On the discrepancy estimate of normal numbers

by

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Dedicated to Professor N. M. Korobov
on the occasion of his 80th birthday

1. Introduction

1.1. A number \( \alpha \in (0, 1) \) is said to be normal to base \( q \) if in the \( q \)-ary expansion of \( \alpha, \ \alpha = d_1d_2\ldots \) \( (d_i \in \Delta = \{0, 1, \ldots, q-1\}, \ i = 1, 2, \ldots) \), each fixed finite block of digits of length \( k \) appears with an asymptotic frequency of \( q^{-k} \) along the sequence \( (d_i)_{i \geq 1} \). Normal numbers were introduced by Borel (1909).

1.1.1. Let \( (x_n)_{n \geq 1} \) be an arbitrary sequence of real numbers. The quantity

\[
D(N) = D(N, (x_n)_{n \geq 1}) = \sup_{\gamma \in (0, 1)} |\#\{0 \leq n < N \mid \{x_n\} < \gamma\}/N - \gamma|
\]

is called the discrepancy of \( (x_n)_{n=1}^{N} \), where \( \{x\} = x - [x] \) is the fractional part of \( x \). The sequence \( \{x_n\}_{n \geq 1} \) is said to be uniformly distributed (u.d.) in \([0, 1)\) if \( D(N) \to 0 \).

1.2. In \([K1]\) Korobov posed the problem of finding a function \( \psi \) with maximum decay, such that

\[
\exists \alpha : D(N, \{\alpha q^n\}_{n \geq 0}) \leq \psi(N), \quad N = 1, 2, \ldots
\]

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He showed that \( \psi(N) = O(N^{-1/2}) \) (see [K1]). The lower bound of the discrepancy for the Champernowne and Davenport–Erdős normal numbers was found by Schiffer [S]:

\[
D(N, \{a q^n\}_{n \geq 1}) \geq K/\log N \quad \text{with} \quad K > 0, \quad N = 2, 3, \ldots
\]

For a bibliography on Korobov’s problem see [Po, L1].

1.3. In [L2] we proposed using small discrepancy sequences (van der Corput type sequences and \( \{n\alpha\}_{n \geq 0} \)) to construct normal numbers, and announced that

\[
\psi(N) = O(N^{-1} \log N).
\]

This result is proved below. The estimate of \( \psi(N) \) was previously known to be \( O(N^{-2/3} \log^{4/3} N) \) (Korobov [K2] for \( q \) prime, and Levin [L1] for arbitrary integer \( q \)). We note that the estimate obtained cannot be improved essentially, since according to W. Schmidt, 1972 (see [N, p. 24]), for any sequence of reals,

\[
\lim_{N \to \infty} ND(N)/\log N > 0.
\]

1.4. Let \( x = [a_0(x); a_1(x), a_2(x), \ldots] \) be the continued fraction expansion of \( x \), with partial quotients \( a_i(x) \). For an integer \( b \) and \( Q > 1 \) let \( \sum a_i(b/Q) \) denote the sum of all partial quotients of \( b/Q \). Following [P] we prove (see Lemma 3) that there exists an integer sequence \( b_m \) and a constant \( K > 0 \) with

\[
\sum_{r=1}^{m} \sum_{0 \leq k < q^m} a_i(\{b_m/q^r\}) \leq Km^3, \quad m = 1, 2, \ldots
\]

**Theorem 1.** Let

\[
\alpha = \sum_{m \geq 1} \frac{1}{q^{nm}} \sum_{0 \leq k < q^m} \left\{ \frac{b_m k}{q^m} \right\} \frac{1}{q^{mk}}
\]

where \( b_m \) satisfy (2),

\[
n_1 = 0 \quad \text{and} \quad n_k = \sum_{1 \leq r < k} rq^r, \quad k = 2, 3, \ldots
\]

Then the number \( \alpha \) is normal to base \( q \), and

\[
D(N, \{a q^n\}_{n \geq 0}) = O(N^{-1} \log^3 N).
\]

1.5. Let \( (p_{i,j})_{i,j \geq 1} \) be Pascal’s triangle:

\[
p_{i,1} = p_{i,i} = 1, \quad i = 1, 2, \ldots, \quad p_{i,j} = p_{i,j-1} + p_{i-1,j}, \quad i, j = 2, 3, \ldots,
\]

and \( (p_{i,j})_{i,j \geq 1} \) be Pascal’s triangle mod 2:

\[
p_{i,j} \equiv p_{i,j}' \mod 2, \quad i, j = 1, 2, \ldots
\]
Every integer \(n \geq 0\) has a unique digit expansion in base \(q\),
\[
n = \sum_{j \geq 1} e_j(n) q^{j-1} \quad \text{with } e_j(n) \in \Delta = \{0, \ldots, q - 1\},
\]
\(j = 1, 2, \ldots\), and \(e_j(n) = 0\) for all sufficiently large \(j\).

