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A family of elliptic $\mathbb{Q}$-curves defined over biquadratic fields and their modularity

by

Takeshi Hibino and Atsuki Umegaki (Tokyo)

1. Introduction

Definition 1.1. Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$. Then E is called an (elliptic) $\mathbb{Q}$-*curve* if E and its Galois conjugate E^σ are isogenous over $\overline{\mathbb{Q}}$ for any σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

In Gross [4], E is assumed to have complex multiplication, but we do not assume that in this paper.

Many versions of modularity problems are known for elliptic curves over number fields. The most famous and typical is the Taniyama–Shimura conjecture, shown by Taylor and Wiles in the case where E is semi-stable. This conjecture says that every elliptic curve E over $\mathbb{Q}$ is modular, i.e. E is isogenous over $\mathbb{Q}$ to a $\mathbb{Q}$-simple factor of the jacobian variety of the modular curve $X_0(N)$ for a positive integer N. Similarly, a modular $\mathbb{Q}$-curve is defined as follows:

Definition 1.2. Let E be a $\mathbb{Q}$-curve. Then we say that E is *modular* if E is isogenous over $\overline{\mathbb{Q}}$ to a factor of the jacobian variety of the modular curve $X_1(N)$ for a positive integer N.

The following conjecture which was mentioned by Ribet is known as a *generalized Taniyama–Shimura conjecture* (cf. [7]):

Conjecture 1.3. *Every* $\mathbb{Q}$*-curve is modular.*

We have shown a method to construct families of $\mathbb{Q}$-curves defined over biquadratic fields in [6]. In this paper, by supposing a few more conditions on $\mathbb{Q}$-curves, we give a family of modular $\mathbb{Q}$-curves defined over biquadratic fields, which are either totally real fields or CM-fields (both cases can happen).

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This paper is organized as follows. In Section 2, we introduce our main theorem. In Section 3, we give a modular equation of $X_0(22)$ and give some of its properties. Using this equation, we shall prove the main theorem in Section 4. In Section 5, using our main theorem, we give some interesting examples.

2. Results. In order to state our main theorem, we need some notation. We define the rational function $j(x,y)$ by

$$j(x,y) = (A(x) + B(x)y)/x^{22}, \tag{2.1}$$

where

$$\begin{aligned} A(x) = 64(&16x^{33} + 176x^{32} + 836x^{31} + 2288x^{30} + 4048x^{29} + 4873x^{28} \\ &+ 4048x^{27} + 2288x^{26} + 836x^{25} + 176x^{24} + 16x^{23} + 12x^{22} \\ &+ 768x^{21} + 41216x^{20} + 761024x^{19} + 7499008x^{18} \\ &+ 47232768x^{17} + 209361328x^{16} + 692209408x^{15} \\ &+ 1772657920x^{14} + 3605725376x^{13} + 5924557056x^{12} \\ &+ 7948915456x^{11} + 8761456704x^{10} + 7948914688x^9 \\ &+ 5924548608x^8 + 3605685248x^7 + 1772548096x^6 \\ &+ 692015104x^5 + 209127424x^4 + 47038464x^3 \\ &+ 7389184x^2 + 720896x + 32768) \end{aligned}$$

and

$$\begin{aligned} B(x) = {} & 128(x+1)(x+2)(2x+1) \\ & \times (x^2+4)(x^2+2x+2)(x^2+3x+1) \\ & \times (2x^2+3x+2)(x^3-4x-4)(x^3-4x^2+4x+2) \\ & \times (x^3+2x^2+4x+2)(x^4-2x^3+2x^2+4x+4) \\ & \times (x^6+2x^5+4x^4+12x^3+20x^2+16x+4). \end{aligned}$$

