 2$,

$$u_k(i, j) := \begin{cases} 
\chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{i/2}f_{k-3} & (i + j = 0), \\
\chi(k : 0, 3, 4)\chi(k : 0, 3)^i f_{k-3} & (j = -f_{k-1}), \\
\chi(k : 3, 5)\chi(k : 2, 3, 4)^i(-1)^{i/2}f_{k-3} & (i + j = f_{k-1} - 1), \\
\chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-3} & (j = -1), \\
0 & \text{(otherwise)}, 
\end{cases}$$

$$v_k(i, j) := \begin{cases} 
\chi(k : 1, 2, 4)\chi(k : 0, 1, 2)^i(-1)^{i/2}f_{k-2} & (i + j = 0), \\
\chi(k : 2, 3, 5)\chi(k : 2, 5)^i f_{k-2} & (j = -f_{k-2}), \\
\chi(k : 0, 1, 2, 3)\chi(k : 1, 2, 3)^i(-1)^{i/2}f_{k-2} & (i + j = f_{k-2} - 1), \\
\chi(k : 0, 1)\chi(k : 1, 4)^i f_{k-2} & (j = -1), \\
0 & \text{(otherwise)}, 
\end{cases}$$

and with $\overline{u}_k : U_k \rightarrow \mathbb{Z}$ and $\overline{v}_k : V_k \rightarrow \mathbb{Z}$ given $\overline{u}_1(1, -1) := 1$, $\overline{v}_1(1, -1) := 0$, and for $k \geq 2$,

$$\overline{u}_k(i, j) := \begin{cases} 
\chi(k : 1, 4)\chi(k : 0, 1, 2)^i(-1)^{i/2}f_{k-4} & (i + j = 0), \\
\chi(k : 0, 1, 2)\chi(k : 0, 3)^i f_{k-4} & (j = -f_{k-1}), \\
\chi(k : 1, 2, 3, 4)\chi(k : 0, 1, 5)^i(-1)^{i/2}f_{k-4} & (i + j = f_{k-1} - 1), \\
\chi(k : 0, 1)\chi(k : 1, 4)^i f_{k-4} & (j = -1), \\
0 & \text{(otherwise)}, 
\end{cases}$$

$$\overline{v}_k(i, j) := \begin{cases} 
\chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{i/2}f_{k-3} & (i + j = 0), \\
\chi(k : 3)\chi(k : 2, 5)^i f_{k-3} & (j = -f_{k-2}), \\
\chi(k : 2, 4)\chi(k : 0, 4, 5)^i(-1)^{i/2}f_{k-3} & (i + j = f_{k-2} - 1), \\
\chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-3} & (j = -1), \\
0 & \text{(otherwise)}. 
\end{cases}$$

For $k \geq 1$ let

$$U_k := \{(n, f_k) : n \in \mathbb{N} \text{ and } n \equiv_k 1\},$$

$$V_k := \{(n, f_k) : n \in \mathbb{N} \text{ and } n \equiv_k 2f_{k+1} + f_k - 1\},$$

$$T_k := (V_k + (-f_{k-2}, f_k)) \cap \Omega,$$

where $V + (x, y) := \{(v + x, w + y) : (v, w) \in V\}$ for $V \subset \mathbb{Z}^2, (x, y) \in \mathbb{Z}^2$. 


Theorem 5. We have
\[ \Omega = \bigcup_{k=1}^{\infty} \left( \bigcup_{(i,j) \in U_k} (U_k + (i,j)) \cup \bigcup_{(i,j) \in V_k} (V_k + (i,j)) \cup T_k \right), \]
where the right hand side is a disjoint union, so that \( \Omega \) is tiled by the \( U_k \)'s, \( V_k \)'s and \( T_k \)'s. Moreover, for any \((n,m) \in \Omega\), if \((n,m) = (i,j) + (i',j')\) with \((i,j) \in U_k\) and \((i',j') \in U_k\), then \( H_{n,m} = u_k(i,j) \) and \( \overline{H}_{n,m} = \overline{u}_k(i,j) \). Also, if \((n,m) = (i,j) + (i',j')\) with \((i,j) \in V_k\) and either \((i',j') \in V_k\) or \((i',j') = (-f_{k-2},f_k)\), then \( H_{n,m} = v_k(i,j) \) and \( \overline{H}_{n,m} = \overline{v}_k(i,j) \). Furthermore, in this tiling, the tiles \( U_k, V_k \) and \( T_k \) with \( k \geq 2 \) are followed by the sequences of smaller tiles \( U_{k-1}V_{k-1}U_{k-1}, U_{k-1} \) and \( U_{k-1} \), respectively, as shown in Figure 1.

\[ \text{Fig. 1. Tiling for } H_{n,m}. \]

Proof. Take an arbitrary point \((n,m) \in \Omega\). Let \( f_{k-1} \leq m < f_k \). If \( n + m - f_k \geq 0 \), define \( 0 \leq i < f_{k+2} \) by \( i \equiv_{k+2} n \).