**Theorem 2.** Let
\[
\alpha = \sum_{m \geq 1} q^{b_m} \sum_{0 \leq n < q^{2m}} \frac{1}{q^n} \sum_{i=1}^{2^m} d_i(n) q^i
\]
where
\[
d_i(n) \equiv \sum_{j \geq 1} p_{i,j} e_j(n) \mod q, \quad d_i(n) \in \Delta, \quad i = 1, \ldots, 2^m, \quad n \in [0, q^{2m}),
\]
\[
n_1 = 0 \quad \text{and} \quad n_m = \sum_{1 \leq r < m} 2^r q^{2^r}, \quad m = 2, 3, \ldots
\]

Then the number \(\alpha\) is normal to base \(q\) and
\[
D(N, \{\alpha q^n\}_{n \geq 0}) = O(N^{-1} \log^2 N).
\]

**Remark 1.** We use here the sequence of \(2^m \times 2^m\) matrices of Pascal’s triangle mod 2. A similar result is valid for the sequence of \(m \times m\) matrices of Pascal’s triangle (or \(m \times m\) matrices of Pascal’s triangle mod \(p\)) but with
\[
D(N, \{\alpha q^n\}_{n \geq 0}) = O(N^{-1} \log^3 N),
\]
where \(\alpha\) is denoted by a concatenation of blocks \(\omega_m\):
\[
\alpha = \omega_1 \ldots \omega_m \ldots,
\]
where
\[
\omega_m = (d_1(1) \ldots d_m(1) \ldots d_1(q^m) \ldots d_m(q^m)), \quad m = 1, 2, \ldots,
\]
and
\[
d_i(n) \equiv \sum_{j \geq 1} p_{i,j} e_j(n) \mod q.
\]

**Remark 2.** Let \((\sigma_i)_{i \geq 1}\) be any sequence of substitutions of the set \(\Delta = \{0, 1, \ldots, q - 1\}\). The proof of Theorem 2 does not change if in (8) we use the functions \(\sigma_i(e_i(n))\) instead of the functions \(e_i(n)\) (see [B], [N, p. 25]).

**2. Proof of the theorems.** Let \(m \geq 1, \ b, \ i\) be integers, \(0 \leq i < m, \ (b, q) = 1, \)
\[
\alpha_m = \alpha_m(b) = \sum_{0 \leq k < q^m} \left\{ \frac{bk}{q^m} \right\} \frac{1}{q^{mk}},
\]
\[
\alpha_{mni} = [q^{2m-i} \{\alpha_m q^{i+mn}\}] / q^{2m-i}.
\]
It is easy to see that \( \{ \{ bn/q \} m \} q^i = \{ bn/q \} m \), and
\[
\{ \alpha_m q^{i+mn} \} = \left\{ \left\{ \frac{bn}{q^m} \right\} q^i + \left\{ \frac{b(n+1)}{q^m} \right\} \frac{1}{q^{m-i}} + \left\{ \frac{b(n+2)}{q^m} \right\} \frac{1}{q^{2m-i}} + \ldots \right\} \\
= \left\{ \frac{bn}{q^{m-i}} \right\} + \left\{ \frac{b(n+1)}{q^m} \right\} \frac{1}{q^{m-i}} + \left\{ \frac{b(n+2)}{q^m} \right\} \frac{1}{q^{2m-i}} + \ldots
\]

Therefore
\[
\alpha_{mni} = \left\{ \frac{bn}{q^{m-i}} \right\} + \left\{ \frac{b(n+1)}{q^m} \right\} \frac{1}{q^{m-i}}.
\]

Let \( N \in [1, mq^m] \) be an integer, \( \gamma \in (0, 1] \),
\[
A(\gamma, N, (x_n)) = \begin{cases} 
\#\{0 \leq n < N \mid \{x_n\} < \gamma\} & \text{for } \gamma > 0, \\
0 & \text{for } \gamma \leq 0,
\end{cases}
\]
and
\[
A(\gamma, Q, P, (x_n)) = \#\{Q \leq n < Q + P \mid \{x_n\} < \gamma\}.
\]
Hence and from (10) we obtain
\[
A(\gamma, N, \{ \alpha_m q^n \}_{n \geq 0}) = \sum_{i=0}^{m-1} A(\gamma, [N/m], \{ \alpha_m q^n \}_{n \geq 0}) + \theta m
\]
with \( \theta \in [0, 1] \).

