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Hankel determinants for the Fibonacci word and Padé approximation

by

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1. Introduction. The aim of this paper is to give a concrete and interesting example of the Padé approximation theory as well as to develop the general theory so as to find a quantitative relation between the Hankel determinant and the Padé pair. Our example is the formal power series related to the Fibonacci word.

The *Fibonacci word* $\varepsilon(a,b)$ on an alphabet $\{a,b\}$ is the infinite sequence

(1) $$\begin{aligned}\varepsilon(a,b) &= \widehat{\varepsilon}_0\widehat{\varepsilon}_1\ldots\widehat{\varepsilon}_n\ldots\\ &:= abaababaabaab\ldots \qquad (\widehat{\varepsilon}_n\in\{a,b\}),\end{aligned}$$

which is the fixed point of the substitution

(2) $$\sigma:\quad a\to ab,\quad b\to a.$$

The *Hankel determinants* for an infinite word (or sequence) $\varphi=\varphi_0\varphi_1\ldots$ $(\varphi_n\in\mathbb{K})$ over a field $\mathbb{K}$ are

(3) $$H_{n,m}(\varphi):=\det(\varphi_{n+i+j})_{0\le i,j\le m-1}\qquad (n=0,1,\ldots;\ m=1,2,\ldots).$$

It is known [2] that the Hankel determinants play an important role in the theory of Padé approximation for the formal Laurent series

(4) $$\varphi(z)=\sum_{k=0}^{\infty}\varphi_k z^{-k+h}.$$

Let $\mathbb{K}((z^{-1}))$ be the set of formal Laurent series φ as above of z with coefficients in $\mathbb{K}$ and $h\in\mathbb{Z}$ providing a nonarchimedean norm $\|\varphi\|:=\exp(-k_0+h)$ with $k_0=\inf\{k:\varphi_k\neq 0\}$. Let φ be as above with $h=-1$. We say that a

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pair $(P,Q) \in \mathbb{K}[z]^2$ of polynomials of z over $\mathbb{K}$ is a *Padé pair* of order m for φ if

$$\|Q\varphi - P\| < \exp(-m), \qquad Q \neq 0, \qquad \deg Q \leq m. \tag{5}$$

A Padé pair (P,Q) of order m for φ always exists and the rational function $P/Q \in \mathbb{K}(z)$ is uniquely determined for each $m = 0, 1, \ldots$ The element $P/Q \in \mathbb{K}(z)$ with P, Q satisfying (5) is called the mth *diagonal Padé approximation* for φ. A number m is called a *normal index* if (5) implies $\deg Q = m$. Note that P/Q is irreducible if m is a normal index, although it can be reducible for a general m. A normal Padé pair (P,Q), i.e., $\deg Q$ is a normal index, is said to be *normalized* if the leading coefficient of Q is equal to 1. It is a classical result that m is a normal index for φ if and only if the Hankel determinant $\det(\varphi_{i+j})_{0 \leq i,j \leq m-1}$ is nonzero. Note that 0 is always a normal index and the determinant for the empty matrix is considered to be 1, so that the above statement remains valid for $m = 0$.

We succeed in obtaining a quantitative relation between the Hankel determinant and the normalized Padé pair. Namely,

$$\det(\varphi_{i+j})_{0 \leq i,j \leq m-1} = (-1)^{[m/2]} \prod_{z;\, Q(z)=0} P(z) \tag{6}$$

for any normal index m with the normalized Padé pair (P,Q), where $\prod_{z;\, Q(z)=0}$ indicates a product taken over all zeros z of Q with their multiplicity (Theorem 6).

We are specially interested in the Padé approximation theory applied to the Fibonacci words $\varepsilon := \varepsilon(1,0)$ and $\overline{\varepsilon} := \varepsilon(0,1)$, where 0, 1 are considered as elements in the field $\mathbb{Q}$, since we have the following remark.

REMARK 1. *Let M be a matrix of size $m \times m$ with entries consisting of two independent variables a and b. Then* $\det M = (a-b)^{m-1}(pa + (-1)^{m-1}qb)$, *where p and q are integers defined by*

$$p = \det M|_{a=1,\, b=0}, \qquad q = \det M|_{a=0,\, b=1}.$$

Proof. Subtracting the first column vector from all the other column vectors of M, we see that $\det M$ is divisible by $(a-b)^{m-1}$ as a polynomial in $\mathbb{Z}[a,b]$. Hence, $\det M = (a-b)^{m-1}(xa + yb)$ for integers x, y. Setting $(a,b) = (1,0), (0,1)$, we get the assertion.

In Section 2, we study the structure of the Fibonacci word, in particular, its repetition property. The notion of singular words introduced in Z.-X. Wen and Z.-Y. Wen [5] plays an important role.

In Section 3, we give the value of the Hankel determinants $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words in some closed forms. It is a rare case where the Hankel determinants are determined completely. Another such case is for the Thue–Morse sequence φ consisting of 0 and 1, where the Hankel determinants $H_{m,n}(\varphi)$ modulo 2 are obtained, and the function $H_{m,n}(\varphi)$ of (m,n) is proved to be 2-dimensionally automatic (see [1]).

In Section 4, we consider the self-similar property of the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words. The quarter plane $\{(n,m) : n \geq 0, m \geq 1\}$ is tiled by 3 kinds of tiles with the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ on it with various scales.

In Section 5, we develop a general theory of Padé approximation. We also obtain the admissible continued fraction expansion of φ_ε and $\varphi_{\overline{\varepsilon}}$, the formal Laurent series (4) with $h = -1$ for the sequences ε and $\overline{\varepsilon}$, and determine all the convergents p_k/q_k of the continued fractions. It is known in general that the set of the convergents p_k/q_k for φ is the set of diagonal Padé approximations and the set of degrees of q_k's in z coincides with the set of normal indices for φ.

2. Structure of the Fibonacci word. In what follows, σ denotes the substitution defined by (2), and

$$\widehat{\varepsilon} = \widehat{\varepsilon}_0\widehat{\varepsilon}_1\ldots\widehat{\varepsilon}_n\ldots \qquad (\widehat{\varepsilon}_n \in \{a,b\})$$

is the (infinite) Fibonacci word (1). A finite word over $\{a,b\}$ is sometimes considered to be an element of the free group generated by a and b with inverses a^{-1} and b^{-1}. For $n = 0,1,\ldots$, we define the nth *Fibonacci word* F_n and the nth *singular word* W_n as follows:

$$F_n := \sigma^n(a) = \sigma^{n+1}(b), \qquad W_n := \beta_n F_n \alpha_n^{-1}, \tag{7}$$

where we put

$$\alpha_n = \beta_m = \begin{cases} a & (n \text{ even}, m \text{ odd}), \\ b & (n \text{ odd}, m \text{ even}), \end{cases} \tag{8}$$

and we define W_{-2} to be the empty word and $W_{-1} := a$ for convenience. Let $(f_n; n \in \mathbb{Z})$ be the *Fibonacci sequence*:

$$f_{n+2} = f_{n+1} + f_n \quad (n \in \mathbb{Z}), \qquad f_{-1} = f_0 = 1. \tag{9}$$

Then $|F_n| = |W_n| = f_n\,(n \geq 0)$, where $|\xi|$ denotes the *length* of a finite word ξ.

For a finite word $\xi = \xi_0\xi_1\ldots\xi_{n-1}$ and a finite or infinite word $\eta = \eta_0\eta_1\ldots$ over an alphabet, we denote

$$\xi \prec_k \eta \tag{10}$$

if $\xi = \eta_k\eta_{k+1}\ldots\eta_{k+n-1}$. We simply write

$$\xi \prec \eta \tag{11}$$

and say that ξ is a *subword* of η if $\xi \prec_k \eta$ for some k. For a finite word $\xi = \xi_0\xi_1\ldots\xi_{n-1}$ and i with $0 \leq i < n$, we denote the ith *cyclic permutation* of ξ by $C_i(\xi) := \xi_i\xi_{i+1}\ldots\xi_{n-1}\xi_0\xi_1\ldots\xi_{i-1}$. We also define $C_i(\xi) := C_{i'}(\xi)$ with $i' := i - n[i/n]$ for any $i \in \mathbb{Z}$.

In this section, we study the structure of the Fibonacci word $\widehat{\varepsilon}$ and discuss the repetition property. The following two lemmas were obtained by Z.-X. Wen and Z.-Y. Wen [5] and we omit the proofs.

Lemma 1. *We have the following statements*:

(1) $\widehat{\varepsilon} = F_nF_{n-1}F_nF_{n+1}F_{n+2}\ldots \ (n \geq 1)$,
(2) $F_n = F_{n-1}F_{n-2} = F_{n-2}F_{n-1}\beta_n^{-1}\alpha_n^{-1}\beta_n\alpha_n \ (n \geq 2)$,
(3) $F_nF_n \prec \widehat{\varepsilon} \ (n \geq 3)$,
(4) $\widehat{\varepsilon} = W_{-1}W_0W_1W_2W_3\ldots$,
(5) $W_n = W_{n-2}W_{n-3}W_{n-2} \ (n \geq 1)$,
(6) W_n *is a palindrome*, *that is*, W_n *stays invariant under reading the letters from the end* $(n \geq -2)$,
(7) $C_i(F_n) \prec \widehat{\varepsilon} \ (n \geq 0,\ 0 \leq i < f_n)$,
(8) $C_i(F_n) \neq C_j(F_n)$ *for any* $i \neq j$, *moreover*, *they are different already before their last places* $(n \geq 1,\ 0 \leq i < f_n)$,
(9) $W_n \neq C_i(F_n) \ (n \geq 0,\ 0 \leq i < f_n)$,
(10) $\xi \prec \widehat{\varepsilon}$ *and* $|\xi| = f_n$ *imply that either* $\xi = C_i(F_n)$ *for some* i *with* $0 \leq i < f_n$ *or* $\xi = W_n \ (n \geq 0)$.

Lemma 2. *For any* $k \geq -1$, *we have the decomposition of* $\widehat{\varepsilon}$ *as follows*:

$$\widehat{\varepsilon} = (W_{-1}W_0\ldots W_{k-1})W_k\gamma_0W_k\gamma_1\ldots W_k\gamma_n\ldots,$$

where all the occurrences of W_k *in* $\widehat{\varepsilon}$ *are picked up and* γ_n *is either* W_{k+1} *or* W_{k-1} *corresponding to* $\widehat{\varepsilon}_n$ *is* a *or* b, *respectively. That is*, *any two different occurrences of* W_k *do not overlap and are separated by* W_{k+1} *or* W_{k-1}.

We introduce another method to discuss the repetition property of $\widehat{\varepsilon}$. Let $\mathbb{N}$ be the set of nonnegative integers. For $n \in \mathbb{N}$, let

$$n = \sum_{i=0}^{\infty} \tau_i(n)f_i, \qquad \tau_i(n) \in \{0,1\} \quad \text{and} \quad \tau_i(n)\tau_{i+1}(n) = 0 \quad (i \in \mathbb{N}) \tag{12}$$

be the *regular expression* of n in the Fibonacci base due to Zeckendorf. For $m, n \in \mathbb{N}$ and a positive integer k, we define

$$m \equiv_k n \tag{13}$$

if $\tau_i(m) = \tau_i(n)$ for all $i < k$.

Lemma 3. *We have* $\widehat{\varepsilon}_n = a$ *if and only if* $\tau_0(n) = 0$.

Proof. We use induction on n. The lemma holds for $n = 0, 1, 2$. Assume that it holds for any $n \in \mathbb{N}$ with $n < f_k$ for some $k \geq 2$. Take any $n \in \mathbb{N}$ with $f_k \leq n < f_{k+1}$. Then, since $0 \leq n - f_k < f_{k-1}$, we have

$$n = \sum_{i=0}^{k-1} \tau_i(n-f_k)f_i + f_k,$$

which gives the regular expression if $\tau_{k-1}(n-f_k)=0$. If $\tau_{k-1}(n-f_k)=1$, then we have the regular expression $n=\sum_{i=0}^{k-2}\tau_i(n-f_k)f_i+f_{k+1}$. In any case, we have $\tau_0(n)=\tau_0(n-f_k)$. On the other hand, since $\widehat{\varepsilon}$ starts with F_kF_{k-1} by Lemma 1, we have $\widehat{\varepsilon}_n=\widehat{\varepsilon}_{n-f_k}$. Hence, $\widehat{\varepsilon}_n=a$ if and only if $\tau_0(n)=0$ by the induction hypothesis. Thus, we have the assertion for any $n<f_{k+1}$, and by induction, we complete the proof. ∎

Lemma 4. *Let $n=\sum_{i=0}^{\infty}n_if_i$ with $n_i\in\{0,1\}$ $(i\in\mathbb{N})$. Assume that $n_in_{i+1}=0$ for $0\le i<k$. Then $n_i=\tau_i(n)$ for $0\le i<k$.*

Proof. If there exists $i\in\mathbb{N}$ such that $n_in_{i+1}=1$, let i_0 be the maximum such i. Take the maximum j such that $n_{i_0+1}=n_{i_0+3}=n_{i_0+5}=\ldots=n_j=1$. Then, replacing $f_{i_0}+f_{i_0+1}+f_{i_0+3}+f_{i_0+5}+\ldots+f_j$ by f_{j+1}, we have a new expression of n:

$$n=\sum_{i=0}^{\infty}n_i'f_i := \sum_{i=0}^{i_0-1}n_if_i+f_{j+1}+\sum_{i=j+3}^{\infty}n_if_i.$$

This new expression is unchanged at the indices less than k, and is either regular or has a smaller maximum index i with $n_i'n_{i+1}'=1$. By continuing this procedure, we finally get the regular expression of n, which does not differ from the original expression at the indices less than k. Thus, $n_i=\tau_i(n)$ for any $0\le i<k$. ∎

Lemma 5. *For any $n\in\mathbb{N}$ and $k\ge 0$, $\tau_0(n+f_k)\ne\tau_0(n)$ if and only if either $n\equiv_{k+2} f_{k+1}-2$ or $n\equiv_{k+2} f_{k+1}-1$. Moreover,*

$$\widehat{\varepsilon}_{n+f_k}-\widehat{\varepsilon}_n=\begin{cases}(-1)^{k-1}(a-b) & (n\equiv_{k+2} f_{k+1}-2),\\ (-1)^k(a-b) & (n\equiv_{k+2} f_{k+1}-1),\end{cases}$$

where a and b are considered as independent variables.

Proof. If $k=0$, we can verify the statement by a direct calculation.

Assume that $k\ge1$ and $\tau_k(n)=0$. Then

$$n+f_k=\sum_{i=0}^{k-1}\tau_i(n)f_i+f_k+\sum_{i=k+1}^{\infty}\tau_i(n)f_i.$$

By Lemma 4, we have $\tau_0(n+f_k)=\tau_0(n)$ if $k\ge2$ or if $k=1$ and $\tau_0(n)=0$. In the case where $k=1$, $\tau_0(n)=1$ and $\tau_2(n)=0$, since

$$n+f_k=1+2+\sum_{i=3}^{\infty}\tau_i(n)f_i=f_2+\sum_{i=3}^{\infty}\tau_i(n)f_i,$$

we have $\tau_0(n+f_k)=0$ by Lemma 4. On the other hand, in the case where $k=1$, $\tau_0(n)=1$ and $\tau_2(n)=1$, since

$$n + f_k = 1 + 2 + 3 + \sum_{i=4}^{\infty} \tau_i(n) f_i = f_0 + f_3 + \sum_{i=4}^{\infty} \tau_i(n) f_i,$$

we have $\tau_0(n + f_k) = 1$ by Lemma 4.