Case 1: \( n + m - f_k < 0 \). We get \((n,m) \in T_k\).

Case 2: \( 0 \leq i < f_{k-1} \). We get \((n,m) \in U_k + (n + m - i - f_k, f_k)\).
Case 3: $f_{k-1} \leq i < f_{k+1}$. We get $(n,m) \in U_{k+1} + (n+m-i-f_{k+1}, f_{k+1})$.

Case 4: $f_{k+1} \leq i < f_{k+1} + f_{k-1}$. We get $(n,m) \in U_k + (n+m-i+f_{k-1}, f_k)$.

Case 5: $f_{k+1} + f_{k-1} \leq i < f_{k+2}$. We get $(n,m) \in V_k + (n+m-i+2f_{k-1}, f_k)$.

The fact that the written-in values coincide with $H_{n,m}$ and $\bar{H}_{n,m}$ follows from Lemma 18 (first case in $u_k$ and $\pi_k$), Theorem 3 (second case), Lemma 21 (third case), Theorem 4 (fourth case), Corollary 3 (fifth case), Lemma 19 (first case in $v_k$ and $\bar{v}_k$), Theorem 3 (second case), Lemma 20 (third case), Lemma 20 (fourth case) and Corollary 3 (fifth case). The $m$ in the preceding lemmas and theorems coincides with $f_k + j$ in Theorem 5 while the meaning of the symbols $k,i,n$ is not necessarily the same. ■

5. Padé approximation. Let $\varphi = \varphi_0 \varphi_1 \ldots$ be an infinite sequence over a field $\mathbb{K}$, $\hat{H}_{n,m} := H_{n,m}(\varphi)$ be the Hankel determinant (3), and $\varphi(z)$ the formal Laurent series (4) with $h = -1$. We also denote the Hankel matrices

\[
\hat{M}_{n,m} := (\varphi_{n+i+j})_{i,j=0,1,...,m-1} \quad (n = 0,1,...; m = 1,2,...),
\]

so that $\hat{H}_{n,m} = \det \hat{M}_{n,m}$.

The following proposition is well known ([1], for example). But we give a proof for self-containment.

Proposition 1. (1) For any $m = 1,2,...$, a Padé pair $(P,Q)$ of order $m$ for $\varphi$ exists. Moreover, for each $m$, the rational function $P/Q \in \mathbb{K}(z)$ is determined uniquely for such Padé pairs $(P,Q)$.

(2) For any $m = 1,2,...$, $m$ is a normal index for $\varphi$ if and only if $\hat{H}_{0,m}(\varphi) \neq 0$.

Proof. Let

\[
P = p_0 + p_1 z + p_2 z^2 + \ldots + p_m z^m,
Q = q_0 + q_1 z + q_2 z^2 + \ldots + q_m z^m.
\]

Then the condition $\| Q \varphi - P \| < \exp(-m)$ is equivalent to

\[
-q_m \varphi_0 = 0, \quad -p_m = 0,
q_m \varphi_0 - p_m = 0, \quad \ldots
\]

(26)

\[
q_0 \varphi_0 + \ldots + q_m \varphi_{m-1} - p_0 = 0, \quad q_0 \varphi_1 + \ldots + q_m \varphi_m = 0, \quad \ldots
\]

\[
q_0 \varphi_{m-1} + q_1 \varphi_{m-2} + \ldots + q_m \varphi_{2m-1} = 0.
\]
Furthermore, the system (26) for \((q_0 q_1 \ldots q_m)\) is equivalent to

\[
(q_0 q_1 \ldots q_{m-1}) \hat{M}_{0,m} + q_m (\varphi_m \varphi_{m+1} \ldots \varphi_{2m-1}) = (00 \ldots 0),
\]

where \((p_0 p_1 \ldots p_m)\) is determined by \((q_0 q_1 \ldots q_m)\) by the upper half of (26).

There are two cases.