Let \( c = \lceil q^m \gamma \rceil, N_1 \in [1, q^m] \) and \( 0 \leq i < m \). From (11) and (13) we deduce
\[
A\left( \frac{c-1}{q^m}, N_1, (\alpha_{mni})_{n \geq 0} \right) \leq A(\gamma, N_1, \{ \alpha_m q^{mn+i} \}_{n \geq 0}) \leq A\left( \frac{c+1}{q^m}, N_1, (\alpha_{mni})_{n \geq 0} \right).
\]

**Lemma 1.** Let \( N \in [1, mq^m] \) be an integer, \( \gamma \in (0, 1], (b, q) = 1 \). Then
\[
A(\gamma, N, \{ \alpha_m q^n \}_{n \geq 0}) = \gamma N + \epsilon_1 \left( 4m + 3 \sum_{i=1}^{m} \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0}) \right),
\]
\[
A(\gamma, mq^m, \{ \alpha_m q^n \}_{n \geq 0}) = \gamma mq^m + 3\epsilon_2 m
\]
with \( |\epsilon_j| < 1, j = 1, 2 \).
Proof. Let \( 0 \leq i < m, d, d_1, \) and \( d_2 \) be integers, \( d = d_1 q^i + d_2, d_1 \in \[0, q^{m-i}) \), \( d_2 \in \[0, q^i) \). By (12) and (13) we get
\[
A \left( \frac{d}{q^m}, N_1, (\alpha_{mn})_{n \geq 0} \right)
= \# \left\{ 0 \leq n < N_1 \left| \left\{ \frac{b n}{q^{m-i}} \right\} + \left\{ \frac{b(n+1)}{q^m} \right\} q^{m-i} \leq \frac{d_1}{q^{m-i}} + \frac{1}{q^{m-i}} \cdot \frac{d_2}{q^i} \right. \right\}.
\]
Consequently,
\[
A(d/q^m, N_1, (\alpha_{mn})_{n \geq 0}) = T_1(N_1) + T_2(N_1),
\]
where
\[
T_1(N) = \# \left\{ 0 \leq n < N \left| \left\{ \frac{b n}{q^{m-i}} \right\} < \frac{d_1}{q^{m-i}} \right. \right\},
\]
\[
T_2(N) = \# \left\{ 0 \leq n < N \left| \left\{ \frac{b n}{q^{m-i}} \right\} = \frac{d_1}{q^{m-i}} \text{ and } \left\{ \frac{b(n+1)}{q^m} \right\} q^{m-i} < \frac{d_2}{q^i} \right. \right\}.
\]
Let \( N_1 = N_2 q^{m-i} + N_3 \) with \( N_3 \in \[0, q^{m-i}) \) and \( N_2 \in \[0, q^i) \). It is easy to see that
\[
T_1(N_1) = T_1(q^{m-i} N_2) + T_1(N_3).
\]
We see from (20) and (1) that
\[
T_1(N_2 q^{m-i}) = N_2 d_1,
\]
and
\[
T_1(N_3) = \frac{d_1}{q^{m-i}} N_3 + \varepsilon N_3 D \left( N_3, \left\{ \frac{b n}{q^{m-i}} \right\}_{n \geq 0} \right) \quad \text{with } |\varepsilon| \leq 1.
\]
This yields
\[
T_1(N_1) = \frac{d_1}{q^{m-i}} N_1 + \varepsilon \max_{1 \leq N < q^{m-i}} N D \left( N, \left\{ \frac{b n}{q^{m-i}} \right\}_{n \geq 0} \right) \quad \text{with } |\varepsilon| \leq 1.
\]
Now we compute \( T_2(N) \). Let \( d_0 \) be an integer, \( d_0 \equiv d_1 b^{-1} \mod q^{m-i} \) with \( d_0 \in \[0, q^{m-i}) \), and
\[
Y = \{ 0 \leq n < N_1 \mid \{ b n / q^{m-i} \} = d_1 / q^{m-i} \}.
\]
Clearly if \( \{ b n / q^{m-i} \} = d_1 / q^{m-i} \), then \( b n \equiv d_1 \mod q^{m-i}, n \equiv d_0 \mod q^{m-i} \), and
\[
Y = \{ d_0 + r q^{m-i} \mid 0 \leq r < N_4 \} \quad \text{with } N_4 = \left\lceil \frac{N_1 - d_0 - 1}{q^{m-i}} \right\rceil + 1.
\]
Combining (21) and (1) we obtain
\begin{equation}
T_2(N_1) = \# \left\{ n \in Y \left| \frac{b(n + 1)}{q^m} < \frac{d_2}{q^i} \right. \right\}
= \# \left\{ 0 \leq r < N_4 \left| \frac{b(d_0 + 1)}{q^m} + \frac{br}{q^i} < \frac{d_2}{q^i} \right. \right\}
= N_4 \frac{d_2}{q^i} + \varepsilon_2 N_4 D \left( N_4, \left\{ \frac{bn}{q^i} + \theta \right\}_{n \geq 0} \right)
\end{equation}

with \( \theta = b(d_0 + 1)/q^m \), \(|\varepsilon_2| \leq 1 \).