For any rational number r, we put

$$x(r) = r + \sqrt{r^2-1}, \tag{2.2}$$

$$y(r) = \left(2r - 1 + \frac{2r+1}{r+1}\sqrt{r^2-1}\right)\sqrt{w(r)}, \tag{2.3}$$

where $w(r) = (r+1)(16r^3+48r^2+44r+13)$. Let K_r be the extension over $\mathbb{Q}$ generated by $x(r)$ and $y(r)$. Then

$$K_r = \mathbb{Q}(\sqrt{r^2-1}, \sqrt{w(r)}).$$

We put $j_r = j(x(r), y(r))$, and define the elliptic curve E_r with j-invariant

j_r by

$$(2.4)\quad E_r : \begin{cases} Y^2 + XY = X^3 - \dfrac{36}{j_r - 1728}X - \dfrac{1}{j_r - 1728} & \text{if } j_r \neq 0, 1728, \\ Y^2 + Y = X^3 & \text{if } j_r = 0, \\ Y^2 = X^3 + X & \text{if } j_r = 1728. \end{cases}$$

Now we state our main theorem:

THEOREM 2.1. *If the denominator of r is prime to 11 and r is not congruent to 1 or 9 modulo 11, then the elliptic curve E_r is a modular $\mathbb{Q}$-curve defined over K_r.*

In the case $[K_r : \mathbb{Q}] = 4$, K_r is a biquadratic extension. We denote by σ and τ the elements in the Galois group $\mathrm{Gal}(K_r/\mathbb{Q})$ which fix $\mathbb{Q}(\sqrt{r^2-1})$ and $\mathbb{Q}(\sqrt{w(r)})$, respectively. The elliptic curve E_r is isogenous via ϕ and ψ respectively to its conjugates $(E_r)^\tau$ and $(E_r)^\sigma$, whose degrees are equal to 2 and 11, respectively. Then we have the following diagram:

$$\begin{array}{ccc} E_r & \xrightarrow{\phi} & (E_r)^\tau \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi'} \\ (E_r)^\sigma & \xrightarrow[\phi']{} & (E_r)^{\sigma\tau}, \end{array}$$

where ϕ', ψ' are Galois conjugates of ϕ, ψ.

3. The modular curve $X_0(22)$. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For a positive integer N, we define the modular subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ of Γ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, a \equiv d \equiv 1,\ c \equiv 0 \pmod{N} \right\}.$$

We denote by $X, X_0(N)$ and $X_1(N)$ the modular curves defined over $\mathbb{Q}$ corresponding to Γ, $\Gamma_0(N)$ and $\Gamma_1(N)$, respectively.

In this section, we prepare some data on the modular curve $X_0(22)$ to obtain our main theorem. We assume that N is a square-free positive integer. For any positive integer d dividing N, we define the automorphism w_d on $X_0(N)$ which corresponds to the matrix

$$\begin{pmatrix} xd & y \\ zN & wd \end{pmatrix},$$

where $x, y, z, w \in \mathbb{Z}$ satisfy $xwd^2 - yzN = d$. If d is not equal to 1, then w_d is an Atkin–Lehner involution. Moreover we put

$$W(N) = \{w_d \mid d \,|\, N\},$$

which is the subgroup of the automorphism group of $X_0(N)$, and define the quotient curve $X_0^*(N)$ by

$$X_0^*(N) = X_0(N)/W(N), \tag{3.5}$$

which is defined over $\mathbb{Q}$. From Proposition 2.1 of [6], we have the following result:

LEMMA 3.1. *The equation*

$$y^2 = 2(x^3 + 4x^2 + 4x + 2)(2x^3 + 4x^2 + 4x + 1) \tag{3.6}$$

is a non-singular model of $X_0(22)$ *over* $\mathbb{Q}$ *and a covering map* $j : X_0(22) \to X$ *is given by* (2.1). *Moreover the Atkin–Lehner involutions* w_2 *and* w_{11} *act on the modular curve* $X_0(22)$ *by*

$$\begin{cases} (w_2^*x, w_2^*y) = \left(\dfrac{1}{x}, -\dfrac{y}{x^3}\right), \\ (w_{11}^*x, w_{11}^*y) = (x, -y) \end{cases} \tag{3.7}$$

in equation (3.6).