**Case 1:** \(\hat{H}_{0,m} = 0\). In this case, since \(\det \hat{M}_{0,m} = \hat{H}_{0,m} = 0\), there exists a nonzero vector \((q_0 q_1 \ldots q_{m-1})\) such that \((q_0 q_1 \ldots q_{m-1}) \hat{M}_{0,m} = 0\). Then (27) is satisfied with this \((q_0 q_1 \ldots q_{m-1})\) and \(q_m = 0\).

**Case 2:** \(\hat{H}_{0,m} \neq 0\). In this case, since \(\det \hat{M}_{0,m} = \hat{H}_{0,m} \neq 0\), there exists a unique vector \((q_0 q_1 \ldots q_{m-1})\) such that

\[
(q_0 q_1 \ldots q_{m-1}) \hat{M}_{0,m} = -(\varphi_m \varphi_{m+1} \ldots \varphi_{2m-1}).
\]

Then (27) is satisfied with this \((q_0 q_1 \ldots q_{m-1})\) and \(q_m = 1\).

Thus, a Padé pair of order \(m\) exists. Moreover, by the above arguments, a Padé pair \((P, Q)\) of order \(m\) with \(\deg Q < m\) exists if and only if \(\hat{H}_{0,m} = 0\), since if \(\hat{H}_{0,m} \neq 0\), then by (27), \(q_m = 0\) implies \((q_0 q_1 \ldots q_{m-1}) = (00 \ldots 0)\), and hence \(Q = 0\).

Now we prove that for any Padé pairs \((P, Q)\) and \((P', Q')\) of order \(m\), we have \(P/Q = P'/Q'\). By (5), we have

\[
\|\varphi - P/Q\| < \exp(-n - \deg Q), \quad \|\varphi - P'/Q'\| < \exp(-m - \deg Q').
\]

Hence,

\[
\|P/Q - P'/Q'\| < \exp(-m - \deg Q \land \deg Q').
\]

Therefore,

\[
\|PQ' - P'Q\| < \exp(-m + \deg Q \lor \deg Q') \leq 1.
\]

Since \(PQ' - P'Q\) is a polynomial of \(z\), \(\|PQ' - P'Q\|\) is either 0 or not less than 1. Hence, the above inequality implies \(PQ' - P'Q = 0\). ■

In view of (26), without loss of generality, we can put

\[
P = p_0 + p_1 z + p_2 z^2 + \ldots + p_{m-1} z^{m-1},
Q = q_0 + q_1 z + q_2 z^2 + \ldots + q_m z^m.
\]

**Theorem 6.** Let \((P, Q)\) be the normalized Padé pair for \(\varphi\) with \(\deg Q\) as its normal index \(m\) with \(P, Q\) given by (29). Then
(1) \( Q(z) = \hat{H}_{0,m}^{-1} \det(z \hat{M}_{0,m} - \hat{M}_{1,m}). \)

(2) \( \det(zI - \hat{M}_{0,m}) \) is equal to

\[
\begin{array}{ccccccccccccccccccccccccccc}
z & z & 1 \\
\vdots & \vdots & \vdots \\
z & z & 1 \\
p_0 & \ldots & p_{m-2} & q_0 & \ldots & q_{m-1} & 1 \\
p_1 & \ldots & p_{m-1} & q_2 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_0 & \ldots & : & : & : & : & : & : & : & : & : & 1
\end{array}
\]

where \( I \) is the unit matrix of size \( m \).

(3) We have

\[
\hat{H}_{0,m} = (-1)^{\lfloor m/2 \rfloor} \prod_{z; Q(z) = 0} P(z) = (-1)^{\lfloor m/2 \rfloor} p_k^m \prod_{z; P(z) = 0} Q(z),
\]

where \( \prod_{z; R(z) = 0} \) denotes the product over all the roots of the polynomial \( R(z) \) with their multiplicity and \( p_k \) is the leading coefficient of \( P(z) \), that is, \( p_{m-1} = \ldots = p_{k+1} = 0, p_k \neq 0 \) if \( P(z) \) is not the zero polynomial, otherwise \( p_k = 0 \).