It follows from (1) that for every real \( \theta \),
\begin{equation}
D(N, \{ x_n + \theta \}_{n \geq 0}) \leq 2D(N, \{ x_n \}_{n \geq 0}).
\end{equation}

By (24) and (25), this yields
\begin{equation}
T_2(N_1) = \left[ \frac{N_1 + q^{m-i} - d_0 - 1}{q^{m-i}} \right] \frac{d_2}{q^i} + 2\varepsilon_2 \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0})
\end{equation}

\begin{equation}
= N_1 \frac{d_2}{q^m} + \varepsilon_3 + 2\varepsilon_2 \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0})
\end{equation}

with \(|\varepsilon_j| \leq 1, j = 2, 3 \).

If \( N_1 = q^m \), then \( N_4 = q^i \), and \( N_4 D(N_4, \{ bn/q^i \}_{n \geq 0}) = 1 \). Hence and from (25) and (26) we obtain
\begin{equation}
T_2(q^m) = d_2 + 2\varepsilon_4 \quad \text{with } |\varepsilon_4| \leq 1.
\end{equation}

Substituting (23) and (27) into (19), we obtain
\begin{equation}
A(d/q^m, N_1, (\alpha_{mn})_{n \geq 0}) = N_1 d/q^m + \varepsilon_5 (1 + \max_{1 \leq N < q^m-i} ND(N, \{ bn/q^{m-i} \}_{n \geq 0})
\end{equation}

\begin{equation}
+ 2 \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0})) \quad \text{with } |\varepsilon_5| \leq 1.
\end{equation}

Using (16) and (15) we get
\begin{equation}
A(\gamma, N_1, (\alpha_m q^{m+n+i})_{n \geq 0}) = \gamma N_1 + \varepsilon_6 (2 + \max_{1 \leq N < q^{m+i}} ND(N, \{ bn/q^{m+i} \}_{n \geq 0})
\end{equation}

\begin{equation}
+ 2 \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0})) \quad \text{with } |\varepsilon_6| \leq 1,
\end{equation}

and
\begin{equation}
A(\gamma, N, (\alpha_m q^n)_{n \geq 0}) = \theta m + \sum_{i=1}^m \gamma \lfloor N/m \rfloor
\end{equation}

\begin{equation}
+ \varepsilon_7 \left( 2m + 3 \sum_{i=1}^m \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0}) \right)
\end{equation}

\begin{equation}
= \gamma N + \varepsilon_8 \left( 4m + 3 \sum_{i=1}^m \max_{1 \leq N \leq q^i} ND(N, \{ bn/q^i \}_{n \geq 0}) \right),
\end{equation}
where $|\varepsilon_j| \leq 1$, $j = 7, 8$. Assertion (17) is proved. Assertion (18) follows analogously from (22) and (28).

**Lemma 2.** Let $j \geq 1$, $1 \leq N \leq q^j$, $(b, q) = 1$, and $a_i(x)$ be partial quotients of $\{x\}$. Then

$$ND(N, \{bn/q^j\}_{n \geq 0}) \leq \sum a_i(b/q^j).$$

For the proof of this well-known theorem, see for example [N, p. 26].

**Lemma 3.** There exists a constant $K > 0$ and integers $c_m \in [0, q^m)$ such that

$$\sum_{1 \leq c \leq q^m, (c, q) = 1} \sum_{r=1}^{m} a_i(c/q^r) \leq K q^m r^2, \quad r = 1, 2, \ldots$$

**Proof.** According to [P, p. 2144] there exist constants $K_q$ such that

$$\sum_{1 \leq c \leq q^m, (c, q) = 1} \sum_{r=1}^{m} a_i(c/q^r) \leq K q^m r^2 \leq K q^m m^3.$$ 

Let $\phi(q^m) = \#\{1 \leq c \leq q^m \mid (c, q) = 1\}$ and $K = K_q q/\phi(q)$. It is known that $\phi(q^m) = q^{m-1}\phi(q)$. Now the assertion of Lemma 3 follows from (29).

**Corollary.** Let $1 \leq N \leq mq^m$. Then

$$A(\gamma, N, \{\alpha_m(b_m)q^n\}_{n \geq 0}) = \gamma N + O(m^3).$$

The statement follows from (1), (2), (10), and Lemmas 1–3.