Now we can regard model (3.6) over $\mathbb{Q}$ as a model over the local ring $\mathbb{Z}_{(11)}$ of $\mathbb{Z}$ at 11. Hence we define a scheme $\mathcal{C}$ over $\mathbb{Z}_{(11)}$ by equation (3.6). Then the special fibre C of $\mathcal{C}$ is given by the model

$$C : \quad y^2 = 4(x-1)^2(x-3)^2(x-4)^2 \tag{3.8}$$

over $\mathbb{F}_{11}$. Then C has the following important property. Recall that a point of $X_0(N)$ over a field with finite characteristic is called *supersingular* if the elliptic curve corresponding to the point is supersingular.

LEMMA 3.2. *The supersingular points in characteristic* 11 *of the modular curve* $X_0(22)$ *correspond to the points on* C *with* $y = 0$, *i.e.* $\{(1,0), (3,0), (4,0)\}$.

Proof. It is known that the supersingular j-invariants in characteristic 11 are $j = 0$ and $j = 1728 = 1$. Let $\mathcal{M}$ be the modular curve $X_0(22)$ over $\mathbb{Z}_{(11)}$. The special fibre of $\mathcal{M}$ has two irreducible components, which we denote by Z and Z', and these two components intersect in exactly three points, since the genus of $X_0(22)$ is equal to 2. Then we see that the intersection points correspond to the supersingular points ([2]). Moreover $\mathcal{M}$ has only one non-regular point. In fact, one of the three intersection points corresponds to the pair (e, α) of an elliptic curve e and its subgroup α of

order 2 such that the j-invariant of e is equal to 1728 and the order of the automorphism group which fixes α is equal to 4, and the others correspond to the pairs with the automorphism group of order 2.

Next we consider the minimal model of $\mathcal{M}$. Let $\widetilde{\mathcal{M}}$ be the scheme over $\mathbb{Z}_{(11)}$ which is obtained by blowing-up $\mathcal{M}$ at the non-regular point. Then $\widetilde{\mathcal{M}}$ is regular, and the special fibre of $\widetilde{\mathcal{M}}$ has three components Z, Z' and E. We can check that the self-intersection number of Z is equal to -3, similarly ${Z'}^2 = -3$, and that the self-intersection number of E is equal to -2, so $\widetilde{\mathcal{M}}$ is the minimal model over $\mathbb{Z}_{(11)}$. It is easy to see that C also has two irreducible components and three intersection points. The minimal model of $\mathcal{C}$ over $\mathbb{Z}_{(11)}$ is obtained by blowing up along the ideal $(x-1, y, 11)$. Because of the universality of the minimal model, $\widetilde{\mathcal{M}}$ is also the minimal model corresponding to $\mathcal{C}$. Thus we see that the supersingular points on $\mathcal{C}$ correspond to the three intersection points, namely the points with $y = 0$ of C (see Figure 1).

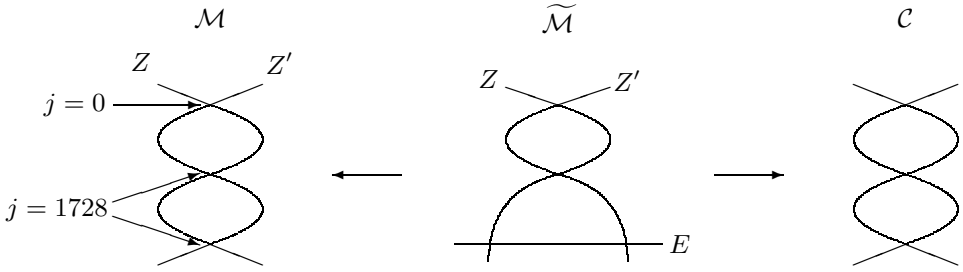

Fig. 1

This completes the proof. ∎

4. Proofs. In this section we shall prove Theorem 2.1. First, we introduce the criterion for the modularity of $\mathbb{Q}$-curves which is given in [5]. Let N be a square-free positive integer, and p a rational prime which divides N. We consider two quotient curves of $X_0(N)$. One of them is $X_0^*(N)$, which is defined by (3.5), and the other is

$$X_0^*(N;p) = X_0(N)/W(N;p), \tag{4.9}$$

where

$$W(N;p) = \left\{ w_d \;\middle|\; d \,|\, \tfrac{N}{p} \right\} \tag{4.10}$$

is the subgroup in the automorphism group of $X_0(N)$. Then we have a covering map

$$f : X_0^*(N;p) \to X_0^*(N)$$

of degree 2. By Theorem C of [5], we know the following:

THEOREM 4.1. *Let x be a point on $X_0(N)$ such that the image y in $X_0^*(N)$ of x is a $\mathbb{Q}$-rational point, but not a cuspidal point or a CM point. Moreover we assume that some prime divisor p of N satisfies the following conditions*:

(a) *the reduction of y at p is not a supersingular point*,
(b) *the fibre $X_0^*(N;p)_y$ over y has no $\mathbb{Q}$-rational points*,
(c) *$p \geq 5$, $p \neq 1+2^n$ and $p \neq 1+3\cdot 2^n$, $n \in \mathbb{Z}$, $n > 0$.*

Then the $\mathbb{Q}$-curve corresponding to x is modular.

REMARK 4.2. We note that the rational prime 11 is the minimum prime satisfying condition (c) of Theorem 4.1.

REMARK 4.3. In Hasegawa–Hashimoto–Momose [5], they obtained similar results for all $p \geq 5$ under some conditions modulo p. In general, it is not easy to obtain a result similar to our main theorem, because we must check the additional conditions.

Now we recall the elliptic curve E_r which is given in (2.4).

LEMMA 4.4. *The elliptic curve E_r is a $\mathbb{Q}$-curve defined over K_r.*

P r o o f. As this lemma has been proved in Theorem 3.3 of [6], we only give a sketch of the proof. We make use of the result by Elkies [3] that the $\mathbb{Q}$-curves of "degree N", i.e. $\mathbb{Q}$-curves which have isogenies to their conjugates with degree dividing N, are parameterized by the $\mathbb{Q}$-rational points of $X_0^*(N)$. We recall that model (3.6) is a defining equation of $X_0(22)$ over $\mathbb{Q}$ and the Atkin–Lehner involutions w_2 and w_{11} act on the points (x,y) of $X_0(22)$ in the manner which is given by (3.7). Then it is easy to check that the rational function $t = \frac{1}{2}(x+1/x)$ parameterizes the points of $X_0^*(22)$ (which is of genus 0) by calculating the pole divisors of x and t. Now we specialize t to a rational number r in order to use the result of Elkies. Then it follows that

$$x = r \pm \sqrt{r^2-1}, \tag{4.11}$$

so that we choose the point $(x(r), y(r))$ on $X_0(22)$ which belongs to the fibre of the point r on $X_0^*(22)$, where $x(r)$ and $y(r)$ are given in (2.2) and (2.3). Since the relation between a point on $X_0(22)$ and the corresponding j-invariant is given in (2.1), we see that the elliptic curve E_r is a $\mathbb{Q}$-curve defined over K_r. ∎

Finally, we prove our main theorem.

Proof of Theorem 2.1. We note that the rational prime 11 satisfies condition (c) in Theorem 4.1.

Using defining equation (3.6) of $X_0(22)$ over $\mathbb{Q}$, we obtain model (3.8) over $\mathbb{F}_{11}$. From Lemma 3.2, the points with $y = 0$ in model (3.8) correspond

to the supersingular points on $X_0(22)$ over $\mathbb{F}_{11}$. Moreover, by using relation (2.1), we see that

$$j \equiv \begin{cases} 0 \pmod{11} & \text{if and only if } x \equiv 4 \pmod{11}, \\ 1728 \pmod{11} & \text{if and only if } x \equiv 1 \text{ or } 3 \pmod{11}. \end{cases}$$

Therefore if x is not congruent to $1, 3$ or 4 modulo 11, the image of (x, y) in $X_0^*(22)$ is not a supersingular point in characteristic 11.