Proof. (1) Note that \( q_m = 1 \) by the assumption that \((P,Q)\) is the normalized Padé pair. By (28), we have

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & \ddots \\
-q_0 & -q_1 & \ldots & -q_{m-2} & -q_{m-1} & 1
\end{pmatrix}
\]

\[
\hat{M}_{0,m} = \hat{M}_{1,m}.
\]

Since

\[
\hat{H}_{0,m} = \det \hat{M}_{0,m} \neq 0
\]
by the normality of the index $m$, it follows that

$$Q(z) = \det \left( zI - \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ -q_0 & -q_1 & \ldots & -q_{m-2} & -q_{m-1} \end{pmatrix} \right)$$

$$= \det(zI - \hat{M}_{1,m} \hat{M}_{0,m}^{-1})$$

$$= \hat{H}_{0,m}^{-1} \det(z \hat{M}_{0,m} - \hat{M}_{1,m}).$$

(2) We define the matrices:

$$P_m := \begin{pmatrix} p_{m-1} & p_{m-2} & \cdots & p_1 & p_0 \\ p_{m-2} & & \cdots & p_0 & \\ \vdots & & \ddots & \vdots & \\ p_1 & & \cdots & 0 & \\ p_0 & & & & \end{pmatrix},$$

$$P'_{m-1} := \begin{pmatrix} p_{m-1} & p_{m-2} & \cdots & p_2 & p_1 \\ & & \ddots & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ & & & & p_2 \\ & & & & \end{pmatrix},$$

$$Q_m := \begin{pmatrix} 1 & q_{m-1} & 1 & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ q_1 & q_2 & \cdots & q_{m-1} & 1 \end{pmatrix},$$

$$Q'_m := \begin{pmatrix} 0 & 1 & q_{m-1} \\ & \ddots & \vdots & \vdots \\ & \ddots & \vdots & \vdots \\ 1 & q_{m-1} & \cdots & q_2 & q_1 \end{pmatrix}.$$
\[
Q''_{m-1} := \begin{pmatrix}
1 & 1 & 0 \\
q_{m-1} & \ddots & \\
\vdots & \ddots & \ddots \\
q_2 & \cdots & q_{m-1} & 1
\end{pmatrix},
\]

\[
Q_{m,m-1} := \begin{pmatrix}
q_1 & q_2 & \cdots & q_{m-2} & q_{m-1} \\
q_0 & q_1 & \cdots & q_{m-3} & q_{m-2} \\
q_0 & q_1 & \cdots & q_{m-3} & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & \cdots & q_1 & q_0
\end{pmatrix},
\]

\[
\Phi_{m-1} := \begin{pmatrix}
0 & \varphi_0 & \varphi_1 \\
\varphi_0 & \varphi_1 & \cdots & \cdots & \varphi_{m-3} \\
\varphi_1 & \cdots & \cdots & \cdots & \varphi_{m-2}
\end{pmatrix},
\]

We denote by \(O\) the zero matrices of various sizes. We also denote by \(I_n\) the unit matrix of size \(n\). By (26), we have

\[
\det(zI - \hat{M}_{0,m}) = \det \left( z \begin{pmatrix} O & O & O \\ O & I_m & \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \\ Q_{m}^{-1}Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \right)
\]

\[
= \det \left( \begin{pmatrix} I_{m-1} & O \\ O & Q_{m} \end{pmatrix} \begin{pmatrix} z \begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \\ Q_{m}^{-1}Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \end{pmatrix} \right)
\]

\[
= \det \left( z \begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \\ Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \right)
\]

\[
= \det \left( \begin{pmatrix} z \begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \\ Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_{m-1} & O \Phi_{m-1} \\ O & I_m \end{pmatrix} \right)
\]

\[
= \det \left( z \begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right),
\]

where we use (26) to get the last equality. Hence

\[
\det(zI - \hat{M}_{0,m}) = \det \left( z \begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & 0 \Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right)
\]
Hankel determinants and Padé approximation

\[
= \det \left( \begin{pmatrix} Q''_{m-1} & O \\ O & I_m \end{pmatrix} \right) \left( \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & \Phi_{m-1} \end{pmatrix} \right)
\]

\[
= \det \left( z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -Q''_{m-1} & O \\ Q_{m,m-1} & -P'_{m-1} \end{pmatrix} \right)
\]

\[
= (-1)^m \det \left( \begin{pmatrix} Q''_{m-1} & P'_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} - z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} \right)
\]

\[
= (-1)^m \det \left( \begin{pmatrix} I_m & O & zI_m \\ O & Q''_{m-1} & O \\ Q'_{m} & Q_{m,m-1} & P_{m} \end{pmatrix} \right),
\]

which implies (2).