Thus, in the case where $k \geq 1$ and $\tau_k(n) = 0$, $\tau_0(n + f_k) \neq \tau_0(n)$ if and only if $k = 1$, $\tau_0(n) = 1$ and $\tau_2(n) = 0$, or equivalently, if and only if $n \equiv_{k+2} f_{k+1} - 2$ with $k = 1$. Note that $n \equiv_{k+1} f_{k+1} - 1$ with $k = 1$ contradicts $\tau_k(n) = 0$.

Now assume that $k \geq 1$ and $\tau_k(n) = 1$. Take the minimum $j \geq 0$ such that $\tau_k(n) = \tau_{k-2}(n) = \tau_{k-4}(n) = \ldots = \tau_j(n) = 1$. Then since $2f_i = f_{i+1} + f_{i-2}$ for any $i \in \mathbb{N}$, we have

$$n + f_k = \sum_{i=0}^{j-3} \tau_i(n) f_i + f_{j-2} + f_{j+1} + f_{j+3} + f_{j+5} + \ldots + f_{k+1} + \sum_{i=k+2}^{\infty} \tau_i(n) f_i, \tag{14}$$

where the first term on the right-hand side vanishes if $j = 0, 1, 2$. Hence by Lemma 4, $\tau_0(n + f_k) = \tau_0(n)$ if $j \geq 4$.

In the case where $j = 3$, $\tau_0(n + f_k) = \tau_0(n)$ holds if $\tau_0(n) = 0$ by (14) and Lemma 4. If $\tau_0(n) = 1$, then by (14) and Lemma 4, $\tau_0(n + f_k) = 0$. Thus, for $j = 3$, $\tau_0(n + f_k) \neq \tau_0(n)$ if and only if $\tau_0(n) = 1$.

If $j = 2$, then by the assumption on j, we have $\tau_0(n) = 0$. On the other hand, since $f_0 = 1$, by (14) and Lemma 4, we have $\tau_0(n + f_k) = 1$. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

If $j = 1$, then $\tau_0(n) = 0$ since $\tau_1(n) = 1$ by the assumption on j. On the other hand, since $f_{-1} = 1$, we have $\tau_0(n + f_k) = 1$ by (14) and Lemma 4. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

If $j = 0$, then by the assumption on j, $\tau_0(n) = 1$. On the other hand, since $f_{-2} = 0$, we have $\tau_0(n + f_k) = 0$ by (14) and Lemma 4. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

By combining all the results as above, we get the first part.

The second part follows from Lemma 3 and the fact that for any $k \geq 0$,

$$f_{k+1} - 1 = f_k + f_{k-2} + \ldots + f_i$$

with $i = 0$ if k is even and $i = 1$ if k is odd. Hence,

$$\tau_0(f_{k+1} - 1) = \tau_0(f_{h+1} - 2) = \begin{cases} a & (k \text{ odd}, h \text{ even}), \\ b & (k \text{ even}, h \text{ odd}). \end{cases} \quad \blacksquare$$

LEMMA 6. *For any $k \geq 0$, $W_k \prec_n \widehat{\varepsilon}$ if and only if $n \equiv_{k+2} f_{k+1} - 1$.*

Proof. By Lemma 2, the smallest $n \in \mathbb{N}$ such that $W_k \prec_n \widehat{\varepsilon}$ is

$$f_{-1} + f_0 + f_1 + \ldots + f_{k-1} = f_{k+1} - 1,$$

which is the smallest $n \in \mathbb{N}$ such that $n \equiv_{k+2} f_{k+1} - 1$. Let $n_0 := f_{k+1} - 1$. Then the regular expression of n_0 is

$$n_0 = f_k + f_{k-2} + f_{k-4} + \ldots + f_d,$$

where $d = (1 - (-1)^k)/2$. The next n with $n \equiv_{k+2} n_0$ is clearly

$$n = f_{k+2} + f_k + f_{k-2} + \ldots + f_d,$$

which is, by Lemma 2, the next n such that $W_k \prec_n \widehat{\varepsilon}$ since $f_k + f_{k+1} = f_{k+2}$.

For $i = 1, 2, \ldots$, let

$$n_i = n_0 + \sum_{j=0}^{\infty} \tau_j(i) f_{k+2+j}.$$

Then it is easy to see that n_i is the ith n after n_0 such that $n \equiv_{k+2} f_{k+1} - 1$. We prove by induction on i that n_i is the ith n after n_0 such that $W_k \prec_n \widehat{\varepsilon}$. Assume that it is so for i. Then by Lemma 4, $W_k \gamma_i W_k \prec_{n_i} \widehat{\varepsilon}$. Hence, the next n after n_i such that $W_k \prec_n \widehat{\varepsilon}$ is $n_i + f_k + |\gamma_i|$. Thus, we have

$$\begin{aligned} n_i + f_k + |\gamma_i| &= n_i + f_k + f_{k+1} 1_{\widehat{\varepsilon}_i = a} + f_{k-1} 1_{\widehat{\varepsilon}_i = b} \\ &= n_i + f_{k+2} 1_{\tau_0(i)=0} + f_{k+1} 1_{\tau_0(i)=1} = n_{i+1}, \end{aligned}$$

which completes the proof. ∎

LEMMA 7. *Let $k \ge 0$ and $n, i \in \mathbb{N}$ satisfy $n \equiv_{k+1} i$.*

(1) *If $0 \le i < f_k$, then $\tau_0(n + j) = \tau_0(i + j)$ for any $j = 0, 1, \ldots, f_{k+2} - i - 3$.*

(2) *If $f_k \le i < f_{k+1}$, then $\tau_0(n + j) = \tau_0(i + j)$ for any $j = 0, 1, \ldots, f_{k+3} - i - 3$.*

Proof. (1) We prove the lemma by induction on k. The assertion holds for $k = 0$. Let $k \ge 1$ and assume that the assertion is valid for $k - 1$. For $j = 0, 1, \ldots, f_k - i$, we have $n + j \equiv_k i + j$ and hence, $\tau_0(n + j) = \tau_0(i + j)$. Let $j_0 = f_k - i$. Then, since $n + j_0 \equiv_k i + j_0 \equiv_k 0$, we have $\tau_0(n + j_0 + j) = \tau_0(i + j_0 + j) = \tau_0(j)$ for any $j = 0, 1, \ldots, f_{k+1} - 3$ by the induction hypothesis. Thus, $\tau_0(n + j) = \tau_0(i + j)$ for any $j = 0, 1, \ldots, f_{k+2} - i - 3$. This proves (1).

(2) In this case, $\tau_{k+1}(n) = 0$. Hence, $n \equiv_{k+2} i$. Therefore, we can apply (1) with $k + 1$ for k. Thus, we get (2). ∎

Let $n, m, i \in \mathbb{N}$ with $m \ge 2$ and $0 < i < m$. We call n an (m, i)-*shift invariant place* in $\widehat{\varepsilon}$ if

$$\widehat{\varepsilon}_n \widehat{\varepsilon}_{n+1} \ldots \widehat{\varepsilon}_{n+m-1} = \widehat{\varepsilon}_{n+i} \widehat{\varepsilon}_{n+i+1} \ldots \widehat{\varepsilon}_{n+i+m-1}.$$

We call n an m-*repetitive place* in $\widehat{\varepsilon}$ if there exist $i, j \in \mathbb{N}$ with $i > 0$ and $i + j < m$ such that $n + j$ is an (m, i)-shift invariant place in $\widehat{\varepsilon}$. Let $\mathcal{R}_m$ be the set of m-repetitive places in $\widehat{\varepsilon}$.

LEMMA 8. (1) *Let* $n \equiv_{k+1} 0$ *for some* $k \geq 1$. *Then* n *is an* $(f_{k+1}-2, f_k)$-*shift invariant place in* $\widehat{\varepsilon}$.

(2) *Let* $n \equiv_{k+1} f_k$ *for some* $k \geq 2$. *Then* n *is an* $(f_{k+1}-2, f_{k-1})$-*shift invariant place in* $\widehat{\varepsilon}$.

Proof. (1) Since the least $i \geq n$ such that either $i \equiv_{k+2} f_{k+1}-1$ or $i \equiv_{k+2} f_{k+1}-2$ is not less than $n+f_{k+1}-2$, by Lemma 5, we have

$$\widehat{\varepsilon}_n \widehat{\varepsilon}_{n+1} \ldots \widehat{\varepsilon}_{n+f_{k+1}-3} = \widehat{\varepsilon}_{n+f_k} \widehat{\varepsilon}_{n+f_k+1} \ldots \widehat{\varepsilon}_{n+f_k+f_{k+1}-3}.$$

(2) Since the minimum $i \geq n$ such that either $i \equiv_{k+1} f_k - 1$ or $i \equiv_{k+1} f_k - 2$ is $n + f_{k+1} - 2$, by Lemma 5, we have

$$\widehat{\varepsilon}_n \widehat{\varepsilon}_{n+1} \ldots \widehat{\varepsilon}_{n+f_{k+1}-3} = \widehat{\varepsilon}_{n+f_{k-1}} \widehat{\varepsilon}_{n+f_{k-1}+1} \ldots \widehat{\varepsilon}_{n+f_{k-1}+f_{k+1}-3}. \blacksquare$$

THEOREM 1. *The pair* (n,m) *of nonnegative integers satisfies* $n \in \mathcal{R}_m$ *if one of the following two conditions holds*:

(1) $f_k + 1 \leq m \leq f_{k+1} - 2$, $n - i \equiv_{k+1} 0$ *and* $i \leq n$ *for some* $k \geq 1$ *and* $i \in \mathbb{Z}$ *with* $f_k + 1 \leq m + i \leq f_{k+1} - 2$.

(2) $f_{k-1} + 1 \leq m \leq f_{k+1} - 2$, $i \leq n$ *and* $n - i \equiv_{k+1} f_k$ *for some* $k \geq 2$ *and* $i \in \mathbb{Z}$ *with* $f_{k-1} + 1 \leq m + i \leq f_{k+1} - 2$.

REMARK 2. The "if and only if" statement actually holds in Theorem 1 in place of "if" since we will prove later that $H_{n,m} \neq 0$ if none of the conditions (1) and (2) hold.

Proof (of Theorem 1). Assume (1) and $i \geq 0$. By Lemma 8(1), $n-i$ is an $(f_{k+1}-2, f_k)$-shift invariant place. Then n is an (m, f_k)-shift invariant place since $i + m \leq f_{k+1} - 2$. Thus, $n \in \mathcal{R}_m$ as $f_k < m$.

Assume (1) and $i < 0$. Then, since $n-i$ is an $(f_{k+1}-2, f_k)$-shift invariant place and $m \leq f_{k+2}-2$, it is an (m, f_k)-shift invariant place. Moreover, since $f_k - i < m$, n is an m-repetitive place.

Assume (2) and $i \geq 0$. Then, $n-i$ is an $(f_{k+1}-2, f_{k-1})$-shift invariant place by Lemma 8(2). Then, n is an (m, f_{k-1})-shift invariant place since $i + m \leq f_{k+1} - 2$. Thus, n is an m-repetitive place as $f_{k-1} < m$.

Assume (2) and $i < 0$. Then, since $n - i$ is an $(f_{k+1} - 2, f_{k-1})$-shift invariant place and $m \leq f_{k+1} - 2$, it is an (m, f_{k-1})-shift invariant place. Then n is an m-repetitive place, since $f_{k-1} - i < m$. Thus, $n \in \mathcal{R}_m$. ■

COROLLARY 1. *The place* 0 *is* m-*repetitive for an* $m \geq 2$ *if* $m \notin \bigcup_{k=1}^{\infty}\{f_k - 1, f_k\}$.

REMARK 3. The "if and only if" statement actually holds in Corollary 1 in place of "if" since we prove later that $H_{0,m} \neq 0$ if $m \in \bigcup_{k=1}^{\infty}\{f_k - 1, f_k\}$.

Proof (of Corollary 1). Let $i = 0$ in (1) of Theorem 1. Then 0 is m-repetitive if $f_k + 1 \leq m \leq f_{k+1} - 2$ for some $k \geq 1$. ■

Corollary 2. *Let $k \geq 2$. The place n is f_k-repetitive if*

$$W_k \prec \widehat{\varepsilon}_{n+1}\widehat{\varepsilon}_{n+2}\ldots\widehat{\varepsilon}_{n+2f_k-3}.$$

Proof. By (2) of Theorem 1, for any $k \geq 2$, n is an f_k-repetitive place if $n-i \equiv_{k+1} f_k$ for some i with $i \leq n$ and $-f_{k-2}+1 \leq i \leq f_{k-1}-2$. Since the condition $n-i \equiv_{k+1} f_k$ is equivalent to $n-i \equiv_{k+2} f_k$ and there is no carry in addition of $-i$ to both sides of $n \equiv_{k+2} f_k+i$, the condition $n-i \equiv_{k+1} f_k$ is equivalent to $n \equiv_{k+2} f_k+i$. Hence, the place n is f_k-repetitive if $n \equiv_{k+2} j$ for some j with $f_{k-1}+1 \leq j \leq f_{k+1}-2$. By Lemma 6, this condition is equivalent to W_k starting at one of the places in $\{n+1, n+2, \ldots, f_k-2\}$, which completes the proof. ∎

3. Hankel determinants. The aim of this section is to find the value of the Hankel determinants

$$H_{n,m} := H_{n,m}(\varepsilon) = \det(\varepsilon_{n+i+j})_{0\leq i,j\leq m-1},$$
$$\overline{H}_{n,m} := H_{n,m}(\overline{\varepsilon}) = \det(\overline{\varepsilon}_{n+i+j})_{0\leq i,j\leq m-1}$$
$$(n = 0, 1, \ldots; m = 1, 2, \ldots)$$

for the Fibonacci word $\varepsilon(a,b)$ at $(a,b) = (1,0)$ and $(a,b) = (0,1)$:

$$\varepsilon := \varepsilon(1,0) = 10110101101101\ldots,$$
$$\overline{\varepsilon} := \varepsilon(0,1) = 01001010010010\ldots$$

It is clear that $H_{n,m}(\varepsilon(a,b)) = 0$ if n is the m-repetitive place in $\varepsilon(a,b)$, where a,b are considered to be two independent variables, and that, in general, $H_{n,m}(\varepsilon(a,b))$ becomes a polynomial in a and b as stated in Remark 1.

In the following lemmas, theorems and corollary, we give parallel statements for ε and $\overline{\varepsilon}$, while we give the proofs only for ε since those for $\overline{\varepsilon}$ are similar. The only difference is the starting point, Lemma 5, where $a-b$ on the right-hand side is 1 for ε and -1 for $\overline{\varepsilon}$.

