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# On consecutive integers of the form $ax^2, by^2$ and $cz^2$

by

Michael A. Bennett (Princeton, N.J.)

**1. Introduction.** In the problem session of the Fifth Conference of the Canadian Number Theory Association (CNTA5), Herman J. J. te Riele posed the following:

> When I became 49, I realized that this square is preceded by 3 times a square and followed by 2 times a square. Are there more (nontrivial) such squares?

In other words, we would like to know if the simultaneous equations

$$2x^2 - y^2 = 1, \qquad y^2 - 3z^2 = 1 \tag{1}$$

have a solution in positive integers $(x, y, z)$ other than that given by $x = 5$, $y = 7$ and $z = 4$. A negative answer to this question follows from a classical result of Ljunggren [8], as recently refined by Cohn [4]:

Theorem 1.1. *Let the fundamental solution of the equation $v^2 - Du^2 = 1$ be $a + b\sqrt{D}$ (i.e. $(v, u) = (a, b)$ is the smallest positive solution). Then the only possible solutions of the equation $x^4 - Dy^2 = 1$ are given by $x^2 = a$ and $x^2 = 2a^2 - 1$; both solutions occur in only one case, $D = 1785$.*

To see this, note that (1) implies that $y^4 - 6(xz)^2 = 1$. More generally, if $a, b$ and $c$ are positive integers, one may consider the simultaneous Diophantine equations

$$ax^2 - by^2 = 1, \qquad by^2 - cz^2 = 1. \tag{2}$$

In this paper, we prove

Theorem 1.2. *If $a, b$ and $c$ are positive integers, then the simultaneous equations (2) possess at most one solution $(x, y, z)$ in positive integers.*

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The special cases where $b=1$ correspond to the aforementioned work of Ljunggren and Cohn, upon noting that, if $(x,y,z)$ is a positive solution to (2), then

$$b^2y^4 - ac(xz)^2 = 1.$$

The equations in (2) fit into the broader framework of *simultaneous Pell equations*, defined, more generally, for $a,b,c,d,e$ and $f$ integers, by

$$ax^2 - by^2 = c, \qquad dx^2 - ez^2 = f.$$

Under fairly mild restrictions upon the coefficients, such a system of equations defines a curve of genus one and hence has at most finitely many integral solutions, by work of Siegel. The literature associated with determining these solutions (or bounding their number) is an extensive one (see e.g. [1], [2], [7], [10] and [12]).

For comparison to Theorem 1.2, in [3] the author, extending a result of Masser and Rickert [9], obtained

Theorem 1.3. *If $a$ and $b$ are distinct nonzero integers, then the simultaneous equations*

$$x^2 - az^2 = 1, \qquad y^2 - bz^2 = 1$$

*possess at most three solutions $(x,y,z)$ in positive integers.*

Along these lines, if we take $a = 2A$, $b = C$ and $c = 2B$, Theorem 1.2 immediately implies

Corollary 1.4. *If $A, B$ and $C$ are nonzero integers, then the simultaneous equations*

$$Ax^2 - Bz^2 = 1, \qquad Cy^2 - 2Bz^2 = 1$$

*possess at most one solution $(x,y,z)$ in positive integers.*

A like result in the special case $A = C = 1$ has been obtained by Walsh [13] through application of Theorem 1.1. While the proofs of Cohn and Walsh are elementary, our approach to proving Theorem 1.2 utilizes lower bounds for linear forms in logarithms of algebraic numbers.

In Section 2, we will derive a result which ensures that if (2) has two positive solutions, then their heights cannot be too close together. In Section 3, we combine this with estimates from the theory of linear forms in logarithms of algebraic numbers to obtain Theorem 1.2 in all but a few exceptional cases. Finally, in Section 4, we treat these remaining cases.

For the remainder of the paper, we will assume that the system of equations (2) is solvable in positive integers $(x,y,z)$. Under this hypothesis, it is readily observed that the three fields $\mathbb{Q}(\sqrt{a})$, $\mathbb{Q}(\sqrt{b})$ and $\mathbb{Q}(\sqrt{c})$ are necessarily distinct (i.e. $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$ are linearly independent over $\mathbb{Q}$). We further suppose, without loss of generality, that $a$, $b$ and $c$ are squarefree.