(3) By (2), we have

\[
\hat{H}_{0,m} = (-1)^m \det (0I - \hat{M}_{0,m})
\]

\[
= (-1)^{[m/2]} \left| \begin{array}{cccc}
p_{m-1} & p_{m-2} & \cdots & q_{m-1} \\
p_{m-2} & p_{m-1} & \cdots & \cdots \\
p_1 & \cdots & p_{m-1} & q_2 \\
p_0 & \cdots & p_{m-2} & p_{m-1} & q_1 & \cdots & q_{m-1} \\
p_0 & \cdots & p_{m-2} & q_0 & \cdots & q_{m-2} \\
p_0 & p_1 & \cdots & \cdots & q_1 \\
p_0 & \cdots & p_0 & q_0 \\
\end{array} \right|
\]

which completes the proof since the last determinant is Sylvester’s determinant for \(P(z)\) and \(Q(z)\).

For a finite or infinite sequence \(a_0(z), a_1(z), \ldots\) of elements in \(K((z^{-1}))\), we use the notation

\[
[a_0(z); a_1(z), a_2(z), \ldots, a_n(z)] := a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \cdots + \frac{1}{a_n(z)}}}
\]

and

\[
[a_0(z); a_1(z), a_2(z), \ldots] := \lim_{n \to \infty} [a_0(z); a_1(z), a_2(z), \ldots, a_n(z)]
\]

provided that the limit exists, where the limit is taken with respect to the metric induced by the nonarchimedean norm in \(K((z^{-1}))\).
We denote by $T$ admissible. We say a continued fraction is admissible if and only if (32) holds. We remark that a continued fraction (33) is admissible if and only if $g_n(z) = 0$, and

$$p_n(z) = |a_0(z); a_1(z), a_2(z), \ldots, a_n(z)| \in \mathbb{K}((z^{-1})) \cup \{\infty\} \quad (n \geq 0),$$

where we mean $\psi/0 := \infty$ for $\psi \in \mathbb{K}((z^{-1})) \setminus \{0\}$, and $\psi + \infty := \infty, \psi/\infty := 0$ for $\psi \in \mathbb{K}((z^{-1}))$. By using (31), it can be shown that the limit (30) always exists in the set $\mathbb{K}((z^{-1}))$ as far as

$$a_n(z) \in \mathbb{K}[z] \quad (n \geq 0), \quad \deg a_n(z) \geq 1 \quad (n \geq 1).$$

For $\varphi(z) \in \mathbb{K}((z^{-1}))$ given by (4), we denote by $[\varphi(z)]$ the polynomial part of $\varphi(z)$, which is defined as follows:

$$[\varphi(z)] := \sum_{k=0}^{h} \varphi_h z^{-k+h} \in \mathbb{K}[z].$$

We denote by $T$ the mapping $T : \mathbb{K}((z^{-1})) \setminus \{0\} \to \mathbb{K}((z^{-1}))$ defined by

$$T(\psi(z)) := \frac{1}{\psi(z)} - \left[ \frac{1}{\psi(z)} \right] \quad (\psi(z) \in \mathbb{K}((z^{-1})) \setminus \{0\}).$$

Then, for any given $\varphi(z) \in \mathbb{K}((z^{-1}))$, we can define the continued fraction expansion of $\varphi(z)$:

$$\varphi(z) = \left\{ \begin{array}{ll} [a_0(z); a_1(z), a_2(z), \ldots, a_{N-1}(z)] & \text{if } \varphi(z) \in \mathbb{K}(z), \\ [a_0(z); a_1(z), a_2(z), a_3(z), \ldots] & \text{otherwise} \end{array} \right.$$  

with $a_n(z)$ satisfying (32) according to the following algorithm.

Continued Fraction Algorithm:

$$a_0(z) = [\varphi(z)], \quad a_n(z) = \left\lfloor \frac{1}{T^{n-1}(\varphi(z) - a_0(z))} \right\rfloor,$$

$$N = N(\varphi(z)) := \inf\{m : T^{m-1}(\varphi(z)) = 0\} \quad (\inf \emptyset := \infty).$$

We note that if $\varphi(z) \in \mathbb{K}(z)$, then $N < \infty$; if $\varphi(z) \in \mathbb{K}((z^{-1})) \setminus \mathbb{K}(z)$, then $N = \infty$ and the continued fraction (33) converges to the given $\varphi(z) \in \mathbb{K}(z)$. We say a continued fraction is admissible if it is obtained by the algorithm. We remark that a continued fraction (33) is admissible if and only if (32) holds.