We use the following notation: for every subset S of $\{0,1,2,3,4,5\}$, $\chi(k:S)$ is the function on $k \in \mathbb{Z}$ such that

$$\chi(k:S) = \begin{cases} -1 & \text{if } k \equiv s \pmod 6 \text{ for some } s \in S, \\ 1 & \text{otherwise.} \end{cases}$$

The following corollary follows from Theorem 1.

Corollary 3. $H_{n,m} = 0$ *if one of the conditions* (1), (2) *in Theorem* 1 *is satisfied. The same statement holds for* $\overline{H}_{n,m}$.

Lemma 9. *For any $k \geq 2$, we have*

$$H_{0,f_k} = \chi(k:2,3)(H_{0,f_{k-1}} - (-1)^{f_{k-1}} H_{f_k-1,f_{k-1}}),$$
$$\overline{H}_{0,f_k} = \chi(k:1,3,4,5)(\overline{H}_{0,f_{k-1}} - (-1)^{f_{k-1}} \overline{H}_{f_k-1,f_{k-1}}).$$

P r o o f. The matrix $(\varepsilon_{i+j})_{0\le i,j<f_k}$ is decomposed into three parts:

$$(\varepsilon_{i+j})_{0\le i,j<f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},$$

where

$$\begin{aligned} A &= (\varepsilon_{i+j})_{0\le i<f_{k-2},\,0\le j<f_k},\\ A' &= (\varepsilon_{f_{k-2}+i+j})_{0\le i<f_{k-3},\,0\le j<f_k},\\ B &= (\varepsilon_{f_{k-1}+i+j})_{0\le i<f_{k-2},\,0\le j<f_k}. \end{aligned}$$

By Lemma 5, the following two subwords of ε:

$$\varepsilon_0\varepsilon_1\ldots\varepsilon_{f_{k-2}+f_k-2} \quad \text{and} \quad \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1}\ldots\varepsilon_{f_{k-1}+f_{k-2}+f_k-2}$$

differ only at two places, namely, $\varepsilon_{f_k-2} \ne \varepsilon_{f_{k-1}+f_k-2}$ and $\varepsilon_{f_k-1} \ne \varepsilon_{f_{k-1}+f_k-1}$. Thus, we get

$$(15)\quad B - A = \begin{pmatrix} & & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^k & (-1)^{k-1} & \\ 0 & & & \cdots & & \\ & (-1)^k & (-1)^{k-1} & & 0 & \end{pmatrix}.$$

Let $A_0, A_1, \ldots, A_{f_k-1}$ be the columns of the matrix $\binom{A}{A'}$ in order from the left. Since

$$\begin{aligned} (A_0A_1\ldots A_{f_{k-2}-2}) &= (\varepsilon_{i+j})_{0\le i<f_{k-1},\,0\le j<f_{k-2}-1},\\ (A_{f_{k-1}}A_{f_{k-1}+1}\ldots A_{f_k-2}) &= (\varepsilon_{f_{k-1}+i+j})_{0\le i<f_{k-1},\,0\le j<f_{k-2}-1} \end{aligned}$$

and

$$\varepsilon_0\varepsilon_1\ldots\varepsilon_{f_{k-2}+f_{k-1}-3} = \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1}\ldots\varepsilon_{f_{k-1}+f_{k-2}+f_{k-1}-3}$$

by Lemma 5, we get

$$(16)\qquad (A_0A_1\ldots A_{f_{k-2}-2}) = (A_{f_{k-1}}A_{f_{k-1}+1}\ldots A_{f_k-2}).$$

Thus, from (15) and (16) we obtain

$$(17)\quad H_{0,f_k} = \det\begin{pmatrix} A_0 & \ldots & A_{f_{k-1}-1} & A_{f_{k-1}} & \ldots & A_{f_k-2} & A_{f_k-1} \\ & & & & & (-1)^k & (-1)^{k-1} \\ & & & & (-1)^k & (-1)^{k-1} & \\ & 0 & & \ldots & \ldots & & \\ & & (-1)^k & (-1)^{k-1} & & 0 & \end{pmatrix}$$

$$= \det\begin{pmatrix} A_0 & \ldots & A_{f_{k-1}-1} & 0 & \ldots & 0 & A_{f_k-1} \\ & & & & & (-1)^k & (-1)^{k-1} \\ & & & & (-1)^k & (-1)^{k-1} & \\ & 0 & & \ldots & \ldots & & \\ & & (-1)^k & (-1)^{k-1} & & 0 & \end{pmatrix}$$

$$= (-1)^{(k-1)f_{k-2}}(-1)^{[f_{k-2}/2]}\det(A_0A_1\dots A_{f_{k-1}-1})$$
$$+ (-1)^{kf_{k-2}}(-1)^{[f_{k-2}/2]+f_k-1}\det(A_{f_k-1}A_0A_1\dots A_{f_{k-1}-2}).$$

Since

$$\varepsilon_0\varepsilon_1\dots\varepsilon_{2f_{k-1}-3} = \varepsilon_{f_k}\varepsilon_{f_k+1}\dots\varepsilon_{f_k+2f_{k-1}-3}$$

by Lemma 5, we get

$$\det(A_{f_k-1}A_0A_1\dots A_{f_{k-1}-2}) = \det(\varepsilon_{f_k-1+i+j})_{0\le i,j<f_{k-1}} = H_{f_k-1,f_{k-1}}.$$

Thus we get

$$\begin{aligned} H_{0,f_k} &= (-1)^{(k-1)f_{k-2}}(-1)^{[f_{k-2}/2]}H_{0,f_{k-1}} \\ &\quad + (-1)^{kf_{k-2}}(-1)^{[f_{k-2}/2]+f_k-1}H_{f_k-1,f_{k-1}} \\ &= \chi(k:2,3)(H_{0,f_{k-1}} - (-1)^{f_{k-1}}H_{f_k-1,f_{k-1}}), \end{aligned}$$

where we have used the fact that

$$(-1)^{(k-1)f_{k-2}}(-1)^{[f_{k-2}/2]} = \chi(k:2,3). \ \blacksquare$$

LEMMA 10. *For $k \ge 2$, we have*

$$H_{f_{k+1}-1,f_k} = \chi(k:1,3,4,5)H_{f_{k+1}-1,f_{k-1}},$$
$$\overline{H}_{f_{k+1}-1,f_k} = \chi(k:2,3)\overline{H}_{f_{k+1}-1,f_{k-1}}.$$

Proof. Just as in the proof of Lemma 9, we decompose the matrix $(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k}$ into three parts:

$$(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},$$

where

$$\begin{aligned} A &= (\varepsilon_{f_{k+1}-1+i+j})_{0\le i<f_{k-2},\,0\le j<f_k}, \\ A' &= (\varepsilon_{f_{k+1}-1+f_{k-2}+i+j})_{0\le i<f_{k-3},\,0\le j<f_k}, \\ B &= (\varepsilon_{f_{k+1}-1+f_{k-1}+i+j})_{0\le i<f_{k-2},\,0\le j<f_k}. \end{aligned}$$

By Lemma 5, the following two subwords of ε:

$$\varepsilon_{f_{k+1}-1}\varepsilon_{f_{k+1}}\dots\varepsilon_{f_{k+1}+f_{k-2}+f_k-3} \quad \text{and}$$
$$\varepsilon_{f_{k+1}-1+f_{k-1}}\varepsilon_{f_{k+1}+f_{k-1}}\dots\varepsilon_{f_{k+1}+f_{k-1}+f_{k-2}+f_k-3}$$

differ only at two places. Namely, $\varepsilon_{f_{k+1}+f_k-2} \ne \varepsilon_{f_{k+1}+f_{k-1}+f_k-2}$ and $\varepsilon_{f_{k+1}+f_k-1} \ne \varepsilon_{f_{k+1}+f_{k-1}+f_k-1}$. Therefore, we get

$$B - A = \begin{pmatrix} & & & & (-1)^k \\ & & & (-1)^k & (-1)^{k-1} \\ \mathbf{0} & & \dots & & \\ & \dots & & & \\ (-1)^k & (-1)^{k-1} & & & \mathbf{0} \end{pmatrix}.$$

Thus, we have

$$(18)\qquad \det(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k}$$

$$= \det\begin{pmatrix} A_0 & A_1 & \dots & A_{f_{k-1}-1} & A_{f_{k-1}} & \dots & A_{f_k-2} & A_{f_k-1} \\ & & & & & & & (-1)^k \\ & & & & & & (-1)^k & (-1)^{k-1} \\ & & 0 & & & \dots & \dots & \\ & & & & (-1)^k & (-1)^{k-1} & & 0 \end{pmatrix}$$

$$= (-1)^{kf_{k-2}}(-1)^{[f_{k-2}/2]}\det(A_0A_1\dots A_{f_{k-1}-1})$$

$$= \chi(k:1,3,4,5)H_{f_{k+1}-1,f_{k-1}}. \ \blacksquare$$

LEMMA 11. *For any $k \ge 2$, we have*

$$H_{f_{k+1}-1,f_{k-1}} = \chi(k:2,5)H_{0,f_{k-1}},$$
$$\overline{H}_{f_{k+1}-1,f_{k-1}} = \chi(k:2,5)\overline{H}_{0,f_{k-1}}.$$

Proof. Since, by Lemma 5,

$$\varepsilon_{f_{k+1}-1}\varepsilon_{f_{k+1}}\dots\varepsilon_{f_{k+1}+f_{k-1}-2} = \varepsilon_{f_{k+1}+f_{k-1}-1}\varepsilon_{f_{k+1}+f_{k-1}}\dots\varepsilon_{f_{k+1}+2f_{k-1}-2},$$

we get

$$(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_{k-1}} = \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & \cdot & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 1 & 0 \end{pmatrix}(\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_{k-1}}.$$

Also, by Lemma 5,

$$(\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_k} = (\varepsilon_{i+j})_{0\le i,j<f_k}.$$

Thus we obtain

$$\begin{aligned} H_{f_{k+1}-1,f_{k-1}} &= \det(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_{k-1}} \\ &= (-1)^{f_{k-1}-1}\det(\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_{k-1}} \\ &= \chi(k:2,5)H_{0,f_{k-1}}. \ \blacksquare \end{aligned}$$

LEMMA 12. *For any $k \ge 3$, we have*

$$H_{0,f_k} = \chi(k:2,3)H_{0,f_{k-1}} + \chi(k:2,4)H_{0,f_{k-2}},$$
$$\overline{H}_{0,f_k} = \chi(k:1,3,4,5)\overline{H}_{0,f_{k-1}} + \chi(k:0,1,2,3)\overline{H}_{0,f_{k-2}}.$$

Proof. Clear from Lemmas 9–11. ■

LEMMA 13. *For any $k \ge 0$, we have*

$$H_{0,f_k} = \chi(k:2)f_{k-1},$$
$$\overline{H}_{0,f_k} = \chi(k:1,2,4)f_{k-2}.$$

Proof. We have

$$\begin{aligned} H_{0,f_0} &= 1, & H_{0,f_1} &= 1, & H_{0,f_2} &= -2,\\ \overline{H}_{0,f_0} &= 0, & \overline{H}_{0,f_1} &= -1, & \overline{H}_{0,f_2} &= -1. \end{aligned}$$

Thus, the assertion holds for $k = 0, 1, 2$. For $k \geq 3$, we can prove it by induction on k using Lemma 12. ∎

Lemma 14. *For any* $k \geq 1$, *we have*

$$\begin{aligned} H_{0,f_k-1} &= \chi(k:0,4) f_{k-2},\\ \overline{H}_{0,f_k-1} &= \chi(k:2,3,4,5) f_{k-3}. \end{aligned}$$

Proof. Since the matrix $(\varepsilon_{i+j})_{0\le i,j<f_k-1}$ is obtained from $(\varepsilon_{i+j})_{0\le i,j<f_k}$ by removing the last row and the last column, for any $k \geq 2$ we have by (17),

$$(19)\qquad \begin{aligned} &H_{0,f_k-1}\\ &= \det\begin{pmatrix} A_0 & A_1 & \ldots & A_{f_{k-1}-1} & 0 & \ldots & 0 & 0\\ & & & & & & & (-1)^k\\ & & & & & & (-1)^k & (-1)^{k-1}\\ & & \mathbf{0} & & & \ldots & \ldots & \\ & & & & (-1)^k & (-1)^{k-1} & & \mathbf{0} \end{pmatrix}\\ &= (-1)^{k(f_{k-2}-1)}(-1)^{[(f_{k-2}-1)/2]}\det(A_0A_1\ldots A_{f_{k-1}-1})\\ &= (-1)^{k(f_{k-2}-1)}(-1)^{[(f_{k-2}-1)/2]}H_{0,f_{k-1}}. \end{aligned}$$

Hence, in view of Lemma 13, we obtain the formula for H_{0,f_k-1}. ∎

Theorem 2. *For any* $m, k \geq 1$ *with* $f_{k-1} < m \leq f_k$ *and* $n \in \mathbb{N}$ *with* $n \equiv_{k+1} 0$, *we have*

$$H_{n,m} = \begin{cases} \chi(k:2) f_{k-1} & \text{if } m = f_k,\\ \chi(k:0,4) f_{k-2} & \text{if } m = f_k - 1,\\ 0 & \text{otherwise}, \end{cases}$$

$$\overline{H}_{n,m} = \begin{cases} \chi(k:1,2,4) f_{k-2} & \text{if } m = f_k,\\ \chi(k:2,3,4,5) f_{k-3} & \text{if } m = f_k - 1,\\ 0 & \text{otherwise}. \end{cases}$$

Proof. By Lemmas 3 and 7, the matrix for $H_{n,m}$ coincides with that for $H_{0,m}$ so that $H_{n,m} = H_{0,m}$. Thus, the first two cases follow from Lemmas 13 and 14. For the last case, by Corollary 1, there exist two identical rows in the matrix $(\varepsilon_{i+j})_{0\le i,j<m}$, so that $H_{0,m} = 0$. ∎

THEOREM 3. *For any $k, n, i \in \mathbb{N}$ with $n \equiv_{k+1} i$ and $0 \le i \le f_{k+1} - 1$, we have*

$$H_{n,f_k} = \begin{cases} \chi(k:2)\chi(k:1,4)^i f_{k-1} \\ \qquad \text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \le i < f_{k-1} \\ \qquad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \le i < f_k, \\ \chi(k:1,2,4) f_{k-2} \\ \qquad \text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\ \qquad \text{or } i = f_{k+1} - 1, \\ 0 \quad \text{otherwise}, \end{cases}$$

$$\overline{H}_{n,f_k} = \begin{cases} \chi(k:1,2,4)\chi(k:1,4)^i f_{k-2} \\ \qquad \text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \le i < f_{k-1} \\ \qquad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \le i < f_k, \\ \chi(k:2) f_{k-3} \\ \qquad \text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\ \qquad \text{or } i = f_{k+1} - 1, \\ 0 \quad \text{otherwise.} \end{cases}$$

Proof. The assertion holds for $k = 0$. Let $k \ge 1$.