We recall that the point $(x(r), y(r))$ of $X_0(22)$ belongs to the fibre of the $\mathbb{Q}$-rational point r of $X_0^*(22)$. Then it follows that

$$r \equiv \begin{cases} 1 \pmod{11} & \text{if } x \equiv 1 \pmod{11}, \\ 9 \pmod{11} & \text{if } x \equiv 3 \text{ or } 4 \pmod{11}. \end{cases}$$

Therefore if the denominator of r is prime to 11 and $r \not\equiv 1, 9 \pmod{11}$, then the point on $X_0^*(22)$ corresponding to r is not a supersingular point in characteristic 11, and hence the point $(x(r), y(r))$ satisfies condition (a) in Theorem 4.1.

Next we consider the $\mathbb{Q}$-rational points of $X_0(22;11) = X_0(22)/\langle w_2 \rangle$ in order to verify condition (b) in Theorem 4.1. We put

$$S = \frac{2 \cdot 11^2 \cdot x}{(x-1)^2}, \qquad T = \frac{2 \cdot 11^2 \cdot y}{(x-1)^3}. \tag{4.12}$$

Since the Atkin–Lehner involution w_2 acts on the points (x, y) as in (3.7), we note that S and T are invariant under w_2. Moreover as $Sx^2 - (2S + 2 \cdot 11^2)x + S = 0$ and $y = (x-1)^3 T/(2 \cdot 11^2)$, S and T generate a subfield which is of index 2 in $\mathbb{Q}(X_0(22))$. Consequently, S and T generate the function field of $X_0(22;11)$, and we can take S and T as parameters of $X_0(22;11)$. Then we obtain the following defining equation of $X_0(22;11)$:

$$T^2 = S^3 + 188S^2 + 11616S + 234256, \tag{4.13}$$

since the defining equation of $X_0(22)$ is written as (3.6). Hence $X_0(22;11)$ is isomorphic to $X_0(11)$, whose Mordell–Weil rank over $\mathbb{Q}$ is equal to zero. In particular, the number of $\mathbb{Q}$-rational points of $X_0(11)$ is 5 (cf. [1]), and hence we see that the set of $\mathbb{Q}$-rational points of this curve (4.13) is $\{\infty, (0, \pm 484), (-44, \pm 44)\}$. In fact, we can give a defining equation of $X_0(11)$ as

$$Y^2 + Y = X^3 - X^2 - 10X - 20, \tag{4.14}$$

and an isomorphism from $X_0(22;11)$ to $X_0(11)$ as follows:

$$X = \frac{S}{4} + 16, \qquad Y = \frac{T}{8} - \frac{1}{2}. \tag{4.15}$$

Hence if $r \neq 1, -7/4$, then the point of $X_0(22;11)$ corresponding to r is not a $\mathbb{Q}$-rational point. This is condition (b) in Theorem 4.1.

Finally, we summarize the conditions for the modularity of E_r in terms of a rational number r. Elliptic curves of CM type are modular by [8], and hence we may assume that E_r does not have complex multiplication. If the denominator of r is prime to 11 and r is not congruent to 1 or 9 modulo 11, then the $\mathbb{Q}$-curve E_r corresponding to r is modular from Theorem 4.1. This completes the proof of Theorem 2.1. ∎

Remark 4.5. For a square-free positive integer N and a prime number p dividing N, if $X_0^*(N)$ is isomorphic to the projective line $\mathbb{P}^1$ and $X_0^*(N;p)$ has finite $\mathbb{Q}$-rational points, then we can get a similar theorem. In fact, we have results for the cases $(N,p) = (33,11), (46,23)$.

5. Examples

Example 5.1. Let $r = 11/5$. Then $K_r = \mathbb{Q}(\sqrt{6}, \sqrt{29})$ has class number 1. The $\mathbb{Q}$-curve E_r has j-invariant

$$\begin{aligned} j(E_r) = \frac{1}{5^{22}}(&998269691281725129260266540119630 4704 \\ &- 40754189488135321090109133597561154 56\sqrt{6} \\ &+ 18537402791159630521518878692955412 48\sqrt{29} \\ &- 75678629992478957693784269242729267 2\sqrt{174}\,). \end{aligned}$$