**2. A gap principle.** Suppose, for $i$ an integer, that $(x_i, y_i, z_i)$ is a positive solution to (2). From the theory of Pellian equations (see e.g. Walker [11]), it follows that

$$y_i = \frac{\alpha^{j_i} - \alpha^{-j_i}}{2\sqrt{b}} = \frac{\beta^{k_i} + \beta^{-k_i}}{2\sqrt{b}} \tag{3}$$

where $\alpha$ and $\beta$ are the fundamental solutions to the equations $ax^2 - by^2 = 1$ and $by^2 - cz^2 = 1$ (i.e. $\alpha = \sqrt{a}u_0 + \sqrt{b}v_0$ and $\beta = \sqrt{b}u_1 + \sqrt{c}v_1$ where $(u_0, v_0)$ and $(u_1, v_1)$ are the smallest solutions in positive integers to $ax^2 - by^2 = 1$ and $by^2 - cz^2 = 1$ respectively). Here $j_i$ and $k_i$ are positive integers satisfying

$$\begin{cases} k_i \equiv 1 \pmod 2 & \text{if } a = 1, \\ j_i \equiv 1 \pmod 2 & \text{if } b = 1, \\ j_i \equiv k_i \equiv 1 \pmod 2 & \text{otherwise.} \end{cases}$$

It follows that there exists an integer $m \geq 2$ such that

$$\alpha^{j_1} = \sqrt{m} + \sqrt{m+1} \quad \text{and} \quad \beta^{k_1} = \sqrt{m-1} + \sqrt{m}. \tag{4}$$

Let us define $[n]$ to be the *square class* of $n$ (i.e. the unique integer $s$ such that $s$ is squarefree and $n = st^2$ for some integer $t$). Since we assume $a$, $b$ and $c$ to be squarefree, for a fixed choice of $m$ in (4), we therefore have

$$(a, b, c) = ([m+1], [m], [m-1]).$$

LEMMA 2.1. *Suppose that $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are two positive solutions to* (2) *with corresponding $\alpha$, $\beta$, $j_1$, $j_2$, $k_1$ and $k_2$. If $y_2 > y_1$, then*

$$j_2 > \frac{\log \beta}{2.1}\, \alpha^{2j_1}.$$

Proof. Let us first note that (3) implies

$$\beta^{k_i} = \alpha^{j_i}(1 - \alpha^{-2j_i} - \beta^{-k_i}\alpha^{-j_i}) \qquad (1 \leq i \leq 2).$$

If we suppose that $\alpha^{j_i} > 20$ (whence $\beta^{k_i} > 19$), we therefore have

$$\beta^{k_i} > 0.994\alpha^{j_i}.$$

Applying this to (3) yields the inequalities

$$2\alpha^{-j_i} < \alpha^{j_i} - \beta^{k_i} < 2.007\alpha^{-j_i}.$$

Considering the Taylor series expansion for $e^{\Lambda}$ where we take

$$\Lambda = j_i \log \alpha - k_i \log \beta,$$

we therefore have

$$2\alpha^{-2j_i} < j_i \log \alpha - k_i \log \beta < 2.02\alpha^{-2j_i} \tag{5}$$

or, roughly equivalently,

$$\frac{2}{j_i \log \beta}\alpha^{-2j_i} < \frac{\log \alpha}{\log \beta} - \frac{k_i}{j_i} < \frac{2.02}{j_i \log \beta}\alpha^{-2j_i}. \tag{6}$$

Since $\alpha$ and $\beta$ are each no less than $1+\sqrt{2}$ and $\alpha^{j_i} > \beta^{k_i} > 19$ (for $1 \le i \le 2$), we may conclude from (6) that $k_i/j_i$ is a convergent in the continued fraction expansion to $\log\alpha/\log\beta$. Also, $y_2 > y_1$ implies $j_2 > j_1$ and so $k_1/j_1 \ne k_2/j_2$ (since otherwise (6) implies that

$$\frac{2}{j_1\log\beta}\alpha^{-2j_1} < \frac{2.02}{j_2\log\beta}\alpha^{-2j_2} < \frac{2.02}{j_1\log\beta}\alpha^{-2j_1-2}$$

and so $\alpha^2 < 1.01$, contradicting $\alpha \ge 1+\sqrt{2}$ ).