Assume that either $\tau_{k+1}(n) = 0$ and $0 \le i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \le i < f_k$. Then by Lemmas 3 and 7 we have

$$\begin{aligned} \varepsilon_{i+j} &= \varepsilon_{n+j} \qquad (j = 0, 1, \ldots, f_k - i - 1), \\ \varepsilon_{i+j-f_k} &= \varepsilon_{n+j} \qquad (j = f_k - i, f_k, \ldots, 2f_k - 2), \\ \varepsilon_j &= \varepsilon_{j+f_k} \qquad (j = 0, 1, \ldots, f_k - 1). \end{aligned}$$

Hence, the columns of the matrix $(\varepsilon_{n+h+j})_{0\le h,j\le f_k}$ coincide with those of $(\varepsilon_{h+j})_{0\le h,j\le f_k}$. The jth column of the former is the $(i + j) \pmod{f_k}$th column of the latter for $j = 0, \ldots, f_k - 1$. Therefore, we get $H_{n,f_k} = (-1)^{i(f_k-i)} H_{0,f_k}$, which leads to the first case of our theorem by Theorem 2.

Assume that $i = f_{k+1} - 1$. Then $H_{n,f_k} = H_{f_{k+1}-1,f_k}$ by Lemmas 3 and 7. Thus, by Lemmas 10–12 we get

$$H_{n,f_k} = \chi(k:1,2,4) f_{k-2}.$$

Assume that $\tau_{k+1}(n) = 0$ and $i = f_{k-1}$. Then, since $n \equiv_{k+2} i$, we have $H_{n,f_k} = H_{f_{k-1},f_k}$ by Lemmas 3 and 7. By Lemma 1,

$$\begin{aligned} \xi &:= \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1}\ldots\varepsilon_{f_{k-1}+2f_k-2} \prec_1 W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}, \\ \eta &:= \varepsilon_{f_{k+1}-1}\varepsilon_{f_{k+1}}\ldots\varepsilon_{f_{k+1}+2f_k-3} \prec_{f_k} W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}. \end{aligned}$$

Since the last letter of η comes one letter before the last letter of the palindrome word $W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}$, it follows that ξ is the mirror image of η, so that

$$(\varepsilon_{f_{k-1}+i+j})_{0\le i,j<f_k} = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot & & \\ 1 & & 0 & \end{pmatrix} (\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k} \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot & & \\ 1 & & 0 & \end{pmatrix}.$$

Thus, we obtain $H_{f_{k-1},f_k} = H_{f_{k+1}-1,f_k}$ and

$$H_{n,f_k} = \chi(k:1,2,4)f_{k-2}.$$

Assume that n does not belong to the above two cases. Then, since $\tau_{k+1}(n) = 1$ implies $i < f_k$, we have the following condition:

$$\tau_{k+1}(n) = 0 \quad \text{and} \quad f_{k-1}+1 \le i \le f_{k+1}-2.$$

This condition is nonempty only if $k \ge 2$, which we assume. Then the condition (2) of Theorem 1 is satisfied with f_k (resp. $i - f_k$) in place of m (resp. i). Thus, by Corollary 3, $H_{n,f_k} = 0$. ∎

LEMMA 15. *For any $k, n, i \in \mathbb{N}$ with $k \ge 1$ and $n \equiv_{k+1} i$, assume that either $\tau_{k+1}(n) = 0$ and $0 \le i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \le i < f_k$. Then*

$$H_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-2} & (i=0), \\ \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} & \\ \quad + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} & (0<i\le f_{k-2}), \\ \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} & (f_{k-2}<i\le f_{k-1}), \\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & (f_{k-1}<i<f_k), \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5)f_{k-3} & (i=0), \\ \chi(k:1,3,4,5)\chi(k:1,2,4,5)^i \overline{H}_{i+f_k,f_{k-1}-1} & \\ \quad + \chi(k:0,1)\chi(k:1,4)^i f_{k-3} & (0<i\le f_{k-2}), \\ \chi(k:1,3,4,5)\chi(k:1,2,4,5)^i \overline{H}_{i+f_k,f_{k-1}-1} & (f_{k-2}<i\le f_{k-1}), \\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & (f_{k-1}<i<f_k). \end{cases}$$

Proof. If $i = 0$, then the statement follows from Theorem 2. Let

$$\begin{aligned} A_j &= {}^t(\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_{k-1}-1}), \\ A'_j &= {}^t(\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_{k-1}-2}), \\ B'_j &= {}^t(\varepsilon_{j+f_{k-1}}, \varepsilon_{j+f_{k-1}+1}, \ldots, \varepsilon_{j+f_k-1}) \quad (j = 0,1,\ldots). \end{aligned} \tag{20}$$

Then, by the same argument as in the proof of Theorem 3, we obtain

$$\begin{aligned} H_{n,f_k-1} &= \det\begin{pmatrix} A_i \ldots A_{f_k-1} A_0 \ldots A_{i-2} \\ B'_i \ldots B'_{f_k-1} B'_0 \ldots B'_{i-2} \end{pmatrix} \\ &= (-1)^{(i-1)(f_k-i)} \det\begin{pmatrix} A_0 \ldots A_{i-2} A_i \ldots A_{f_k-1} \\ B'_0 \ldots B'_{i-2} B'_i \ldots B'_{f_k-1} \end{pmatrix}. \end{aligned}$$

Therefore, if $f_{k-2} < i \le f_{k-1}$, then by the same argument as for (17), we obtain

$$(-1)^{(i-1)(f_k-i)} H_{n,f_k-1}$$
$$= \det \begin{pmatrix} A_0 \dots A_{i-2} A_i \dots A_{f_{k-1}-1} & 0 & \dots & 0 & A_{f_k-1} \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \dots & & \mathbf{0} \\ & (-1)^k & (-1)^{k-1} & & \end{pmatrix}.$$

Since by Lemma 5,

$$A_{f_k-1} - A_{f_{k-2}-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^k \end{pmatrix},$$

we get

$$(-1)^{(i-1)(f_k-i)} H_{n,f_k-1}$$
$$= \det \begin{pmatrix} A'_0 \dots A'_{i-2} A'_i \dots A'_{f_{k-1}-1} & 0 & \dots & 0 & 0 \\ * \dots * * \dots * & 0 & \dots & 0 & (-1)^k \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \dots & & \mathbf{0} \\ & (-1)^k & (-1)^{k-1} & & \end{pmatrix}$$
$$= (-1)^{k f_{k-2}} (-1)^{[f_{k-2}/2]} \det(A'_0 \dots A'_{i-2} A'_i \dots A'_{f_{k-1}-1})$$
$$= \chi(k : 1,3,4,5)(-1)^{(i-1)(f_{k-1}-i)} H_{i+f_k, f_{k-1}-1}.$$

Thus we obtain

$$H_{n,f_k-1} = \chi(k : 2,3)\chi(k : 1,2,4,5)^i H_{i+f_k, f_{k-1}-1}.$$

Assume that $f_{k-1} < i < f_k$. Then as above we have

$$(-1)^{(i-1)(f_k-i)} H_{n,f_k-1}$$
$$= \det \begin{pmatrix} A_0 \dots A_{f_{k-1}-1} & 0 & \dots 0 & 0 \dots & 0 & A_{f_k-1} \\ & & & & (-1)^k & (-1)^{k-1} \\ & & & & \dots & \\ & & 0 & (-1)^k & \dots & \\ \mathbf{0} & & 0 & (-1)^{k-1} & & \\ & & (-1)^k & 0 & & \mathbf{0} \\ & & \dots & & & \\ & (-1)^k & \dots & & & \end{pmatrix}$$
$$= (-1)^{k(i-f_{k-1}-1)+(k-1)(f_k-i)+[(f_{k-2}-1)/2]} \det(A_0 \dots A_{f_{k-1}-1}).$$

Hence, by Lemma 13,

$$H_{n,f_k-1} = \chi(k:0,3,4)\chi(k:1,4)^i H_{0,f_{k-1}} = \chi(k:0,4)\chi(k:1,4)^i f_{k-2}.$$

Assume that $0 < i < f_{k-2}$. Then, since $A_{i-1+f_{k-1}} = A_{i-1}$, by the same arguments as above we get

$$\begin{aligned}
&(-1)^{(i-1)(f_k-i)} H_{n,f_k-1}\\
&= \det\begin{pmatrix} A'_0 \ldots A'_{i-2}A'_i \ldots A'_{f_{k-1}-1} & 0 & \ldots A'_{i-1} \ldots & & 0 \\ * \ldots ** \ldots * & 0 & \ldots * \ldots & & (-1)^k \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \ldots & & \mathbf{0} \\ & (-1)^k & & & \end{pmatrix}\\
&= (-1)^{kf_{k-2}}(-1)^{[f_{k-2}/2]} \det(A'_0 \ldots A'_{i-2}A'_i \ldots A'_{f_{k-1}-1})\\
&\quad + (-1)^{k(i-1)+(k-1)(f_{k-2}-i)}(-1)^{i-1+[(f_{k-2}-1)/2]}\\
&\quad \times \det(A_0 \ldots A_{i-2}A_i \ldots A_{f_{k-1}-1}A_{i-1}).
\end{aligned}$$

Since

$$\det(A_0 \ldots A_{i-2}A_i \ldots A_{f_{k-1}-1}A_{i-1}) = (-1)^{f_{k-1}-i} H_{0,f_{k-1}},$$

by Lemma 13 we obtain

$$\begin{aligned} H_{n,f_k-1} = {}& \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} \\ &+ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}. \end{aligned} \tag{21}$$

Note that (21) holds also for $i = f_{k-2}$ since in this case,

$$\begin{aligned} H_{n,f_k-1} = {}& (-1)^{k(f_{k-2}-1)}(-1)^{f_{k-2}-1+[(f_{k-2}-1)/2]} \\ &\times \det(A_0 \ldots A_{f_{k-2}-2}A_{f_{k-2}} \ldots A_{f_{k-1}-2}A_{f_k-1}) \end{aligned}$$

and

$$A_{f_k-1} = A_{f_{k-1}-1} + {}^t(0,\ldots,0,(-1)^k). \ \blacksquare$$

Lemma 16. *For any $k, n, i \in \mathbb{N}$ with $k \ge 1$ and $n \equiv_{k+1} i$, assume that either $\tau_{k+1}(n) = 0$ and $0 \le i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \le i < f_k$. Then*

$$H_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-2} & (i=0), \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (0 < i \le f_{k-1}), \\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & (f_{k-1} < i < f_k), \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5)f_{k-3} & (i=0), \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (0 < i \le f_{k-1}), \\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & (f_{k-1} < i < f_k). \end{cases}$$

Proof. The first and third cases have already been proved in Lemma 15. Consider the second case where $0 < i \le f_{k-1}$. We divide it into two subcases, and use induction on k.

Case 1: $i = 1$. If $k = 1$, then

$$H_{n,f_{k-1}} = H_{n,1} = \varepsilon_n = 0$$

since $n \equiv_2 1$ and $\tau_0(n) = 1$. On the other hand, $f_{k-3} = f_{-2} = 0$, and hence, we get the statement. Assume that $k \ge 2$ and the assertion holds for $k-1$. Then, by Lemma 15 and the induction hypothesis, we get

$$\begin{aligned}H_{n,f_k-1} &= \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1}\\ &\quad + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}\\ &= \chi(k:1,3,4,5)H_{1+f_k,f_{k-1}-1} + \chi(k:2,3,4,5)f_{k-2}\\ &= \chi(k:1,3,4,5)\chi(k-1:2,3,4,5)f_{k-4} + \chi(k:2,3,4,5)f_{k-2}\\ &= \chi(k:0,1)f_{k-4} + \chi(k:2,3,4,5)f_{k-2}\\ &= \chi(k:2,3,4,5)f_{k-3},\end{aligned}$$

which is the desired statement.

Case 2: $i \ge 2$. If $f_{k-2} < i \le f_{k-1}$, then it follows from the third case and then the fourth case of Lemma 15 that

$$\begin{aligned}H_{n,f_k-1} &= \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1}\\ &= \chi(k:2,3)\chi(k:1,2,4,5)^i\chi(k-1:0,4)\chi(k-1:1,4)^i f_{k-3}\\ &= \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3}.\end{aligned}$$

Assume that $i \le f_{k-2}$ and the statement holds for $k-1$. Then by Lemma 15, we get

$$\begin{aligned}H_{n,f_k-1} &= \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1}\\ &\quad + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}\\ &= \chi(k:2,3)\chi(k:1,2,4,5)^i\chi(k-1:1,2,3,5)\chi(k-1:1,4)^i f_{k-4}\\ &\quad + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}\\ &= \chi(k:0,4)\chi(k:1,4)^i f_{k-4} + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}\\ &= \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3}. \ \blacksquare\end{aligned}$$

Lemma 17. *For any $k, n \in \mathbb{N}$ with $k \ge 2$ and $\tau_{k+1}(n) = 0$, we have*

$$H_{n,f_k-1} = \begin{cases}\chi(k:2,3,4,5)f_{k-3} & (n \equiv_{k+1} f_{k-1}),\\ \chi(k:0,4)f_{k-2} & (n \equiv_{k+1} f_{k-1}+1),\end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases}\chi(k:0,4)f_{k-4} & (n \equiv_{k+1} f_{k-1}),\\ \chi(k:2,3,4,5)f_{k-3} & (n \equiv_{k+1} f_{k-1}+1).\end{cases}$$

Proof. Assume that $n \equiv_{k+1} f_{k-1}$. Then since $\tau_{k+1}(n) = 0$, we have $n \equiv_{k+2} f_{k-1}$. Therefore, by Lemmas 3 and 7, we get

$$H_{n,f_k-1} = \det\begin{pmatrix} A_{f_{k-1}} \dots A_{f_k-1} A_{f_k} \dots A_{f_{k+1}-2} \\ B'_{f_{k-1}} \dots B'_{f_k-1} B'_{f_k} \dots B'_{f_{k+1}-2}\end{pmatrix},$$

where we use the notation (20). By Lemma 5, the following two subwords of ε:

$$\varepsilon_n\varepsilon_{n+1}\dots\varepsilon_{n+f_{k-2}+f_k-3} \quad \text{and} \quad \varepsilon_{n+f_{k-1}}\varepsilon_{n+f_{k-1}+1}\dots\varepsilon_{n+f_{k-1}+f_{k-2}+f_k-3}$$

differ only at two places, namely, at the (f_k-2-f_{k-1})th and the (f_k-1-f_{k-1})th places. Hence, we have

$$H_{n,f_k-1} = \det\begin{pmatrix} A_{f_{k-1}} \dots A_{f_k-1} A_{f_k} \dots A_{f_{k+1}-2} \\ B'_{f_{k-1}} \dots B'_{f_k-1} B'_{f_k} \dots B'_{f_{k+1}-2}\end{pmatrix}$$

$$= \det\begin{pmatrix} A_{f_{k-1}} & \dots & \dots & A_{f_k-1} & A_{f_k}\dots A_{f_{k+1}-2} \\ & & (-1)^k & (-1)^{k-1} & \\ & & (-1)^{k-1} & & \\ 0 & & \dots & & \\ & \dots & & & 0 \\ (-1)^k & (-1)^{k-1} & & & \end{pmatrix}.$$