The following proposition is known [2], but we give a proof for completeness.
PROPOSITION 2. The set of all $P/Q \in \mathbb{K}(z)$ for Padé pairs $(P,Q)$ for $\varphi(z) \in \mathbb{K}(z)$ coincides with the set of convergents $p_n(z)/q_n(z)$ ($0 \leq n < N$) of the continued fraction expansion of $\varphi(z)$. Moreover, $m$ is a normal index if and only if $m$ is a degree of $q_n(z)$ for some $n = 0, 1, 2, \ldots$ (with $n < N$ if $\varphi(z) \in \mathbb{K}(z)$).

**Proof.** Note that

$$\varphi(z) = \frac{(a_n(z) + T^n(\varphi(z) - a_0))p_{n-1}(z) + p_{n-2}(z)}{(a_n(z) + T^n(\varphi(z) - a_0))q_{n-1}(z) + q_{n-2}(z)},$$

$$(-1)^n = p_{n-1}(z)q_{n-2}(z) - p_{n-2}(z)q_{n-1}(z).$$

Hence, we have

$$\|q_n(z)\varphi(z) - p_n(z)\| = \left\| \frac{(-1)^nT^n(\varphi(z) - a_0(z))}{q_n(z) + T^n(\varphi(z) - a_0(z))q_{n-1}(z)} \right\|$$

$$= \exp(-\deg a_{n+1}(n) - \deg q_n(z)),$$

so that

$$\|q_n(z)\varphi(z) - p_n(z)\| < \exp(-\deg q_n(z)) \quad (n < N).$$

In the case $N < \infty$, the left-hand side of (34) turns out to be 0 for $n = N - 1$. Therefore, $(p_n(z), q_n(z))$ is a Padé pair of order $m = \deg q_n(z)$ for all $m \in \{\deg q_n(z) : 0 \leq n < N\}$.

Conversely, for any $k = 1, 2, \ldots$, let $(P,Q)$ be a Padé pair of order $k$. Let $\deg q_n(z) \leq k < \deg q_{n+1}(z)$ for some $n = 0, 1, 2, \ldots$ with $n < N$ (deg $q_N(z) := \infty$). Then, since $\deg Q \leq k < \deg q_{n+1}$, it follows from (34) that

$$\|\varphi(z) - p_n(z)/q_n(z)\| = \exp(-\deg q_n(z) - \deg q_{n+1}(z))$$

$$< \exp(-\deg q_n(z) - \deg Q).$$

Since $(P,Q)$ is a Padé pair of order $k$, we have

$$\|\varphi(z) - P/Q\| < \exp(-k - \deg Q) \leq \exp(-\deg q_n(z) - \deg Q).$$

Therefore,

$$\left\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \right\| < \exp(-\deg q_n(z) - \deg Q).$$

On the other hand, if $P/Q \neq p_n(z)/q_n(z)$, then

$$\left\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \right\| = \left\| \frac{Pq_n(z) - Qp_n(z)}{Qq_n(z)} \right\|$$

$$\geq \exp(-\deg q_n(z) - \deg Q),$$

which is a contradiction. Thus $P/Q = p_n(z)/q_n(z)$.

Note that $p_n(z)/q_n(z)$ is irreducible for any $n = 1, 2, \ldots$ with $n < N$, since $p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$. Let $m = \deg q_n(z)$ for some $n = 1, 2, \ldots$ with $n < N$. Take any Padé pair $(P,Q)$ of order $m$. Then $\deg Q \leq m$. On the
other hand, by the above argument, \( P/Q = p_n(z)/q_n(z) \). Since \( p_n(z)/q_n(z) \) is irreducible, this implies that \( \deg Q \geq \deg q_n(z) = m \). Thus, \( m \) is a normal index.

Conversely, let \( m \geq 0 \) be any normal index. Take any Padé pair \((P,Q)\) of order \( m \). Then, by the above argument, there exists \( n = 0,1,2,\ldots \) with \( n < N \) such that \( P/Q = p_n(z)/q_n(z) \). Hence the irreducibility of \( p_n(z)/q_n(z) \) implies \( \deg q_n(z) \leq \deg Q \leq m \). Hence, \((p_n(z),q_n(z))\) is a Padé pair of order \( m \). Since \( m \) is a normal index, \( \deg q_n(z) = m \).