Now if $p_r/q_r$ is the $r$th convergent in the continued fraction expansion to $\log\alpha/\log\beta$, then

$$\left|\frac{\log\alpha}{\log\beta} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2}$$

where $a_{r+1}$ is the $(r+1)$st partial quotient to $\log\alpha/\log\beta$ (see e.g. [5] for details). It follows from (6) that if $k_1/j_1 = p_r/q_r$ then

$$\frac{2.02}{d_1 l_1\log\beta}\alpha^{-2d_1 l_1} > \frac{1}{(a_{r+1}+2)l_1^2}$$

where $\gcd(k_t, j_t) = d_t$ and $j_t = d_t l_t$ for $1 \le t \le 2$, and so

$$a_{r+1} > \frac{d_1\log\beta}{2.02 l_1}\alpha^{2d_1 l_1} - 2.$$

Since $k_2/j_2$ is distinct from $k_1/j_1$ and provides a better approximation to $\log\alpha/\log\beta$, it follows that $l_2 \ge a_{r+1}l_1$ and thus

$$j_2 > \frac{d_1 d_2\log\beta}{2.02}\alpha^{2j_1} - 2d_2 l_1.$$

Since $d_1$ and $d_2$ are positive integers and $\alpha^{j_1} > 20$, we conclude as stated upon noting that

$$\frac{(\log\alpha\log\beta)^{-1}}{\left(\frac{1}{2.02}-\frac{1}{2.1}\right)} < 52.5$$

(since $\max\{\alpha,\beta\} \ge \sqrt{2}+\sqrt{3}$ and $\min\{\alpha,\beta\} \ge 1+\sqrt{2}$ ) while

$$\frac{\alpha^{2j_1}}{\log(\alpha^{2j_1})} > 66.7$$

follows from $\alpha^{2j_1} > 400$.

If, on the other hand, we have $\alpha^{j_1} \le 20$, then we need only consider (4) with $2 \le m \le 100$. For each of these cases, we may readily compute corresponding $(a,b,c)$, $\alpha$, $\beta$, $(x_1,y_1,z_1)$ and $(j_1,k_1)$. In all cases in question, except those with $m = 48, 49$ or $50$, we have $(j_1,k_1) = (1,1)$. In these remaining situations, we have $(j_1,k_1) = (2,1)$, $(3,2)$ and $(1,3)$, respectively. Checking that, for these 99 values of $m$, there are no new solutions $(x_2,y_2,z_2)$ with corresponding $j_2 \le \frac{\log\beta}{2.1}\alpha^{2j_1}$ completes the proof. ∎

**3. Linear forms in two logarithms.** From the recent work of Laurent, Mignotte and Nesterenko [6], we infer

LEMMA 3.1. *If $\alpha$ and $\beta$ are as in* (3), *$j$ and $k$ are positive integers*,

$$\Lambda = k\log\beta - j\log\alpha$$

*and*

$$h = \max\left\{12, 4\log\left(\frac{k}{\log\alpha} + \frac{j}{\log\beta}\right) - 1.8\right\},$$

*then*

$$\begin{aligned}\log|\Lambda| \geq\ & -61.2(\log\alpha\log\beta)h^2 - 24.3(\log\alpha + \log\beta)h - 2h\\ & - 48.1(\log\alpha\log\beta)^{1/2}h^{3/2} - \log(h^2\log\alpha\log\beta) - 7.3.\end{aligned}$$

Proof. This is virtually identical to Lemma 4.1 of [3] and follows readily from Théorème 2 of [6] upon choosing (in the notation of that paper) $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $b_1 = j$, $b_2 = k$, $D = 4$, $\rho = 11$ (so that $\lambda = \log 11$), $a_1 = 18\log\alpha$ and $a_2 = 18\log\beta$. The $\mathbb{Q}$-linear independence of $\log\alpha$ and $\log\beta$ is a consequence of the same property holding for $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$. ∎