By adding the first $f_{k-2}-1$ columns and subtracting the last $f_{k-2}-1$ columns to and from the column beginning by A_{f_k-1}, we get the column

$${}^t(A_{f_k-1}0\dots0) + {}^t((-1)^{k-1}0\dots0(-1)^k0\dots0),$$

where $(-1)^k$ is at the $(f_{k-2}-1)$th place. Since, by Lemma 5,

$$(A_{f_{k-1}}\dots A_{f_k-2}) - (A_{2f_{k-1}}\dots A_{f_k-2})$$

$$= \begin{pmatrix} & & & (-1)^{k-1} \\ 0 & & (-1)^{k-1} & (-1)^k \\ & \dots & \dots & \\ \dots & \dots & & \\ (-1)^{k-1} \quad (-1)^k & & & 0 \\ (-1)^k & & & \\ & 0 & & \end{pmatrix},$$

we get

$$(22)\quad H_{n,f_k-1} = (-1)^{k(f_{k-2}-1)}(-1)^{f_{k-1}(f_{k-2}-1)+[(f_{k-2}-1)/2]}$$
$$\times\{\det(A_{f_k-1}A_{f_k}\dots A_{f_{k+1}-2}) + (-1)^{k-1}\det(A''_{f_k}\dots A''_{f_{k+1}-2})$$
$$+ (-1)^{k+f_{k-2}-1}\det(A'''_{f_k}\dots A'''_{f_{k+1}-2})\},$$

where

$$A''_j := {}^t(\varepsilon_{j+1} \dots \varepsilon_{j+f_{k-1}-1}),$$
$$A'''_j = {}^t(\varepsilon_j \dots \varepsilon_{j+f_{k-2}-2}\varepsilon_{j+f_{k-2}} \dots \varepsilon_{j+f_{k-1}-1}).$$

Here, we have

$$\det(A_{f_k-1}A_{f_k} \dots A_{f_{k+1}-2}) = H_{f_k-1,f_{k-1}}, \qquad (23)$$
$$\det(A''_{f_k} \dots A''_{f_{k+1}-2}) = H_{f_k+1,f_{k-1}-1},$$

and by Lemma 5,

$$\det(A'''_{f_k} \dots A'''_{f_{k+1}-2})$$
$$= \begin{pmatrix} & & \mathbf{0} & & \\ & & & & (-1)^{k-1} \\ & & & (-1)^{k-1} & (-1)^k \\ A'''_{f_k} \dots A'''_{f_k+f_{k-2}-1} & & \cdots & \cdots & \\ & & \cdots & \cdots & \\ & (-1)^{k-1} & \cdots & & \\ & (-1)^k & & & \mathbf{0} \end{pmatrix}$$
$$= \begin{pmatrix} C_{f_k} & & & & \\ \vdots & & \mathbf{0} & & \\ C_{f_k+f_{k-2}-2} & & & & \\ C_{f_k+f_{k-2}} & & & & (-1)^{k-1} \\ & & & (-1)^{k-1} & (-1)^k \\ \vdots & & & & \\ & & \cdots & \cdots & \\ & & \cdots & \cdots & \\ & (-1)^{k-1} & \cdots & & \\ C_{f_{k+1}-1} & (-1)^k & & & \mathbf{0} \end{pmatrix},$$

where we put

$$C_j = (\varepsilon_j\varepsilon_{j+1} \dots \varepsilon_{j+f_{k-2}-1}).$$

Since $C_{f_k+f_{k-2}+j} = C_{f_k+j}$ $(j = 0, 1, \dots, f_{k-3} - 2)$ by Lemma 5, we have

$$\det(A'''_{f_k} \dots A'''_{f_{k+1}-2})$$
$$= (-1)^{(k-1)(f_{k-3}-1)+f_{k-3}-1+[(f_{k-3}-1)/2]} \det \begin{pmatrix} C_{f_k} \\ \vdots \\ C_{f_k+f_{k-2}-2} \\ C_{f_{k+1}-1} \end{pmatrix}.$$

Moreover it follows from Lemma 5 that

$$\det\begin{pmatrix} C_{f_k} \\ \vdots \\ C_{f_k+f_{k-2}-2} \\ C_{f_{k+1}-1}\end{pmatrix} = \det\begin{pmatrix} C_{f_{k+1}} \\ \vdots \\ C_{f_{k+1}+f_{k-2}-2} \\ C_{f_{k+1}-1}\end{pmatrix} = (-1)^{f_{k-2}-1}H_{f_{k+1}-1,f_{k-2}},$$

which implies

$$\det(A'''_{f_k}\dots A'''_{f_{k+1}-2}) = \chi(k:0,3,5)H_{f_{k+1}-1,f_{k-2}}.$$

Thus by (22), (23), Theorem 3 and Lemma 16, we obtain

$$\begin{aligned} H_{n,f_k-1} &= \chi(k:4)H_{f_k-1,f_{k-1}} + \chi(k:0,2)H_{f_k+1,f_{k-1}-1} \\ &\quad + \chi(k:1,3,4)H_{f_{k+1}-1,f_{k-2}} \\ &= \chi(k:2,3,4,5)f_{k-3} + \chi(k:2,3,4,5)f_{k-4} + \chi(k:0,1)f_{k-4} \\ &= \chi(k:2,3,4,5)f_{k-3}, \end{aligned}$$

which is the first case of our lemma.

To prove the second case, assume that $n \equiv_{k+1} f_{k-1}+1$. Then as above we get

$$\begin{aligned} H_{n,f_k-1} &= \det\begin{pmatrix} A_{f_{k-1}+1}\dots A_{f_k-1}A_{f_k}\dots A_{f_{k+1}-1} \\ B'_{f_{k-1}+1}\dots B'_{f_k-1}B'_{f_k}\dots B'_{f_{k+1}-1}\end{pmatrix} \\ &= \det\begin{pmatrix} A_{f_{k-1}+1} & \dots & \dots & A_{f_k-1} & A_{f_k}\dots A_{f_{k+1}-1} \\ & & (-1)^k & (-1)^{k-1} & \\ & & (-1)^{k-1} & & \\ 0 & & \dots & & \\ & \dots & & & 0 \\ (-1)^k & (-1)^{k-1} & & & \\ (-1)^{k-1} & & & & \end{pmatrix} \\ &= (-1)^{(k-1)(f_{k-2}-1)}(-1)^{(f_{k-2}-1)f_{k-1}+[(f_{k-2}-1)/2]} \\ &\quad \times \det(A_{f_k}\dots A_{f_{k+1}-1}). \end{aligned}$$

Therefore, by Theorem 3 we get

$$H_{n,f_k-1} = \chi(k:0,3,4)\chi(k-1:2)f_{k-2} = \chi(k:0,4)f_{k-2}. \ \blacksquare$$

THEOREM 4. *For any $k,n,i\in\mathbb{N}$ with $k\ge 1$, $n\equiv_{k+1} i$ and $0\le i<f_{k+1}$, we have*

$$H_{n,f_k-1}=\begin{cases}\chi(k:0,4)f_{k-2} & (i=0),\\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (0<i\le f_{k-1}),\\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & (f_{k-1}<i<f_k \\ & \quad \textit{and } \tau_{k+1}(n)=1),\\ \chi(k:0,4)f_{k-2} & (i=f_{k-1}+1 \\ & \quad \textit{and } \tau_{k+1}(n)=0),\\ 0 & (\textit{otherwise}),\end{cases}$$

$$\overline{H}_{n,f_k-1}=\begin{cases}\chi(k:2,3,4,5)f_{k-3} & (i=0),\\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (0<i\le f_{k-1}),\\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & (f_{k-1}<i<f_k \\ & \quad \textit{and } \tau_{k+1}(n)=1),\\ \chi(k:2,3,4,5)f_{k-3} & (i=f_{k-1}+1 \\ & \quad \textit{and } \tau_{k+1}(n)=0),\\ 0 & (\textit{otherwise}).\end{cases}$$

Proof. The first four cases follow from Lemmas 16 and 17. Note that for $i=f_{k-1}$, the assertions in these lemmas coincide, so that H_{n,f_k-1} is independent of $\tau_{k+1}(n)$. Consider the last case, where $\tau_{k+1}(n)=0$ and $f_{k-1}+2\le i\le f_{k+1}-1$. We may assume that $k\ge 2$. Then, with $m=f_k-1$ and $i-f_k$ in place of i there, the condition (2) of Theorem 1 is satisfied. Therefore by Theorem 1, $n\in\mathcal{R}_m$, which implies that $H_{n,f_k-1}=0$. ∎

Lemma 18. *For any $n,m\in\mathbb{N}$ such that $f_{k-2}+1\le m\le f_k-2$, $i\le n$ and $n-i\equiv_{k+1}0$ for some $i,k\in\mathbb{Z}$ with $k\ge 2$ and $m+i=f_k$, we have*

$$H_{n,m}=\chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3},$$
$$\overline{H}_{n,m}=\chi(k:1,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-3}.$$

Proof. First, we consider the case $i<f_{k-2}$. By arguments similar to those used in the proof of Lemma 15, we get, with the notation (20),

$$H_{n,m}=\det\begin{pmatrix} A_iA_{i+1} & \ldots & A_{f_{k-1}+i-1} & 0 & \ldots & 0 & A_{f_k-1}\\ & & & & & (-1)^k & (-1)^{k-1}\\ & & & & & \cdots & \\ & \mathbf{0} & & & \ldots & & \mathbf{0}\\ & & (-1)^k & (-1)^{k-1} & & & \end{pmatrix}.$$

Therefore, by Theorems 3 and 4,

$$\begin{aligned}H_{n,m}&=(-1)^{k(f_{k-2}-i+1)+[(f_{k-2}-i+1)/2]}H_{i,f_{k-1}-1}\\ &\quad+(-1)^{(k-1)(f_{k-2}-i)+[(f_{k-2}-i)/2]}H_{i,f_{k-1}}\\ &=\chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}(-f_{k-4}+f_{k-2})\\ &=\chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3}.\end{aligned}$$

If $i=f_{k-2}$, then the statement follows from Theorem 3.

Finally, we consider the case $f_{k-2} < i < f_{k-1}$. Then, setting

$$A_j^r = {}^t(\varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+r-1}), \tag{24}$$

by Theorem 3 we obtain

$$\begin{aligned}
H_{n,m} &= \det(A_i^{f_k-i} A_{i+1}^{f_k-i} \ldots A_{f_k-1}^{f_k-i}) \\
&= \det\begin{pmatrix}
A_i^{f_{k-2}} & A_{i+1}^{f_{k-2}} & \ldots & A_{f_{k-1}-2}^{f_{k-2}} & A_{f_{k-1}-1}^{f_{k-2}} & A_{f_{k-1}}^{f_{k-2}} & \ldots & A_{f_k-1}^{f_{k-2}} \\
 & & & (-1)^{k-1} & (-1)^k & & & \\
 & 0 & & (-1)^k & & & & \\
 & & \ldots & \ldots & & & & \\
 & & \ldots & \ldots & & & 0 & \\
(-1)^{k-1} & (-1)^k & & & & & & \\
(-1)^k & & & & & & &
\end{pmatrix} \\
&= (-1)^{k(f_{k-1}-i)}(-1)^{(f_{k-1}-i)f_{k-2}+[(f_{k-1}-i)/2]} H_{f_{k-1},f_{k-2}} \\
&= \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-3}. \quad \blacksquare
\end{aligned}$$

LEMMA 19. *For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \le m \le f_k - 2$, $i \le n$, $n - i \equiv_k f_{k-1}$ for some $i, k \in \mathbb{Z}$ with $k \ge 2$ and $m + i = f_k$, we have*

$$H_{n,m} = \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-2},$$
$$\overline{H}_{n,m} = \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-3}.$$

P r o o f. By the same arguments and in the same notations as in the second part of the proof of Lemma 18, we obtain

$$\begin{aligned}
H_{n,m} &= \det(A_{f_{k-1}+i}^{f_k-i} \ldots A_{f_k-1}^{f_k-i} A_{f_k}^{f_k-i} \ldots A_{f_{k+1}-1}^{f_k-i}) \\
&= \det\begin{pmatrix}
A_i^{f_{k-1}} & A_{i+1}^{f_{k-1}} & \ldots & A_{f_{k-2}-2}^{f_{k-1}} & A_{f_{k-2}-1}^{f_{k-1}} & A_{f_k}^{f_{k-1}} & \ldots & A_{f_{k+1}-1}^{f_{k-1}} \\
 & & & (-1)^k & (-1)^{k-1} & & & \\
 & 0 & & (-1)^{k-1} & & & & \\
 & & \ldots & \ldots & & & & \\
 & & \ldots & \ldots & & & 0 & \\
(-1)^k & (-1)^{k-1} & & & & & & \\
(-1)^{k-1} & & & & & & &
\end{pmatrix} \\
&= (-1)^{(k-1)(f_{k-2}-i)}(-1)^{(f_{k-2}-i)f_{k-1}+[(f_{k-2}-i)/2]} H_{f_k,f_{k-1}} \\
&= \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-2}. \quad \blacksquare
\end{aligned}$$

LEMMA 20. *For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \le m \le f_k - 2$, $i \le n$ and $n - i \equiv_{k+1} 0$ for some $i, k \in \mathbb{Z}$ with $k \ge 2$ and $m + i = f_k - 1$, we have*

$$H_{n,m} = \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-2},$$
$$\overline{H}_{n,m} = \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-3}.$$

Proof. The proof is similar to the first part of the proof of Lemma 18. With the notation in (20), we get

$$H_{n,m} = \det \begin{pmatrix} A_iA_{i+1} & \dots & A_{f_{k-1}+i-1} & 0 & 0 & \dots & 0 \\ & & & & & & (-1)^k \\ & & & & & \dots & \\ & 0 & & & \dots & & \\ & & & (-1)^k & (-1)^{k-1} & & 0 \end{pmatrix}$$
$$= (-1)^{k(f_{k-2}-1-i)}(-1)^{[(f_{k-2}-1-i)/2]} \det(A_iA_{i+1}\dots A_{f_{k-1}+i-1}).$$

Hence, by Theorem 3

$$H_{n,m} = \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-2}. \ \blacksquare$$

Lemma 21. *For any $n, m \in \mathbb{N}$ such that $f_{k-2}+1 \le m \le f_k - 2$, $i \le n$ and $n - i \equiv_k f_{k-1}$ for some $i, k \in \mathbb{Z}$ with $k \ge 2$ and $m + i = f_k - 1$, we have*

$$H_{n,m} = \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-3},$$
$$\overline{H}_{n,m} = \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-4}.$$

Proof. Since $i = f_k - 1 - m$, we get $1 \le i \le f_{k-1} - 2$.