Applying (3) and (10) we get

$$\{\alpha^{n+m}\} = \{\alpha_m(b_m)q^n\} + \theta q^{n-m} \quad \text{with} \ 0 < \theta < 1 \text{ and } 0 \leq n < mq^m.$$ 

Hence and from (13) we have, for $N \in [1, mq^m]$,

$$A(\gamma - 1/q^m, N - m, \{\alpha_m(b_m)q^n\}_{n \geq 0}) \leq A(\gamma, N, \{\alpha^{n+m}\}_{n \geq 0}) \leq A(\gamma, N, \{\alpha_m(b_m)q^n\}_{n \geq 0}).$$

By using (30) and (14), we obtain

$$A(\gamma, n_m, N, \{\alpha q^n\}_{n \geq 0}) = \gamma N + O(m^3) \quad \text{with} \ 1 \leq N \leq mq^m.$$ 

Similarly, from (18) we deduce that

$$A(\gamma, n_m, mq^m, \{\alpha q^n\}_{n \geq 0}) = \gamma mq^m + O(m).$$
End of the proof of Theorem 1. For every \( N \geq 1 \) there exists an integer \( k \) such that \( N \in [n_k, n_{k+1}) \). By (4) this yields

\[
N = n_k + R \quad \text{with} \quad 0 \leq R < kq^k, \quad N > (k - 1)q^{k-1}, \quad k \leq 2 \log_q N.
\]

Applying (4), (14) and (31)–(33) we obtain

\[
A(\gamma, N, \{\alpha q^n\}_{n \geq 0}) = \sum_{r=1}^{k-1} A(\gamma, n_r, rq^r, \{\alpha q^n\}_{n \geq 0}) + A(\gamma, n_k, R, \{\alpha q^n\}_{n \geq 0})
\]

\[
= \sum_{r=1}^{k-1} (\gamma rq^r + O(r)) + \gamma R + O(k^3)
\]

\[
= \gamma N + O(k^3) = \gamma N + O(\log^3 N).
\]

Thus, by (1), the theorem is proved.

**Proof of Theorem 2.** In [So] Sobol’ proposed the use of Pascal’s triangle mod 2 to construct small discrepancy sequences (see also [F], [N]). Here we use Pascal’s triangle mod 2 to construct normal numbers.

Let \( P_n \) be a sequence of 2\( n \times 2^n \) matrices such that

\[
P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \ldots, \quad P_{n+1} = \begin{pmatrix} P_n & P_n \\ P_n & 0 \end{pmatrix}, \quad \ldots
\]

It is easy to prove by induction that \( P_n \) is the 2\( n \times 2^n \) upper left-hand corner of Pascal’s triangle (5), and \( P_n \) is a triangular-type matrix. The following lemma is proved in [BH] for Pascal’s triangle, and it is clearly valid also for Pascal’s triangle mod 2.

**Lemma 4.** The determinant of any \( n \times n \) array taken with its first row along a row of ones, or with its first column along a column of ones in Pascal’s triangle, written in rectangular form, is one.

From (7) we have

\[
\{\alpha q^n \}_{n \geq m+2^{n+k}} = .d_{k+1} \ldots d_{2^m} \ldots d_1 \ldots d_{k-2^m} n \ldots d_{k+1} \ldots d_{2^m} \ldots d_1 \ldots d_{k-2^m} (n+1) \ldots
\]

Let \( 1 \leq k, i \leq 2^m \) and

\[
\alpha_{ki}(n) = \frac{\{\alpha q^n \}_{n \geq m+2^{n+k}}} {q^i}.
\]

It is easy to see that

\[
\alpha_{ki}(n) = \begin{cases} .d_{k+1} \ldots d_{k+i} & \text{if } k + i \leq 2^m, \\
.d_{k+1} \ldots d_{2^m} \ldots d_{k+i-2^m} \ldots d_{k+1} \ldots d_{2^m} \ldots d_1 \ldots d_{k+1-2^m} (n+1) & \text{otherwise}.
\end{cases}
\]