We prove

PROPOSITION 3.2. *Suppose that $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are positive integral solutions to* (2) *with corresponding $\alpha$, $j_1$ and $j_2$. If $y_1 < y_2$, then $\alpha^{j_1} < 1400$ and $j_2 < 800000$.*

Proof. Note that Lemma 2.1 and $\alpha^{j_1} \geq 1400$ together imply $j_2 > 800000$, so that it suffices to derive the inequality $j_2 < 800000$. Let us suppose the contrary. We apply Lemma 3.1 with $j = j_2$ and $k = k_2$. Since $j_2/\log\beta > k_2/\log\alpha$ and $\beta \geq 1 + \sqrt{2}$, we have

$$h \leq \max\{12, 4\log j_2 + 1.5\}$$

and the lower bound for $j_2$ thus implies

$$4.12\log j_2 > 4\log j_2 + 1.5 > 12,$$

whereby

$$\begin{aligned}\log|\Lambda| \geq\ & -1038.9\log\alpha\log\beta\log^2 j_2 - 100.2(\log\alpha + \log\beta)\log j_2 - 8.3\log j_2\\ & - 402.3(\log\alpha\log\beta)^{1/2}\log^{3/2} j_2 - \log(\log^2 j_2\log\alpha\log\beta) - 10.2.\end{aligned}$$

On the other hand, (5) gives

$$\log|\Lambda| < \log 2.02 - 2j_2\log\alpha$$

and so

$$\begin{aligned}j_2 <\ & 519.5\log\beta\log^2 j_2 + 201.2\log^{1/2}\beta\log^{-1/2}\alpha\log^{3/2} j_2\\ & + 50.1(1 + \log\beta\log^{-1}\alpha)\log j_2 + (4.2\log j_2 + \log\log j_2)\log^{-1}\alpha\\ & + (0.5\log(\log\alpha\log\beta) + 5.5)\log^{-1}\alpha.\end{aligned}$$

Applying the inequalities $\alpha \geq 1+\sqrt{2}$ and $j_2 \geq 800000$ (which implies that $\log\log j_2/\log j_2 < 0.2$) yields

$$j_2 < 519.5 \log\beta \log^2 j_2 + 214.4 \log^{1/2}\beta \log^{3/2} j_2 + 56.9 \log\beta \log j_2 + 55.1 \log j_2 + 0.6 \log\log\beta + 6.2.$$

Since Lemma 2.1 implies

$$j_2 > \frac{\log\beta}{2.1}\alpha^{2j_1} > \frac{\log\beta}{2.1}\beta^{2k_1},$$

the inequalities $k_1 \geq 1$, $\beta \geq 1+\sqrt{2}$ and $j_2 \geq 800000$ yield

$$\log\beta < \tfrac{1}{2}\log j_2 + 0.44 < 0.54 \log j_2.$$

Substituting this implies that

$$j_2 < 280.6 \log^3 j_2 + 188.3 \log^2 j_2 + 55.1 \log j_2 + 0.6 \log\log j_2 + 5.9.$$

This, however, contradicts our initial assumption that $j_2 \geq 800000$, completing the proof. ∎

**4. Small solutions.** To finish the proof of Theorem 1.1, it remains only to deal with those $(a,b,c)$ for which (2) possesses a positive solution $(x_1,y_1,z_1)$ with corresponding $\alpha^{j_1} < 1400$. These coincide with the values $2 \leq m \leq 490000$ in (4), which, as is readily verified using Maple V, define distinct triples $(a,b,c)$. From Proposition 3.2, for each such $m$, we need only show that there fails to exist a second solution $(x_2,y_2,z_2)$ with corresponding $j_2 < 800000$. Assume that such a solution exists. Then Lemma 2.1 implies

$$j_2 > \frac{\log(1+\sqrt{2})}{2.1}(\sqrt{2}+\sqrt{3})^2$$

and so $j_2 \geq 5$ (whence $\alpha^{j_2} > 20$). We therefore find from (4) and (6) that