If $i = f_{k-2} - 1$, then $m = f_{k-1}$ and $n \equiv_k f_k - 1$. Therefore, by Theorem 3, we get

$$H_{n,m} = \chi(k-1:1,2,4) f_{k-3},$$

which coincides with the required identity since

$$\chi(k:0,1,2)^{f_{k-2}-1} = \chi(k:\{0,1,2\}\cap\{0,3\}) = \chi(k:0),$$
$$(-1)^{[(f_{k-2}-1)/2]} = \chi(k:0,4).$$

If $i = f_{k-2}$, then $m = f_{k-1} - 1$ and $n \equiv_k 0$. Therefore, by Theorem 4, we get

$$H_{n,m} = \chi(k-1:0,4) f_{k-3},$$

which coincides with the required statement since

$$\chi(k:0,1,2)^{f_{k-2}} = \chi(k:\{0,1,2\}\cap\{1,2,4,5\}) = \chi(k:1,2),$$
$$(-1)^{[f_{k-2}/2]} = \chi(k:3,4).$$

If $f_{k-2} + 1 \le i \le f_{k-1} - 2$, then $n - i' \equiv_k 0$ with $i' := i - f_{k-2}$. Then, since $m + i' = f_{k-1} - 1$ and $f_{k-2} + 1 \le m \le f_{k-1} - 2$, applying Lemma 20, we obtain

$$\begin{aligned} H_{n,m} &= \chi(k-1:0,4)\chi(k-1:3,4,5)^{i'}(-1)^{[i'/2]} f_{k-3} \\ &= \chi(k:1,5)\chi(k:0,4,5)^i\chi(k:\{0,4,5\}\cap\{1,2,4,5\})(-1)^{[i'/2]} f_{k-3} \\ &= \chi(k:1,4)\chi(k:0,4,5)^i(-1)^{[i/2]}(-1)^{[(f_{k-2}+1)/2]}(-1)^{if_{k-2}} f_{k-3} \\ &= \chi(k:2,3,4,5)\chi(0,1,2)^i(-1)^{[i/2]} f_{k-3}. \end{aligned}$$

Now, we consider the case $1 \le i \le f_{k-2} - 2$. Then, with the notations in (24) and in (20), we get

$$H_{n,m} = \det(A^{f_k-i}_{f_{k-1}+i} \dots A^{f_k-i}_{f_k-1} A^{f_k-i}_{f_k} \dots A^{f_k-i}_{f_{k+1}-2})$$

$$= \det \begin{pmatrix} A_{f_{k-1}+i} & A_{f_{k-1}+i+1} & \dots & A_{f_k-2} & A_{f_k-1} & A_{f_k} & \dots & A_{f_{k+1}-2} \\ & & & (-1)^k & (-1)^{k-1} & & & \\ & 0 & & (-1)^{k-1} & & & & \\ & & \dots & \dots & & & & \\ & & \dots & \dots & & & 0 & \\ & (-1)^k & & & & & & \\ (-1)^k & (-1)^{k-1} & & & & & & \end{pmatrix}.$$

Therefore, by arguments similar to those used in the first part of the proof of Lemma 17, we get

$$\begin{aligned} H_{n,m} = {} & (-1)^{k(f_{k-2}-1-i)}(-1)^{f_{k-1}(f_{k-2}-1-i)+[(f_{k-2}-1-i)/2]} \\ & \times \{\det(A_{f_k-1} A_{f_k} \dots A_{f_{k+1}-2}) + (-1)^{k-1} \det(A''_{f_k} \dots A''_{f_{k+1}-2}) \\ & + (-1)^{k+f_{k-2}-1-i} \det(A'''_{f_k} \dots A'''_{f_{k+1}-2})\}, \end{aligned}$$

where we use the same notations as in the proof of Lemma 17 except for A'''_j's which are defined by

$$A'''_j = {}^t(\varepsilon_j \dots \varepsilon_{j+f_{k-2}-i-2} \varepsilon_{j+f_{k-2}-i} \dots \varepsilon_{j+f_{k-1}-1}).$$

Then, following the arguments there, we get

$$\begin{aligned} H_{n,m} = {} & \chi(k:4)\chi(k:0,1,2)^i(-1)^{[i/2]}\{H_{f_k-1,f_{k-1}} \\ & + (-1)^{k-1} H_{f_k+1,f_{k-1}-1} + (-1)^{k+f_{k-2}-1-i} E\} \end{aligned}$$

with

$$\begin{aligned} E := {} & \det(A'''_{f_k} \dots A'''_{f_{k+1}-2}) \\ = {} & \det(A'_{f_k} \dots A'_{f_k+f_{k-2}-i-2} A'_{f_k+f_{k-2}-i} \dots A'_{f_{k+1}-1}) \\ = {} & \det(A'_{f_{k+1}} \dots A'_{f_{k+1}+f_{k-2}-i-2} A'_{f_k+f_{k-2}-i} \dots A'_{f_{k+1}-1}) \\ = {} & (-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} \det(A'_{f_k+f_{k-2}-i} \dots A'_{f_{k+1}+f_{k-2}-i-2}) \\ = {} & (-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} H_{f_{k-2}-i,f_{k-1}-1}, \end{aligned}$$

where we have used Lemma 5. Therefore, by Theorems 3 and 4, we have

$$\begin{aligned} H_{n,m} = {} & \chi(k:4)\chi(k:0,1,2)^i(-1)^{[i/2]}\{\chi(k-1:1,2,4) f_{k-3} \\ & + (-1)^{k-1}\chi(k-1:2,3,4,5) f_{k-4} \\ & + (-1)^{k+f_{k-2}-1-i}(-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} \\ & \times \chi(k-1:1,2,3,5)\chi(k-1:1,4)^{f_{k-2}-i} f_{k-4}\} \\ = {} & \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-3}. \ \blacksquare \end{aligned}$$

4. Tiling for $H_{n,m}$ and $\overline{H}_{n,m}$. In this section, we collect the values of $H_{n,m}$ and $\overline{H}_{n,m}$ obtained in the last section and arrange them in the quarter plane $\Omega := \{0,1,\ldots\} \times \{1,2,\ldots\}$. We will tile Ω by the following tiles on which the values $H_{n,m}$ are written in. That is, $U_1 := V_1 := \{(1,-1)\}$, and for $k \geq 2$,

$$U_k := \{(i,j) \in \mathbb{Z}^2 : 0 \leq i+j \leq f_{k-1}-1,\ -f_{k-1} \leq j \leq -1\},$$
$$V_k := \{(i,j) \in \mathbb{Z}^2 : 0 \leq i+j \leq f_{k-2}-1,\ -f_{k-2} \leq j \leq -1\},$$

with the written-in values $u_k : U_k \to \mathbb{Z}$ and $v_k : V_k \to \mathbb{Z}$ given by $u_1(1,-1) := 0$, $v_1(1,-1) := 1$, and for $k \geq 2$,

$$u_k(i,j) := \begin{cases} \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=0), \\ \chi(k:0,3,4)\chi(k:0,3)^i f_{k-3} & (j=-f_{k-1}), \\ \chi(k:3,5)\chi(k:2,3,4)^i(-1)^{[i/2]}f_{k-3} & (i+j=f_{k-1}-1), \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (j=-1), \\ 0 & (\text{otherwise}), \end{cases}$$

$$v_k(i,j) := \begin{cases} \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-2} & (i+j=0), \\ \chi(k:2,3,5)\chi(k:2,5)^i f_{k-2} & (j=-f_{k-2}), \\ \chi(k:0,1,2,3)\chi(k:1,2,3)^i(-1)^{[i/2]}f_{k-2} & (i+j=f_{k-2}-1), \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-2} & (j=-1), \\ 0 & (\text{otherwise}), \end{cases}$$

and with $\overline{u}_k : U_k \to \mathbb{Z}$ and $\overline{v}_k : V_k \to \mathbb{Z}$ given $\overline{u}_1(1,-1) := 1$, $\overline{v}_1(1,-1) := 0$, and for $k \geq 2$,

$$\overline{u}_k(i,j) := \begin{cases} \chi(k:1,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-4} & (i+j=0), \\ \chi(k:4)\chi(k:0,3)^i f_{k-4} & (j=-f_{k-1}), \\ \chi(k:1,2,3,4)\chi(k:0,1,5)^i(-1)^{[i/2]}f_{k-4} & (i+j=f_{k-1}-1), \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (j=-1), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\overline{v}_k(i,j) := \begin{cases} \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=0), \\ \chi(k:3)\chi(k:2,5)^i f_{k-3} & (j=-f_{k-2}), \\ \chi(k:2,4)\chi(k:0,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=f_{k-2}-1), \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (j=-1), \\ 0 & (\text{otherwise}). \end{cases}$$

For $k \geq 1$ let

$$\mathcal{U}_k := \{(n,f_k) : n \in \mathbb{N} \text{ and } n \equiv_{k+1} 0\},$$
$$\mathcal{V}_k := \{(n,f_k) : n \in \mathbb{N} \text{ and } n \equiv_{k+2} f_{k+1}+f_{k-1}\},$$
$$T_k := (V_k + (-f_{k-2}, f_k)) \cap \Omega,$$

where $V + (x,y) := \{(v+x, w+y) : (v,w) \in V\}$ for $V \subset \mathbb{Z}^2, (x,y) \in \mathbb{Z}^2$.

Theorem 5. *We have*

$$\Omega=\bigcup_{k=1}^{\infty}\Big(\bigcup_{(i,j)\in\mathcal{U}_k}(U_k+(i,j))\cup\bigcup_{(i,j)\in\mathcal{V}_k}(V_k+(i,j))\cup T_k\Big),$$

where the right hand side is a disjoint union, so that Ω is tiled by the U_k's, V_k's and T_k's. Moreover, for any $(n,m)\in\Omega$, if $(n,m)=(i,j)+(i',j')$ with $(i,j)\in U_k$ and $(i',j')\in\mathcal{U}_k$, then $H_{n,m}=u_k(i,j)$ and $\overline{H}_{n,m}=\overline{u}_k(i,j)$. Also, if $(n,m)=(i,j)+(i',j')$ with $(i,j)\in V_k$ and either $(i',j')\in\mathcal{V}_k$ or $(i',j')=(-f_{k-2},f_k)$, then $H_{n,m}=v_k(i,j)$ and $\overline{H}_{n,m}=\overline{v}_k(i,j)$. Furthermore, in this tiling, the tiles U_k, V_k and T_k with $k\ge 2$ are followed by the sequences of smaller tiles $U_{k-1}V_{k-1}U_{k-1}$, U_{k-1} and U_{k-1}, respectively, as shown in Figure 1.

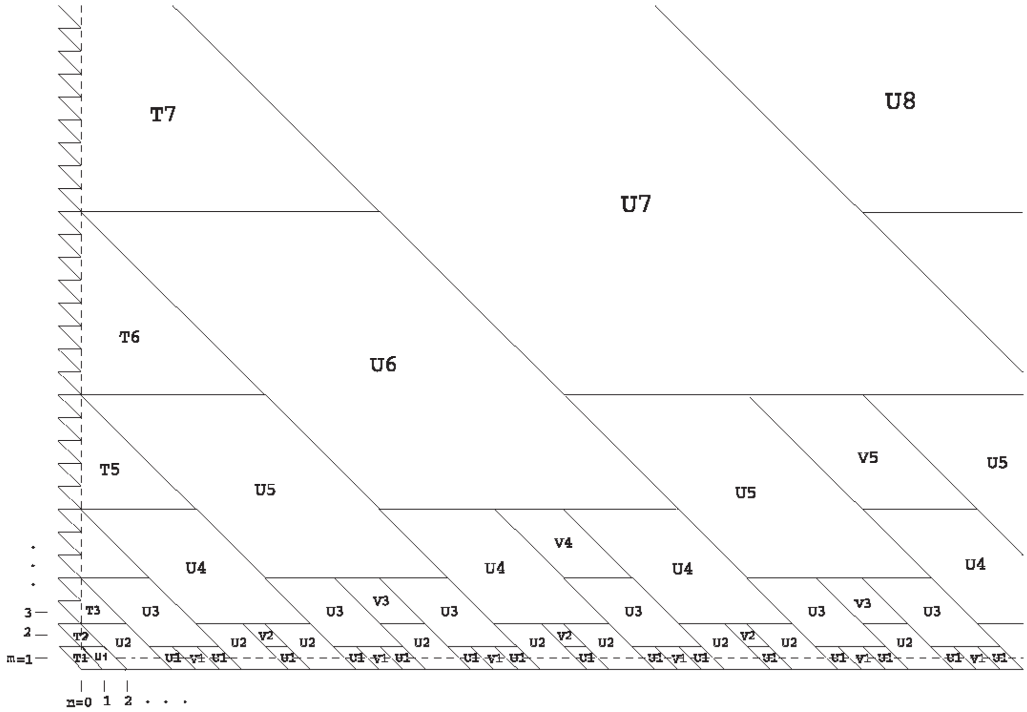

Fig. 1. Tiling for $H_{n,m}$

Proof. Take an arbitrary point $(n,m)\in\Omega$. Let $f_{k-1}\le m<f_k$. If $n+m-f_k\ge 0$, define $0\le i<f_{k+2}$ by $i\equiv_{k+2} n$.

Case 1: $n+m-f_k<0$. We get $(n,m)\in T_k$.

Case 2: $0\le i<f_{k-1}$. We get $(n,m)\in U_k+(n+m-i-f_k,f_k)$.

Case 3: $f_{k-1} \le i < f_{k+1}$. We get $(n,m) \in U_{k+1} + (n+m-i-f_{k+1}, f_{k+1})$.

Case 4: $f_{k+1} \le i < f_{k+1} + f_{k-1}$. We get $(n,m) \in U_k + (n+m-i+f_{k-1}, f_k)$.

Case 5: $f_{k+1} + f_{k-1} \le i < f_{k+2}$. We get $(n,m) \in V_k + (n+m-i+2f_{k-1}, f_k)$.

The fact that the written-in values coincide with $H_{n,m}$ and $\overline{H}_{n,m}$ follows from Lemma 18 (first case in u_k and $\overline{u}_k$), Theorem 3 (second case), Lemma 21 (third case), Theorem 4 (fourth case), Corollary 3 (fifth case), Lemma 19 (first case in v_k and $\overline{v}_k$), Theorem 3 (second case), Lemma 20 (third case), Lemma 20 (fourth case) and Corollary 3 (fifth case). The m in the preceding lemmas and theorems coincides with $f_k + j$ in Theorem 5 while the meaning of the symbols k, i, n is not necessarily the same. ∎

5. Padé approximation. Let $\varphi = \varphi_0\varphi_1\ldots$ be an infinite sequence over a field $\mathbb{K}$, $\widehat{H}_{n,m} := H_{n,m}(\varphi)$ be the Hankel determinant (3), and $\varphi(z)$ the formal Laurent series (4) with $h = -1$. We also denote the *Hankel matrices* by

$$\widehat{M}_{n,m} := (\varphi_{n+i+j})_{i,j=0,1,\ldots,m-1} \qquad (n = 0,1,\ldots;\ m = 1,2,\ldots), \tag{25}$$

so that $\widehat{H}_{n,m} = \det \widehat{M}_{n,m}$.

The following proposition is well known ([1], for example). But we give a proof for self-containment.