**Lemma 5.** Let \( m, k, i, B, f \) be integers, \( 1 \leq i, k \leq 2^m, B \in [0, q^{2^m-i}), f \in [0, q^i) \). Then

\[
A(f/q^i, Bq^i, q^i, \{\alpha_{ki}(n)\}_{n \geq 0}) = f + 2\varepsilon \quad \text{with} \quad |\varepsilon| < 1.
\]
Proof. Case 1. Let \( k + i \leq 2^m \), \( c_j \in \Delta = \{0, 1, \ldots, q - 1\} \) \( (j = 1, \ldots, i) \). We examine the system of equations
\[
(37) \quad d_{k+j}(n) = c_j, \quad j = 1, \ldots, i, \ n \in [Bq^i, (B+1)q^i).
\]
According to (8) this system is equivalent to the system of \( i \) congruences
\[
\sum_{1 \leq \nu \leq 2^m} p_{k+j,\nu} e_\nu(n + Bq^i) \equiv c_j \bmod q, \quad j = 1, \ldots, i, \ n \in [0, q^i).
\]
Applying (6) we see that \( e_\nu(n + Bq^i) = e_\nu(n) + e_\nu(Bq^i), \ \nu = 1, 2, \ldots, \) and
\[
(38) \quad \sum_{1 \leq \nu \leq i} p_{k+j,\nu} e_\nu(n)
\]
\[
\equiv c_j - \sum_{i < \nu \leq 2^m} p_{k+j,\nu} e_\nu(Bq^i) \bmod q, \quad j = 1, \ldots, i,
\]
with \( n \in [0, q^i) \). It follows from Lemma 4 that
\[
(39) \quad |\det(p_{k+j,\nu})_{1 \leq j, \nu \leq i}| = 1.
\]
For any \( c_1, \ldots, c_i \) the system (38) has a unique solution \( (e_1(n), \ldots, e_i(n)) \), and consequently there exists a unique \( n_0 \in [Bq^i, (B+1)q^i) \) satisfying (37).

From (36) and (37) we see that the set \( \{\alpha_{k_i}(n) \mid n \in [Bq^i, (B+1)q^i)\} \) coincides with \( \{j/q^i \mid j \in [0, q^i)\} \). Hence and from (14) we have
\[
(40) \quad A(f/q^i, Bq^i, q^i, (\alpha_{k_i}(n))_{n \geq 0}) = f.
\]
Case 2. Let \( k + i > 2^m \), \( l_1 = 2^m - k \). As in (37) and (38), the system of equations
\[
(41) \quad d_{k+j}(n) = c_j, \quad j = 1, \ldots, l_1,
\]
\[
(42) \quad d_j(n + 1) = c_{j+l_1}, \quad j = 1, \ldots, i - l_1,
\]
with \( n \in [Bq^i, (B+1)q^i) \), is equivalent to the systems of congruences
\[
(43) \quad \sum_{1 \leq \nu \leq i} p_{k+j,\nu} e_\nu(n)
\]
\[
\equiv c_j - \sum_{i < \nu \leq 2^m} p_{k+j,\nu} e_\nu(Bq^i) \bmod q, \quad j = 1, \ldots, l_1,
\]
\[
(44) \quad \sum_{1 \leq \nu \leq i} p_{j,\nu} e_\nu(n + 1)
\]
\[
\equiv c_{j+l_1} - \sum_{i < \nu \leq 2^m+1} p_{j,\nu} e_\nu((B + [(n + 1)/q^i]q^i)) \bmod q,
\]
where \( j = 1, \ldots, i - l_1 \) and \( n \in [0, q^i) \).

Let \( n = n_1 + n_2 q^{l_1} \) with \( n_1 \in [0, q^{l_1}) \) and \( n_2 \in [0, q^{i-l_1}) \). It is evident that \( e_\nu(n) = e_\nu(n_1) \) for \( \nu = 1, \ldots, l_1 \).

The matrix \( P_m \) is triangular. Hence
\[
p_{k+j,\nu} = 0 \quad \text{with} \ \nu > 2^m - k - j = l_1 - j.
\]
The system (43) is equivalent to the following system of congruences:
\[(45) \sum_{1 \leq \nu \leq l_1} p_{k+j,\nu}e_\nu(n_1) = e_j - \sum_{i < \nu \leq 2^m} p_{k+j,\nu}e_\nu(Bq^i) \mod q, \quad j = 1, \ldots, l_1,\]

where \(n_1 \in [0, q^{l_1})\) and \(n_2 \in [0, q^{i-l_1})\).

Applying (39) with \(i = l_1\) shows that this system has a unique solution with \((e_1(n_1), \ldots, e_{l_1}(n_1))\). Consequently, there exists a unique solution \(n_1 = n'_1 \in [0, q^{l_1})\) satisfying (45).

By (41) and (43) we obtain
\[(46) \{ (d_{k+1}(n_1 + n_2q^{l_1} + Bq^i), \ldots, d_{k+l_1}(n_1 + n_2q^{l_1} + Bq^i)) \mid 0 \leq n_1 < q^{l_1} \} = \{(c_1, \ldots, c_{l_1}) \mid c_j \in \Delta, \ j = 1, \ldots, l_1 \}.
\]

Now we examine the system (44) with \(n_1 = n'_1\) the solution of (45).