We now obtain the continued fraction expansions for

\[ \varphi_\varepsilon(z) = \hat{\varepsilon}_0 z^{-1} + \hat{\varepsilon}_1 z^{-2} + \hat{\varepsilon}_2 z^{-3} + \ldots \in \mathbb{Q}(z^{-1}) \]

corresponding to the Fibonacci words \( \hat{\varepsilon} = \varepsilon(a,b) \) with \( (a,b) = (1,0) \) and \( (a,b) = (0,1) \). As in Section 3, we use the notations \( \varepsilon \) and \( \hat{\varepsilon} \) for them. The proofs in the following theorems are given only for \( \varepsilon \), since the proof is similar for \( \hat{\varepsilon} \). In [3], J. Tamura gave the Jacobi–Perron–Parusnikov expansion for a vector consisting of Laurent series with coefficients given by certain substitutions, which contains the following as its special case (cf. the footnote on p. 301 of [3]):

**Proposition 3.** We have

\[ (z-1)\varphi_\varepsilon(z) = [0; z^{f_{-2}}, z^{f_{-1}}, z^{f_0}, z^{f_1}, z^{f_2}, \ldots]. \]

**Theorem 7.** We have the following admissible continued fraction for \( \varphi_\varepsilon(z) \) and \( \varphi_{\hat{\varepsilon}}(z) \):

\[ \varphi_\varepsilon(z) = [0; a_1, a_2, a_3, \ldots], \quad \varphi_{\hat{\varepsilon}}(z) = [0; \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots] \]

with

\[
\begin{align*}
    a_1 &= z, \quad a_2 = -z + 1, \quad a_3 = -\frac{1}{2}(z + 1), \\
    a_{2n+2} &= (-1)^{n-1} f_n^2 (z^{f_n-1} + z^{f_n-2} + \ldots + 1), \\
    a_{2n+3} &= (-1)^{n-1} \frac{1}{f_n f_{n+1}} (z - 1) \quad (n = 1, 2, \ldots),
\end{align*}
\]

and

\[
\begin{align*}
    \bar{a}_1 &= z^2, \quad \bar{a}_2 = -z, \\
    \bar{a}_{2n+1} &= (-1)^{n-1} f_{n-1}^2 (z^{f_{n-1}-1} + z^{f_{n-2}} + \ldots + 1), \\
    \bar{a}_{2n+2} &= (-1)^{n-1} \frac{1}{f_{n-1} f_n} (z - 1) \quad (n = 1, 2, \ldots).
\end{align*}
\]

**Proof.** We put

\[
\begin{align*}
    \theta_n &:= [0; z^{f_n}, z^{f_{n+1}}, z^{f_{n+2}}, \ldots] \quad (n \geq -2), \\
    \xi_n &:= (-1)^{n-1} \frac{f_n^2 z^{f_n} + f_{n-1} f_n + f_{n-1} f_{n+1}}{z - 1} \quad (n \geq 1),
\end{align*}
\]
ηₙ := \((-1)^{n-1} \frac{z - 1}{fₙf_{n+1} + fₙ²\theta_{n+1}} \) (\(n \geq 1\)),
cₙ := \((-1)^{n-1} fₙ²(z fₙ⁻¹ + z fₙ⁻² + \ldots + 1) \) (\(n \geq 1\)),
dₙ := \((-1)^{n-1} \frac{1}{fₙf_{n+1}}(z - 1) \) (\(n \geq 1\)).

Then

\begin{align*}
ξₙ &= [cₙ; ηₙ] \quad (= cₙ + 1/ηₙ),
ηₙ &= [dₙ; ξₙ].
\end{align*}

Using

\begin{align*}
θ⁻¹ &= z fₙ + θ_{n+1}
\end{align*}

and Proposition 3, we get

\(\varphi_ε(z) = \frac{θ₋²}{z - 1} \quad (∥θ₋²/(z - 1)∥ < 1)\)

\[
= [0; (z - 1)θ₋¹]
= [0; z - 1 + (z - 1)θ₋₁] \quad (∥1 + (z - 1)θ₋₁∥ < 1)
= \left[0; z, \frac{θ₋¹}{θ₋₁ + z - 1} \right] = \left[0; z, \frac{z + θ₀}{z - 1 - θ₀} \right]
= \left[0; z, -z + 1 + \frac{1 + (-z + 2)θ₀}{-1 - θ₀} \right] \quad (\left\| \frac{1 + (-z + 2)θ₀}{-1 - θ₀} \right\| < 1)
= \left[0; z, -z + 1, \frac{-1 - θ₀⁻¹}{-z + 2 + θ₀⁻¹} \right]
= \left[0; z, -z + 1, \frac{-z - 1 - θ₁}{2 + θ₁} \right]
= \left[0; z, -z + 1, \frac{1}{2} (z + 1), \frac{4θ₀⁻¹ + 2}{z - 1} \right]
= \left[0; z, -z + 1, \frac{1}{2} (z + 1), \frac{4z + 2 + 4θ₂}{z - 1} \right].
\]