Proposition 1. (1) *For any $m = 1,2,\ldots,$ a Padé pair (P,Q) of order m for φ exists. Moreover, for each m, the rational function $P/Q \in \mathbb{K}(z)$ is determined uniquely for such Padé pairs (P,Q).*

(2) *For any $m = 1,2,\ldots,$ m is a normal index for φ if and only if $\widehat{H}_{0,m}(\varphi) \ne 0$.*

Proof. Let

$$P = p_0 + p_1 z + p_2 z^2 + \ldots + p_m z^m,$$
$$Q = q_0 + q_1 z + q_2 z^2 + \ldots + q_m z^m.$$

Then the condition $\|Q\varphi - P\| < \exp(-m)$ is equivalent to

$$\begin{array}{llllll} & & & & -p_m & = 0, \\ & & & q_m\varphi_0 & -p_{m-1} & = 0, \\ & & & \cdots & \cdots & \\ & q_1\varphi_0 + & \ldots + & q_m\varphi_{m-1} & -p_0 & = 0, \\ q_0\varphi_0 + & q_1\varphi_1 + & \ldots + & q_m\varphi_m & & = 0, \\ & \cdots & \cdots & & & \\ q_0\varphi_{m-1} + & q_1\varphi_{m-2} + & \ldots + & q_m\varphi_{2m-1} & & = 0. \end{array} \tag{26}$$

Furthermore, the system (26) for $(q_0 q_1 \ldots q_m)$ is equivalent to

$$(27) \qquad (q_0 q_1 \ldots q_{m-1})\widehat{M}_{0,m} + q_m(\varphi_m \varphi_{m+1} \ldots \varphi_{2m-1}) = (00\ldots 0),$$

where $(p_0 p_1 \ldots p_m)$ is determined by $(q_0 q_1 \ldots q_m)$ by the upper half of (26). There are two cases.

CASE 1: $\widehat{H}_{0,m} = 0$. In this case, since $\det \widehat{M}_{0,m} = \widehat{H}_{0,m} = 0$, there exists a nonzero vector $(q_0 q_1 \ldots q_{m-1})$ such that $(q_0 q_1 \ldots q_{m-1})\widehat{M}_{0,m} = 0$. Then (27) is satisfied with this $(q_0 q_1 \ldots q_{m-1})$ and $q_n = 0$.

CASE 2: $\widehat{H}_{0,m} \neq 0$. In this case, since $\det \widehat{M}_{0,m} = \widehat{H}_{0,m} \neq 0$, there exists a unique vector $(q_0 q_1 \ldots q_{m-1})$ such that

$$(28) \qquad (q_0 q_1 \ldots q_{m-1})\widehat{M}_{0,m} = -(\varphi_m \varphi_{m+1} \ldots \varphi_{2m-1}).$$

Then (27) is satisfied with this $(q_0 q_1 \ldots q_{m-1})$ and $q_m = 1$.

Thus, a Padé pair of order m exists. Moreover, by the above arguments, a Padé pair (P, Q) of order m with $\deg Q < m$ exists if and only if $\widehat{H}_{0,m} = 0$, since if $\widehat{H}_{0,m} \neq 0$, then by (27), $q_m = 0$ implies $(q_0 q_1 \ldots q_{m-1}) = (00\ldots 0)$, and hence $Q = 0$.

Now we prove that for any Padé pairs (P, Q) and (P', Q') of order m, we have $P/Q = P'/Q'$. By (5), we have

$$\|\varphi - P/Q\| < \exp(-n - \deg Q), \qquad \|\varphi - P'/Q'\| < \exp(-m - \deg Q').$$

Hence,

$$\|P/Q - P'/Q'\| < \exp(-m - \deg Q \wedge \deg Q').$$

Therefore,

$$\|PQ' - P'Q\| < \exp(-m + \deg Q \vee \deg Q') \leq 1.$$

Since $PQ' - P'Q$ is a polynomial of z, $\|PQ' - P'Q\|$ is either 0 or not less than 1. Hence, the above inequality implies $PQ' - P'Q = 0$. ∎

In view of (26), without loss of generality, we can put

$$(29) \qquad \begin{aligned} P &= p_0 + p_1 z + p_2 z^2 + \ldots + p_{m-1} z^{m-1}, \\ Q &= q_0 + q_1 z + q_2 z^2 + \ldots + q_m z^m. \end{aligned}$$

THEOREM 6. *Let (P, Q) be the normalized Padé pair for φ with $\deg Q$ as its normal index m with P, Q given by* (29). *Then*

(1) $Q(z)=\widehat{H}_{0,m}^{-1}\det(z\widehat{M}_{0,m}-\widehat{M}_{1,m})$.

(2) $\det(zI-\widehat{M}_{0,m})$ *is equal to*

$$\begin{vmatrix}
z & & & & & & & & & & & & & 1\\
& z & & & & & & & & & & & 1 & \\
& & \ddots & & & & & & & & & \iddots & & \\
& & & z & & & & & & & 1 & & & \\
& & & & z & & & & & 1 & & & & \\
p_{m-1} & & & & & 1 & & & & & & & & \\
p_{m-2} & p_{m-1} & & & & q_{m-1} & 1 & & & & & & & \\
\vdots & \ddots & \ddots & & & \vdots & \ddots & \ddots & & & & & & \\
p_1 & \cdots & & p_{m-1} & & q_2 & \cdots & & 1 & & & & & \\
p_0 & \cdots & & p_{m-2} & p_{m-1} & q_1 & \cdots & & q_{m-1} & 1 & & & & \\
& p_0 & \cdots & \vdots & p_{m-2} & q_0 & \cdots & \cdots & \vdots & q_{m-1} & 1 & & & \\
& & \ddots & p_1 & \vdots & & q_0 & \cdots & \vdots & \vdots & & \ddots & & \\
& & & p_0 & p_1 & & & \ddots & q_1 & q_2 & \cdots & \cdots & 1 & \\
& & & & p_0 & & & & q_0 & q_1 & \cdots & \cdots & q_{m-1} & 1
\end{vmatrix},$$

where I is the unit matrix of size m.

(3) *We have*

$$\widehat{H}_{0,m}=(-1)^{[m/2]}\prod_{z;\,Q(z)=0}P(z)=(-1)^{[m/2]}p_k^m\prod_{z;\,P(z)=0}Q(z),$$

where $\prod_{z;\,R(z)=0}$ denotes the product over all the roots of the polynomial $R(z)$ with their multiplicity and p_k is the leading coefficient of $P(z)$, that is, $p_{m-1}=\ldots=p_{k+1}=0, p_k\neq 0$ if $P(z)$ is not the zero polynomial, otherwise $p_k=0$.

Proof. (1) Note that $q_m=1$ by the assumption that (P,Q) is the normalized Padé pair. By (28), we have

$$\begin{pmatrix}
0 & 1 & & & & \\
& 0 & 1 & & & \\
& & & \ddots & & \\
& & & & 0 & 1\\
-q_0 & -q_1 & & \ldots & -q_{m-2} & -q_{m-1}
\end{pmatrix}\widehat{M}_{0,m}=\widehat{M}_{1,m}.$$

Since

$$\widehat{H}_{0,m}=\det\widehat{M}_{0,m}\neq 0$$

by the normality of the index m, it follows that

$$Q(z) = \det \left(zI - \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -q_0 & -q_1 & \dots & -q_{m-2} & -q_{m-1} \end{pmatrix} \right)$$
$$= \det(zI - \widehat{M}_{1,m}\widehat{M}_{0,m}^{-1})$$
$$= \widehat{H}_{0,m}^{-1} \det(z\widehat{M}_{0,m} - \widehat{M}_{1,m}).$$

(2) We define the matrices:

$$P_m := \begin{pmatrix} p_{m-1} & p_{m-2} & \dots & p_1 & p_0 \\ p_{m-2} & \dots & \dots & p_0 & \\ \vdots & & \iddots & & \\ p_1 & \iddots & & \mathbf{0} & \\ p_0 & & & & \end{pmatrix},$$

$$P'_{m-1} := \begin{pmatrix} & & & & p_{m-1} \\ & \mathbf{0} & & p_{m-1} & p_{m-2} \\ & & \iddots & \vdots & \vdots \\ & \iddots & & \vdots & p_2 \\ p_{m-1} & p_{m-2} & \dots & p_2 & p_1 \end{pmatrix},$$

$$Q_m := \begin{pmatrix} 1 & & & & \\ q_{m-1} & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ q_1 & q_2 & \dots & q_{m-1} & 1 \end{pmatrix},$$

$$Q'_m := \begin{pmatrix} & & & & 1 \\ & \mathbf{0} & & 1 & q_{m-1} \\ & & \iddots & & \vdots \\ & \iddots & & & \vdots \\ 1 & q_{m-1} & \dots & q_2 & q_1 \end{pmatrix},$$

$$Q''_{m-1} := \begin{pmatrix} 1 & & & & \\ q_{m-1} & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ q_2 & q_3 & \cdots & q_{m-1} & 1 \end{pmatrix},$$

$$Q_{m,m-1} := \begin{pmatrix} q_1 & q_2 & \cdots & q_{m-2} & q_{m-1} \\ q_0 & q_1 & \cdots & q_{m-3} & q_{m-2} \\ & q_0 & q_1 & \cdots & q_{m-3} \\ & & \ddots & \ddots & \vdots \\ & \mathbf{0} & & \ddots & q_1 \\ & & & & q_0 \end{pmatrix},$$

$$\Phi_{m-1} := \begin{pmatrix} & & & & \varphi_0 \\ & \mathbf{0} & & \varphi_0 & \varphi_1 \\ & & \iddots & \vdots & \vdots \\ & \iddots & & \vdots & \varphi_{m-3} \\ \varphi_0 & \varphi_1 & \cdots & \varphi_{m-3} & \varphi_{m-2} \end{pmatrix}.$$

We denote by O the zero matrices of various sizes. We also denote by I_n the unit matrix of size n. By (26), we have

$$\begin{aligned}
&\det(zI - \widehat{M}_{0,m}) \\
&\quad = \det\left(z \begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_m^{-1} Q_{m,m-1} & \widehat{M}_{0,m} \end{pmatrix} \right) \\
&\quad = \det\left(\begin{pmatrix} I_{m-1} & O \\ O & Q_m \end{pmatrix} \left(z \begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_m^{-1} Q_{m,m-1} & \widehat{M}_{0,m} \end{pmatrix} \right) \right) \\
&\quad = \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_m \widehat{M}_{0,m} \end{pmatrix} \right) \\
&\quad = \det\left(\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_m \widehat{M}_{0,m} \end{pmatrix} \right) \begin{pmatrix} I_{m-1} & O \ \Phi_{m-1} \\ O & I_m \end{pmatrix} \right) \\
&\quad = \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \ -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right),
\end{aligned}$$

where we use (26) to get the last equality. Hence

$$\det(zI - \widehat{M}_{0,m}) = \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \ -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right)$$

$$= \det\left(\begin{pmatrix} Q''_{m-1} & O \\ O & I_m \end{pmatrix}\left(z\begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \;\; -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix}\right)\right)$$

$$= \det\left(z\begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -Q''_{m-1} & O \;\; -P'_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix}\right)$$

$$= (-1)^m \det\left(\begin{pmatrix} Q''_{m-1} & O \;\; P'_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} - z\begin{pmatrix} O & O \\ O & Q_m \end{pmatrix}\right)$$

$$= (-1)^m \det\begin{pmatrix} I_m & O & zI_m \\ O & Q''_{m-1} & O \;\; P'_{m-1} \\ Q'_m & Q_{m,m-1} & P_m \end{pmatrix},$$

which implies (2).

(3) By (2), we have

$$\widehat{H}_{0,m} = (-1)^m \det(0I - \widehat{M}_{0,m})$$

$$= (-1)^{[m/2]}\begin{vmatrix}
p_{m-1} & & & & & 1 & & & \\
p_{m-2} & p_{m-1} & & & & q_{m-1} & 1 & & \\
\vdots & & \ddots & & & \vdots & \ddots & \ddots & \\
p_1 & \ddots & & p_{m-1} & & q_2 & \ddots & & 1 \\
p_0 & \ddots & & p_{m-2} & p_{m-1} & q_1 & \ddots & & q_{m-1} \\
 & p_0 & & & p_{m-2} & q_0 & \ddots & & q_{m-2} \\
 & & \ddots & \vdots & \vdots & & \ddots & & \vdots \\
 & & & p_0 & p_1 & & & \ddots & q_1 \\
 & & & & p_0 & & & & q_0
\end{vmatrix},$$

which completes the proof since the last determinant is Sylvester's determinant for $P(z)$ and $Q(z)$. ∎

For a finite or infinite sequence $a_0(z), a_1(z), \ldots$ of elements in $\mathbb{K}((z^{-1}))$, we use the notation

$$[a_0(z); a_1(z), a_2(z), \ldots, a_n(z)] := a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \ddots + \cfrac{1}{a_n(z)}}}$$

and

$$(30) \qquad [a_0(z); a_1(z), a_2(z), \ldots] := \lim_{n\to\infty} [a_0(z); a_1(z), a_2(z), \ldots, a_n(z)]$$

provided that the limit exists, where the limit is taken with respect to the metric induced by the nonarchimedean norm in $\mathbb{K}((z^{-1}))$.

We define

$$p_0(z)=a_0(z),\quad p_{-1}(z)=1,\quad q_0(z)=1,\quad q_{-1}(z)=0, \tag{31}$$
$$p_n(z)=a_n(z)p_{n-1}(z)+p_{n-2}(z),\quad q_n(z)=a_n(z)q_{n-1}(z)+q_{n-2}(z) \quad (n=1,2,3,\ldots)$$

for any given sequence $a_1(z), a_2(z), \ldots \in \mathbb{K}((z^{-1}))$. Then $p_n(z), q_n(z) \in \mathbb{K}((z^{-1})), p_n(z) \neq 0$ if $q_n(z)=0$, and

$$\frac{p_n(z)}{q_n(z)} = [a_0(z); a_1(z), a_2(z), \ldots, a_n(z)] \in \mathbb{K}((z^{-1})) \cup \{\infty\} \quad (n \geq 0),$$

where we mean $\psi/0 := \infty$ for $\psi \in \mathbb{K}((z^{-1}))\setminus\{0\}$, and $\psi+\infty := \infty, \psi/\infty := 0$ for $\psi \in \mathbb{K}((z^{-1}))$. By using (31), it can be shown that the limit (30) always exists in the set $\mathbb{K}((z^{-1}))$ as far as

$$a_n(z) \in \mathbb{K}[z] \quad (n \geq 0), \qquad \deg a_n(z) \geq 1 \quad (n \geq 1). \tag{32}$$

For $\varphi(z) \in \mathbb{K}((z^{-1}))$ given by (4), we denote by $\lfloor \varphi(z) \rfloor$ the polynomial part of $\varphi(z)$, which is defined as follows:

$$\lfloor \varphi(z) \rfloor := \sum_{k=0}^{h} \varphi_h z^{-k+h} \in \mathbb{K}[z].$$

We denote by T the mapping $T : \mathbb{K}((z^{-1})) \setminus \{0\} \to \mathbb{K}((z^{-1}))$ defined by

$$T(\psi(z)) := \frac{1}{\psi(z)} - \left\lfloor \frac{1}{\psi(z)} \right\rfloor \quad (\psi(z) \in \mathbb{K}((z^{-1})) \setminus \{0\}).$$

Then, for any given $\varphi(z) \in \mathbb{K}((z^{-1}))$, we can define the continued fraction expansion of $\varphi(z)$:

$$\varphi(z) = \begin{cases} [a_0(z); a_1(z), a_2(z), \ldots, a_{N-1}(z)] & \text{if } \varphi(z) \in \mathbb{K}(z), \\ [a_0(z); a_1(z), a_2(z), a_3(z), \ldots] & \text{otherwise} \end{cases} \tag{33}$$

with $a_n(z)$ satisfying (32) according to the following algorithm.