**Case 2.1.** Let \(n'_1 \leq q^{l_1} - 2\). Bearing in mind that
\[e_\nu(n + 1) = e_\nu(n'_1 + 1 + q^{l_1}n_2) = e_\nu(n'_1 + 1) + e_\nu(q^{l_1}n_2),\]
we deduce from (44) that
\[\sum_{l_1 < \nu \leq i} p_{j,\nu}e_\nu(q^{l_1}n_2) = e_j - \sum_{1 \leq \nu \leq l_1} p_{j,\nu}e_\nu(n'_1 + 1) - \sum_{i < \nu \leq 2^m} p_{j,\nu}e_\nu(Bq^i) \mod q\]
with \(j = 1, \ldots, i - l_1\) and \(0 \leq n_2 < q^{i-l_1}\).

Applying Lemma 4 we obtain a unique solution for this system with \((e_{l_1+1}(q^{l_1}n_2), \ldots, e_i(q^{l_1}n_2))\).

By (42) and (44) we get
\[(47) \{ (d_1(n'_1 + n_2q^{l_1} + Bq^i + 1), \ldots, d_{i-1}(n'_1 + n_2q^{l_1} + Bq^i + 1)) \mid 0 \leq n_2 < q^{i-l_1} \} = \{(c_{l_1+1}, \ldots, c_i) \mid c_{l_1+j} \in \Delta, \ j = 1, \ldots, i - l_1 \}.
\]

Let
\[(48) F = \{ d_{k+1}(n) \ldots d_{2^m}(n)d_1(n+1) \ldots d_{k+l_1-2^m}(n+1) \mid 0 \leq n_1 < q^{l_1 - 1}, \ 0 \leq n_2 < q^{i-l_1}, \ n = n_1 + n_2q^{l_1} + Bq^i \},\]
and
\[(49) g_\nu = d_{k+\nu}(q^{l_1} - 1 + Bq^i), \quad \nu = 1, \ldots, l_1.\]

From (46) and (47) we have
\[(50) F = \{ (c_1, \ldots, c_i) \mid c_j \in \Delta, \ j = 1, \ldots, i, \ (c_1, \ldots, c_i) \neq (g_1, \ldots, g_1) \}\]
and
\[(51) \#F = q^i - q^{i-l_1}.\]

**Case 2.2.** Let \(n'_1 = q^{l_1} - 1, \ n_2 \in [0, q^{i-l_1} - 2] \) and \(n = n'_1 + n_2q^{l_1}\). Then \(e_\nu(n'_1 + 1) = 0 \) for \(1 \leq \nu \leq l_1\) and \(e_\nu(n + 1) = e_\nu((n_2 + 1)q^{l_1}) \) for \(l_1 < \nu \leq i.\)
The system (44) is equivalent to the following system of congruences:

\[
\sum_{i_1 < i < i_1 + l_1} p_{j,\nu} e_{\nu}(n_2 + 1)q^{l_1}) = c_{j+l_1} - \sum_{i_1 < \nu \leq 2^n} p_{j,\nu} e_{\nu}(Bq^i) \mod q, \quad j = 1, \ldots, i - l_1,
\]

with \(0 \leq n_2 \leq q^{i_l_1 - 2}\).

For \(n_2 \in [0, q^{i_l_1 - 2}]\) we have the \(q^{i_l_1 - 2}\) distinct vectors of

\[
(e_{i_1+1}((n_2 + 1)q^{l_1})), \ldots, e_i((n_2 + 1)q^{l_1})).
\]

Using Lemma 4 and by (52) we obtain for \(n_2 \in [0, q^{i_l_1 - 2}]\) the \(q^{i_l_1 - 2}\) distinct vectors of \((c_{i_1+1}, \ldots, c_i)\).

Let

\[ G = \{(g_1, \ldots, g_i, d_1((n_2 + 1)q^{l_1} + Bq^i), \ldots, d_{i-l_1}, ((n_2 + 1)q^{l_1} + Bq^i)) \mid 0 \leq n_2 \leq q^{i_l_1 - 2}\}. \]

From (42), (44) and (52) we find that \(\#G = q^{i_l_1 - 1}\), and from (46) and (48)–(51) that \(\#(F \cup G) = q^i - 1\). Hence and from (36) the set \(<\alpha_{k_i}(n) \mid n \in [Bq^i, (B + 1)q^i - 2]\rangle \) coincides with \(q^i - 1\) distinct values of \(j/q^i\) with \(j \in [0, q^i]\). By (14) we get

\[
A(f/q^i, Bq^i, q^i, (\alpha_{k_i}(n))_{n \geq 0}) = f + 2\varepsilon \quad \text{with } |\varepsilon| < 1.
\]