Hence, we have

\begin{align*}
f(z) &= [0; z, -z + 1, -\frac{1}{2} (z + 1), ξ₁] \quad (∥ξ⁻¹∥ < 1).
\end{align*}

From (35) and (36), it follows that

\[
f(z) = [0; z, -z + 1, -\frac{1}{2} (z + 1)c₁, d₁, \ldots, cₙ, dₙ, ξₙ₊₁]
= [0; z, -z + 1, -\frac{1}{2} (z + 1)c₁, d₁, c₂, d₂, \ldots]
\]

which completes the proof for \(\varphi_ε(z)\).

Starting from the identity \(\varphi_ε(z) = (1 - θ₋²)/(z - 1)\) instead of \(\varphi_ε(z) = θ₋²/(z - 1)\), we can get the admissible continued fraction for \(\varphi_ε(z)\) in a similar fashion. ■
Theorem 8. The numerator \( p_n := p_n(z) \) (\( \overline{p}_n := \overline{p}_n(z) \), resp.) and the denominator \( q_n := q_n(z) \) (\( \overline{q}_n := \overline{q}_n(z) \), resp.) of the \( n \)-th convergent of the continued fraction expansion for \( \varphi(z) \) (and \( \varphi(x) \), resp.) are given as follows:

\[
\begin{align*}
p_0 &= 0, \quad p_1 = 1, \quad p_2 = -z + 1, \\
q_0 &= 1, \quad q_1 = z, \quad q_2 = -z^2 + z + 1, \\
p_{2n-1} &= \frac{1}{f_{n-1}}(\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}), \\
p_{2n} &= (-1)^n \{ f_{n-1} z f_n (\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}) \\
&\quad - f_{n-2} (\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}) \}/(z - 1), \\
q_{2n-1} &= \frac{1}{f_{n-1}}(z f_n - 1), \\
q_{2n} &= (-1)^n \{ f_{n-1} z f_n (z f_{n-1} + z f_{n-2} + \ldots + 1) \\
&\quad - f_{n-2} (z f_{n-1} + z f_{n-2} + \ldots + 1) \} \quad (n = 2, 3, \ldots),
\end{align*}
\]

and

\[
\begin{align*}
\overline{p}_0 &= 0, \quad \overline{p}_1 = 1, \\
\overline{q}_0 &= 1, \quad \overline{q}_1 = z^2, \\
\overline{p}_{2n-2} &= -\frac{1}{f_{n-2}}(\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}), \\
\overline{p}_{2n-1} &= (-1)^{n-1} \{ f_{n-2} z f_n (\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}) \\
&\quad - f_{n-3} (\varepsilon_0 z f_{n-1} + \varepsilon_1 z f_{n-2} + \ldots + \varepsilon f_{n-1}) \}/(z - 1) + f_{n-2},
\end{align*}
\]

\[
\begin{align*}
\overline{q}_{2n-2} &= -\frac{1}{f_{n-2}}(z f_n - 1), \\
\overline{q}_{2n-1} &= (-1)^n \{ f_{n-2} z f_n (z f_{n-1} + z f_{n-2} + \ldots + 1) \\
&\quad - f_{n-3} (z f_{n-1} + z f_{n-2} + \ldots + 1) \} \quad (n = 2, 3, \ldots),
\end{align*}
\]

where \( p_{2n} \) and \( \overline{p}_{2n-1} \) are polynomials since the numerators are divisible by \( z - 1 \).

Proof. The values for \( p_0, p_1, p_2, q_0, q_1, q_2 \) are obtained from Theorem 7 by direct calculations. For a general \( n \), we can prove the formula for \( p_n, q_n \) by induction on \( n \) using (31) and Theorem 7 without difficulty. ■

Remark 4. From Proposition 2 and Theorem 8, it follows that the set of normal indices for \( \varphi(z) \) (and \( \varphi(x) \), resp.), is \( \{0, f_0 = f_1 - 1, f_1 = f_2 - 1, f_2, f_3 - 1, \ldots\} \) \( \{0, f_1 = f_2 - 1, f_2, f_3 - 1, \ldots\} \), and which together with Proposition 1 gives another proof of the third cases of Theorem 2 with \( n = 0 \).

Remark 5. In [4], the continued fraction expansion for Laurent series corresponding to infinite words over \( \{a, b\} \) generated by substitutions of “Fibonacci type” is considered, where \( a, b \) are viewed as independent variables.
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