Continued Fraction Algorithm:

$$a_0(z) = \lfloor \varphi(z) \rfloor, \qquad a_n(z) = \left\lfloor \frac{1}{T^{n-1}(\varphi(z) - a_0(z))} \right\rfloor,$$
$$N = N(\varphi(z)) := \inf\{m : T^{m-1}(\varphi(z)) = 0\} \quad (\inf \emptyset := \infty).$$

We note that if $\varphi(z) \in \mathbb{K}(z)$, then $N < \infty$; if $\varphi(z) \in \mathbb{K}((z^{-1})) \setminus \mathbb{K}(z)$, then $N = \infty$ and the continued fraction (33) converges to the given $\varphi(z) \in \mathbb{K}(z)$. We say a continued fraction is *admissible* if it is obtained by the algorithm. We remark that a continued fraction (33) is admissible if and only if (32) holds.

The following proposition is known [2], but we give a proof for completeness.

PROPOSITION 2. *The set of all $P/Q \in \mathbb{K}(z)$ for Padé pairs (P,Q) for $\varphi(z) \in \mathbb{K}((z))$ coincides with the set of convergents $p_n(z)/q_n(z)$ $(0 \le n < N)$ of the continued fraction expansion of $\varphi(z)$. Moreover, m is a normal index if and only if m is a degree of $q_n(z)$ for some $n = 0, 1, 2, \ldots$ (with $n < N$ if $\varphi(z) \in \mathbb{K}(z)$).*

Proof. Note that

$$\varphi(z) = \frac{(a_n(z) + T^n(\varphi(z) - a_0))p_{n-1}(z) + p_{n-2}(z)}{(a_n(z) + T^n(\varphi(z) - a_0))q_{n-1}(z) + q_{n-2}(z)},$$
$$(-1)^n = p_{n-1}(z)q_{n-2}(z) - p_{n-2}(z)q_{n-1}(z).$$

Hence, we have

$$\|q_n(z)\varphi(z) - p_n(z)\| = \left\| \frac{(-1)^n T^n(\varphi(z) - a_0(z))}{q_n(z) + T^n(\varphi(z) - a_0(z))q_{n-1}(z)} \right\|$$
$$= \exp(-\deg a_{n+1}(n) - \deg q_n(z)),$$

so that

$$\|q_n(z)\varphi(z) - p_n(z)\| < \exp(-\deg q_n(z)) \quad (n < N). \tag{34}$$

In the case $N < \infty$, the left-hand side of (34) turns out to be 0 for $n = N-1$. Therefore, $(p_n(z), q_n(z))$ is a Padé pair of order $m = \deg q_n(z)$ for all $m \in \{\deg q_n(z) : 0 \le n < N\}$.

Conversely, for any $k = 1, 2, \ldots$, let (P,Q) be a Padé pair of order k. Let $\deg q_n(z) \le k < \deg q_{n+1}(z)$ for some $n = 0, 1, 2, \ldots$ with $n < N$ $(\deg q_N(z) := \infty)$. Then, since $\deg Q \le k < \deg q_{n+1}$, it follows from (34) that

$$\|\varphi(z) - p_n(z)/q_n(z)\| = \exp(-\deg q_n(z) - \deg q_{n+1}(z))$$
$$< \exp(-\deg q_n(z) - \deg Q).$$

Since (P,Q) is a Padé pair of order k, we have

$$\|\varphi(z) - P/Q\| < \exp(-k - \deg Q) \le \exp(-\deg q_n(z) - \deg Q).$$

Therefore,

$$\left\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \right\| < \exp(-\deg q_n(z) - \deg Q).$$

On the other hand, if $P/Q \ne p_n(z)/q_n(z)$, then

$$\left\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \right\| = \left\| \frac{Pq_n(z) - Qp_n(z)}{Qq_n(z)} \right\|$$
$$\ge \exp(-\deg q_n(z) - \deg Q),$$

which is a contradiction. Thus $P/Q = p_n(z)/q_n(z)$.

Note that $p_n(z)/q_n(z)$ is irreducible for any $n = 1, 2, \ldots$ with $n < N$, since $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$. Let $m = \deg q_n(z)$ for some $n = 1, 2, \ldots$ with $n < N$. Take any Padé pair (P,Q) of order m. Then $\deg Q \le m$. On the other hand, by the above argument, $P/Q = p_n(z)/q_n(z)$. Since $p_n(z)/q_n(z)$ is irreducible, this implies that $\deg Q \ge \deg q_n(z) = m$. Thus, m is a normal index.

Conversely, let $m \ge 0$ be any normal index. Take any Padé pair (P, Q) of order m. Then, by the above argument, there exists $n = 0, 1, 2, \ldots$ with $n < N$ such that $P/Q = p_n(z)/q_n(z)$. Hence the irreducibility of $p_n(z)/q_n(z)$ implies $\deg q_n(z) \le \deg Q$ $(\le m)$. Hence, $(p_n(z), q_n(z))$ is a Padé pair of order m. Since m is a normal index, $\deg q_n(z) = m$. ∎

We now obtain the continued fraction expansions for

$$\varphi_{\widehat{\varepsilon}}(z) = \widehat{\varepsilon}_0 z^{-1} + \widehat{\varepsilon}_1 z^{-2} + \widehat{\varepsilon}_2 z^{-3} + \ldots \in \mathbb{Q}((z^{-1}))$$

corresponding to the Fibonacci words $\widehat{\varepsilon} = \varepsilon(a, b)$ with $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$. As in Section 3, we use the notations ε and $\overline{\varepsilon}$ for them. The proofs in the following theorems are given only for ε, since the proof is similar for $\overline{\varepsilon}$. In [3], J. Tamura gave the Jacobi–Perron–Parusnikov expansion for a vector consisting of Laurent series with coefficients given by certain substitutions, which contains the following as its special case (cf. the footnote on p. 301 of [3]):

Proposition 3. *We have*

$$(z-1)\varphi_{\varepsilon}(z) = [0; z^{f_{-2}}, z^{f_{-1}}, z^{f_0}, z^{f_1}, z^{f_2}, \ldots].$$

Theorem 7. *We have the following admissible continued fraction for $\varphi_{\varepsilon}(z)$ and $\varphi_{\overline{\varepsilon}}(z)$:*

$$\varphi_{\varepsilon}(z) = [0; a_1, a_2, a_3, \ldots], \qquad \varphi_{\overline{\varepsilon}}(z) = [0; \overline{a}_1, \overline{a}_2, \overline{a}_3, \ldots]$$

with

$$a_1 = z, \quad a_2 = -z+1, \quad a_3 = -\tfrac{1}{2}(z+1),$$
$$a_{2n+2} = (-1)^{n-1} f_n^2 (z^{f_n - 1} + z^{f_n - 2} + \ldots + 1),$$
$$a_{2n+3} = (-1)^{n-1} \frac{1}{f_n f_{n+1}} (z-1) \quad (n = 1, 2, \ldots),$$

and

$$\overline{a}_1 = z^2, \quad \overline{a}_2 = -z,$$
$$\overline{a}_{2n+1} = (-1)^{n-1} f_{n-1}^2 (z^{f_n - 1} + z^{f_n - 2} + \ldots + 1),$$
$$\overline{a}_{2n+2} = (-1)^{n-1} \frac{1}{f_{n-1} f_n} (z-1) \quad (n = 1, 2, \ldots).$$

Proof. We put

$$\theta_n := [0; z^{f_n}, z^{f_{n+1}}, z^{f_{n+2}}, \ldots] \quad (n \ge -2),$$
$$\xi_n := (-1)^{n-1} \frac{f_n^2 z^{f_n} + f_{n-1} f_n + f_n^2 \theta_{n+1}}{z-1} \quad (n \ge 1),$$

$$\eta_n := (-1)^{n-1}\frac{z-1}{f_n f_{n+1} + f_n^2\theta_{n+1}} \qquad (n \geq 1),$$
$$c_n := (-1)^{n-1} f_n^2 (z^{f_n-1} + z^{f_n-2} + \ldots + 1) \qquad (n \geq 1),$$
$$d_n := (-1)^{n-1}\frac{1}{f_n f_{n+1}}(z-1) \qquad (n \geq 1).$$

Then

(35) $$\xi_n = [c_n; \eta_n] \ (= c_n + 1/\eta_n), \qquad \eta_n = [d_n; \xi_n].$$

Using

$$\theta_n^{-1} = z^{f_n} + \theta_{n+1}$$

and Proposition 3, we get

$$\begin{aligned}
\varphi_\varepsilon(z) &= \frac{\theta_{-2}}{z-1} \qquad (\|\theta_{-2}/(z-1)\| < 1) \\
&= [0; (z-1)\theta_{-2}^{-1}] \\
&= [0; z-1+(z-1)\theta_{-1}] \qquad (\|-1+(z-1)\theta_{-1}\| < 1) \\
&= \left[0; z, \frac{\theta_{-1}^{-1}}{-\theta_{-1}^{-1}+z-1}\right] = \left[0; z, \frac{z+\theta_0}{-1-\theta_0}\right] \\
&= \left[0; z, -z+1+\frac{1+(-z+2)\theta_0}{-1-\theta_0}\right] \qquad \left(\left\|\frac{1+(-z+2)\theta_0}{-1-\theta_0}\right\| < 1\right) \\
&= \left[0; z, -z+1, \frac{-1-\theta_0^{-1}}{-z+2+\theta_0^{-1}}\right] \\
&= \left[0; z, -z+1, \frac{-z-1-\theta_1}{2+\theta_1}\right] \\
&= \left[0; z, -z+1, -\frac{1}{2}(z+1), \frac{4\theta_1^{-1}+2}{z-1}\right] \\
&= \left[0; z, -z+1, -\frac{1}{2}(z+1), \frac{4z+2+4\theta_2}{z-1}\right].
\end{aligned}$$

Hence, we have

(36) $$f(z) = \left[0; z, -z+1, -\tfrac{1}{2}(z+1), \xi_1\right] \qquad (\|\xi_1^{-1}\| < 1).$$

From (35) and (36), it follows that

$$\begin{aligned}
f(z) &= \left[0; z, -z+1, -\tfrac{1}{2}(z+1)c_1, d_1, \ldots, c_n, d_n, \xi_{n+1}\right] \\
&= \left[0; z, -z+1, -\tfrac{1}{2}(z+1)c_1, d_1, c_2, d_2, \ldots\right]
\end{aligned}$$

which completes the proof for $\varphi_\varepsilon(z)$.

Starting from the identity $\varphi_{\bar\varepsilon}(z) = (1-\theta_{-2})/(z-1)$ instead of $\varphi_\varepsilon(z) = \theta_{-2}/(z-1)$, we can get the admissible continued fraction for $\varphi_{\bar\varepsilon}(z)$ in a similar fashion. ■

THEOREM 8. *The numerator* $p_n := p_n(z)$ ($\overline{p}_n := \overline{p}_n(z)$, *resp.*) *and the denominator* $q_n := q_n(z)$ ($\overline{q}_n := \overline{q}_n(z)$, *resp.*) *of the nth convergent of the continued fraction expansion for* $\varphi_\varepsilon(z)$ (*and* $\varphi_{\overline{\varepsilon}}(z)$, *resp.*) *are given as follows*:

$$\begin{aligned}
&p_0 = 0, \quad p_1 = 1, \quad p_2 = -z+1,\\
&q_0 = 1, \quad q_1 = z, \quad q_2 = -z^2+z+1,\\
&p_{2n-1} = \frac{1}{f_{n-1}}(\varepsilon_0 z^{f_n-1} + \varepsilon_1 z^{f_n-2} + \ldots + \varepsilon_{f_n-1}),\\
&p_{2n} = (-1)^n\{f_{n-1}z^{f_n}(\varepsilon_0 z^{f_{n-1}-1} + \varepsilon_1 z^{f_{n-1}-2} + \ldots + \varepsilon_{f_{n-1}-1})\\
&\qquad - f_{n-2}(\varepsilon_0 z^{f_n-1} + \varepsilon_1 z^{f_n-2} + \ldots + \varepsilon_{f_n-1})\}/(z-1),\\
&q_{2n-1} = \frac{1}{f_{n-1}}(z^{f_n} - 1),\\
&q_{2n} = (-1)^n\{f_{n-1}z^{f_n}(z^{f_{n-1}-1} + z^{f_{n-1}-2} + \ldots + 1)\\
&\qquad - f_{n-2}(z^{f_n-1} + z^{f_n-2} + \ldots + 1)\} \quad (n = 2, 3, \ldots),
\end{aligned}$$

and

$$\begin{aligned}
&\overline{p}_0 = 0, \quad \overline{p}_1 = 1,\\
&\overline{q}_0 = 1, \quad \overline{q}_1 = z^2,\\
&\overline{p}_{2n-2} = -\frac{1}{f_{n-2}}(\overline{\varepsilon}_0 z^{f_n-1} + \overline{\varepsilon}_1 z^{f_n-2} + \ldots + \overline{\varepsilon}_{f_n-1}),\\
&\overline{p}_{2n-1} = (-1)^{n-1}\{f_{n-2}z^{f_n}(\overline{\varepsilon}_0 z^{f_{n-1}-1} + \overline{\varepsilon}_1 z^{f_{n-1}-2} + \ldots + \overline{\varepsilon}_{f_{n-1}-1})\\
&\qquad - f_{n-3}(\overline{\varepsilon}_0 z^{f_n-1} + \overline{\varepsilon}_1 z^{f_n-2} + \ldots + \overline{\varepsilon}_{f_n-1})\}/(z-1) + f_{n-2},\\
&\overline{q}_{2n-2} = -\frac{1}{f_{n-2}}(z^{f_n} - 1),\\
&\overline{q}_{2n-1} = (-1)^{n-1}\{f_{n-2}z^{f_n}(z^{f_{n-1}-1} + z^{f_{n-1}-2} + \ldots + 1)\\
&\qquad - f_{n-3}(z^{f_n-1} + z^{f_n-2} + \ldots + 1)\} \quad (n = 2, 3, \ldots),
\end{aligned}$$

where p_{2n} *and* $\overline{p}_{2n-1}$ *are polynomials since the numerators are divisible by* $z-1$.

Proof. The values for $p_0, p_1, p_2, q_0, q_1, q_2$ are obtained from Theorem 7 by direct calculations. For a general n, we can prove the formula for p_n, q_n by induction on n using (31) and Theorem 7 without difficulty. ∎

REMARK 4. From Proposition 2 and Theorem 8, it follows that the set of normal indices for $\varphi_\varepsilon(z)$ (and $\varphi_{\overline{\varepsilon}}(z)$, resp.), is $\{0, f_0 = f_1 - 1, f_1 = f_2 - 1, f_2, f_3 - 1, \ldots\}$ ($\{0, f_1 = f_2 - 1, f_2, f_3 - 1, \ldots\}$, resp.) which together with Proposition 1 gives another proof of the third cases of Theorem 2 with $n = 0$.

REMARK 5. In [4], the continued fraction expansion for Laurent series corresponding to infinite words over $\{a, b\}$ generated by substitutions of "Fibonacci type" is considered, where a, b are viewed as independent variables.

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