Hence and from (40) we have the assertion of Lemma 5. □

**Corollary 1.**

\[
A(\gamma, Bq^i, q^i, \{\alpha q^{m+2^m n+k}\}_{n \geq 0}) = \gamma q^i + 4\varepsilon \quad \text{with } |\varepsilon| < 1.
\]

**Proof.** Analogously to (16), from (14) and (35) we have

\[
A \left( \frac{f-1}{q^i}, Bq^i, q^i, (\alpha_{k_i}(n))_{n \geq 0} \right) \leq A(\gamma, Bq^i, q^i, \{\alpha q^{m+2^m n+k}\}_{n \geq 0}) \leq A((f + 1)/q^i, Bq^i, q^i, (\alpha_{k_i}(n))_{n \geq 0})
\]

with \(f = [\gamma q^i]\). By using Lemma 5 we obtain (53).

**Corollary 2.** Let \(1 \leq N < 2^m q^{2^m}\). Then

\[
A(\gamma, n_m, N, \{\alpha q^n\}_{n \geq 0}) = \gamma N + 5\varepsilon 2^m \quad \text{with } |\varepsilon| < 1,
\]

\[
A(\gamma, n_m, 2^m q^{2^m}, \{\alpha q^n\}_{n \geq 0}) = \gamma 2^m q^{2^m} + 5\varepsilon 2^m \quad \text{with } |\varepsilon| < 1.
\]

**Proof.** Let \(N' = [N/2^m]\), \(N'' = N - 2^m N'\), \(N' = \sum_{i=0}^{2^m-1} b_i q^i\) with \(b_i \in \Delta, \nabla\)

\[
N_0 = 0, \quad N_j = \sum_{i=0}^{j-1} b_{2^m-i} q^{2^m-i}, \quad j = 1, 2, \ldots, \quad b_i = N_{2^m-i-1}/q^i.
\]
It is evident that \( B_i \) \((i = 1, 2, \ldots)\) are integers, and \( N'' \in [0, 2^m)\). As in (15) we see from (14) that

\[ A(\gamma, n_m, N, \{\alpha q^n\}_{n \geq 0}) = \varepsilon N_2 + \sum_{k=1}^{2^m} A(\gamma, N', \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}), \]

and

\[ A(\gamma, N', \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}) \]

\[ = \sum_{i=1}^{2^m} A(\gamma, N, i-1, b_2m-iq^{2m-i}, \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}) \]

\[ = \sum_{i=0}^{2^m-1} A(\gamma, N, 2^m-i-1, b_i q^i, \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}) \]

\[ = \sum_{i=0}^{2^m-1} \sum_{B=0}^{b_i-1} A(\gamma, N, 2^m-i-1 + Bq^i, q^i, \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}). \]

Using (56) we have

\[ A(\gamma, n_m, N, \{\alpha q^n\}_{n \geq 0}) \]

\[ = \varepsilon 2^m + \sum_{k=1}^{2^m} \sum_{i=0}^{2^m-1} \sum_{B=0}^{b_i-1} A(\gamma, (B_i + B)q^i, q^i, \{\alpha q^{n_m+2^m n+k}\}_{n \geq 0}). \]

Applying (53) we obtain

\[ A(\gamma, n_m, N, \{\alpha q^n\}_{n \geq 0}) = \varepsilon 2^m + \sum_{k=1}^{2^m} \sum_{i=0}^{2^m-1} \sum_{B=0}^{b_i-1} (\gamma q^i + 4\varepsilon_i) = \gamma N + 5q\varepsilon_1 2^m \]

with \(|\varepsilon_1| \leq 1\).

Assertion (54) is proved. We prove (55) analogously.

**End of the proof of Theorem 2.** For every \( N \geq q \) there exists an integer \( k \) such that \( N \in [n_k, n_{k+1}) \). By (9), this yields \( N = n_k + R \) with \( 0 \leq R < 2^k q^{2k} \), \( N \geq 2^{(k-1)q^{2k-1}}, 2^k \leq 2 \log_q N \). Applying (9), (13), (14), (54) and (55) we obtain

\[ A(\gamma, N, \{\alpha q^n\}_{n \geq 0}) = \sum_{m=1}^{k-1} A(\gamma, n_m, 2^m q^{2^m}, \{\alpha q^n\}_{n \geq 0}) \]

\[ + A(\gamma, n_k, R, \{\alpha q^n\}_{n \geq 0}) \]

\[ = \sum_{m=1}^{k-1} (\gamma 2^m q^{2^m} + O(2^m)) + \gamma R + O(2^{2k}) \]

\[ = \gamma N + O(2^{2k}) = \gamma N + O(\log^2 N). \]

Thus, by (1), the theorem is proved. \( \blacksquare \)
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References