The quadratic twist E of E_r by

$$\begin{aligned} \alpha = 1585084727553 &- \frac{1248019557557}{2}\sqrt{6} \\ &- 989865700341\sqrt{29} + \frac{826800325581}{2}\sqrt{174} \end{aligned}$$

has the following global minimal model:

$$\begin{aligned} E: \quad y^2 = x^3 &+ \left(9 + \tfrac{1}{2}\sqrt{6} + \tfrac{1}{2}\sqrt{174}\right)x^2 \\ &+ (-383506419653 - 156534506597\sqrt{6} \\ &\quad + 71201118525\sqrt{29} + 29073539873\sqrt{174}\,)x \\ &- 182798829223792711 - 74627160360067580\sqrt{6} \\ &\quad + 33944822557919841\sqrt{29} + 13857943481193026\sqrt{174}. \end{aligned}$$

Then E has discriminant

$$\begin{aligned} \Delta(E) = 77098749869738970221225796 5120 \\ + 31475432831219625624026162688 0\sqrt{6} \\ - 14316878430089111357711373696 0\sqrt{29} \\ - 58448411438624093585994387840\sqrt{174}, \end{aligned}$$

which generates the ideal

$$(\Delta(E)) = \mathfrak{p}_2^{12} \cdot \mathfrak{p}_5^2 \cdot (\mathfrak{p}_5^\sigma)^{11} \cdot (\mathfrak{p}_5^\tau) \cdot (\mathfrak{p}_5^{\sigma\tau})^{22},$$

and conductor

$$\mathrm{cond}(E) = \mathfrak{p}_2^4 \cdot \mathfrak{p}_5 \cdot (\mathfrak{p}_5^\sigma) \cdot (\mathfrak{p}_5^\tau) \cdot (\mathfrak{p}_5^{\sigma\tau}) = 2^2 \cdot 5,$$

where $\mathfrak{p}_2 = (-2+\sqrt{6})$ and $\mathfrak{p}_5 = \left(\frac{1}{2}+\sqrt{6}+\frac{1}{2}\sqrt{29}\right)$. Therefore E is a modular $\mathbb{Q}$-curve by Theorem 2.1.

EXAMPLE 5.2. Let $r = -4/5$. Then $K_r = \mathbb{Q}(\sqrt{-1}, \sqrt{41})$ has class number 4 and a quadratic twist of E_r has the following model:

$$E: \quad y^2 = x^3 - (9720 - 10296\sqrt{-1} - 1260\sqrt{41})x - 326592 + 741312\sqrt{-1} + 90720\sqrt{41}. \tag{5.16}$$

The curve E has j-invariant

$$j = -\frac{1}{5^{22}}(2528188128191313216 - 9524265011230514688\sqrt{-1} - 201763471435658496\sqrt{41} + 1359858285331273728\sqrt{-41}\,)$$

and conductor

$$\mathrm{cond}_K(E) = \mathfrak{p}_2^8 \cdot \mathfrak{p}_3^2 \cdot \mathfrak{p}_5 \cdot (\mathfrak{p}_5^\sigma) \cdot (\mathfrak{p}_5^\tau) \cdot (\mathfrak{p}_5^{\sigma\tau}),$$

where $\mathfrak{p}_2 = \left(2, \frac{1}{2} - \sqrt{-1} - \frac{1}{2}\sqrt{41}\right)$, $\mathfrak{p}_3 = \left(3, \frac{5}{2} + \frac{3}{2}\sqrt{-1} + \frac{3}{2}\sqrt{41} + \frac{1}{2}\sqrt{-41}\right)$ and $\mathfrak{p}_5 = (5, 1 + \sqrt{-1} + \sqrt{41}\,)$. Then this $\mathbb{Q}$-curve is modular.

REMARK 5.3. All the calculations in the above were done by a program with GNU C and PARI-library, ver. 1.39.

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Takeshi Hibino
Advanced Research Institute
for Science and Engineering
Waseda University
3-4-1 Ohkubo, Shinjuku-ku
Tokyo 169, Japan
E-mail: hibino@mse.waseda.ac.jp

Atsuki Umegaki
Department of Mathematics
School of Science and Engineering
Waseda University
Tokyo, Japan
E-mail: umegaki@gm.math.waseda.ac.jp
696m5012@mn.waseda.ac.jp

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