$$0 < \theta_m - \frac{j_1k_2}{k_1j_2} < \frac{2.02j_1}{k_1j_2\log\beta}\alpha^{-2j_2} \tag{7}$$

where

$$\theta_m = \frac{j_1\log\alpha}{k_1\log\beta} = \frac{\log(\sqrt{m}+\sqrt{m+1})}{\log(\sqrt{m}+\sqrt{m-1})}$$

and $k_1/j_1 \neq k_2/j_2$. It follows, therefore, that

$$\frac{j_1k_2}{k_1j_2} = \frac{p_{2i+1}}{q_{2i+1}}$$

for $p_{2i+1}/q_{2i+1}$ the $(2i+1)$st convergent in the continued fraction expansion to $\theta_m$ (with $i \geq 1$). Arguing as in the proof of Lemma 2.1 implies that

$$a_{2i+2} > \frac{k_1\log\beta}{2.02j_1j_2}\alpha^{2j_2} - 2 \tag{8}$$

where $a_{2i+2}$ is the $(2i+2)$nd partial quotient to $\theta_m$.

If $m = 2$, then $j_1 = k_1 = 1$, $\alpha = \sqrt{2} + \sqrt{3}$, $\beta = 1 + \sqrt{2}$ and so Lemma 2.1 implies $j_2 \geq 5$ whence (8) yields $a_{2i+2} \geq 8293$. On the other hand, in this case, $q_{11} = 2030653$ and $\max_{1 \leq i \leq 4} a_{2i+2} = a_4 = 20$, contradicting $j_2 < 800000$.

If $m \geq 3$, then Lemma 2.1 and (8) imply that $a_{2i+2} > 10^8$. Observe that the only values of $m$ with $2 \leq m \leq 490000$ and $k_1 > 1$ are given by

$$\begin{aligned}
&k_1 = 2,\ m = (2n^2 - 1)^2,\ \ 2 \leq n \leq 18,\\
&k_1 = 3,\ m = n(4n - 3)^2,\ \ 2 \leq n \leq 31,\\
&k_1 = 4,\ m \in \{9409, 332929\},\\
&k_1 = 5,\ m \in \{1682, 23763, 131044, 465125\},\\
&k_1 = 7,\ m = 57122.
\end{aligned}$$

We check, using Maple V, that $q_{2i+1} > 800000k_1$ provided $m \geq 64224$ ($m \neq 71825, 82369, 113569$) if $i = 1$, $m \geq 23296$ if $i = 2$, $m \geq 9271$ if $i = 3$, $m \geq 3754$ if $i = 4$, $m \geq 770$ if $i = 5$, $m \geq 50$ if $i = 6$, $m \geq 29$ if $i = 7$ and $m \geq 2$ if $i \geq 8$. It therefore remains to prove that $\max_{1 \leq i \leq t} a_{2i+2} \leq 10^8$ for

$$\begin{array}{rl}
23926 \leq m \leq 64223 \text{ and } m = 71825, 82369, 113569 & \text{if } t = 1,\\
9271 \leq m \leq 23925 & \text{if } t = 2,\\
3754 \leq m \leq 9270 & \text{if } t = 3,\\
770 \leq m \leq 3753 & \text{if } t = 4,\\
50 \leq m \leq 769 & \text{if } t = 5,\\
29 \leq m \leq 49 & \text{if } t = 6,\\
2 \leq m \leq 28 & \text{if } t = 7.
\end{array}$$

To do this, we compute the continued fraction expansion to $\theta_m$ for the 64226 values of $m$ under discussion, again using Maple V. In all cases, we verify that the partial quotients in question never exceed $10^8$. In fact, only three of them exceed $10^5$ : $a_{12} = 138807$ for $m = 1324$, $a_4 = 177667$ for $m = 17878$ and $a_4 = 332360$ for $m = 30962$. This concludes the proof of Theorem 1.1.

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School of Mathematics
Institute for Advanced Study
Princeton, New Jersey 08540
U.S.A.

*Current address*:
Department of Mathematics
University of Illinois
Champaign-Urbana, Illinois 61801
E-mail: mabennet@math.uiuc.edu

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