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Arithmetic progressions of prime-almost-prime twins

by

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1. Introduction. In 1937 I. M. Vinogradov [17] proved that for every sufficiently large odd integer N the equation

$$p_1 + p_2 + p_3 = N$$

has a solution in prime numbers p_1, p_2, p_3.

Two years later van der Corput [15] used the method of Vinogradov and established that there exist infinitely many arithmetic progressions of three different primes. A corresponding result for progressions of four or more primes has not been proved so far. In 1981, however, D. R. Heath-Brown [6] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is P_2 (as usual, P_r denotes an integer with no more than r prime factors, counted according to multiplicity).

A famous and still unsolved problem in Number Theory is the prime-twins conjecture, which states that there exist infinitely many prime numbers p such that $p+2$ is also a prime. This problem has been attacked by many mathematicians in various ways. The reader may refer to Halberstam and Richert's monograph [4] for a detailed information. One of the most important results in this direction belongs to Chen [2]. In 1973 he proved that there exist infinitely many primes p such that $p+2$ is P_2.

In the present paper we study the solvability of the equation $p_1+p_2 = 2p_3$ in different primes p_i, $1 \le i \le 3$, such that $p_i + 2$ are almost-primes. The first step in this direction was made recently by Peneva and the author. It was proved in [13] that there exist infinitely many triples of different primes satisfying $p_1 + p_2 = 2p_3$ and such that $(p_1+2)(p_2+2) = \mathrm{P}_9$.

Suppose that x is a large real number and k_1, k_2 are odd integers. Denote by $D_{k_1,k_2}(x)$ the number of solutions of $p_1 + p_2 = 2p_3$, $x < p_1, p_2, p_3 \le 3x$, in primes such that $p_i + 2 \equiv 0 \pmod{k_i}$, $i = 1, 2$. The main result of [13]

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is a theorem of Bombieri–Vinogradov's type for $D_{k_1,k_2}(x)$ stating that for each $A > 0$ there exists $B = B(A) > 0$ such that

$$\sum_{\substack{k_1,k_2 \le \sqrt{x}/(\log x)^B \\ (k_1k_2,2)=1}}\sum |D_{k_1,k_2}(x) - (\text{expected main term})| \ll \frac{x^2}{(\log x)^A}$$

(see [13] for details). In [13] the Hardy–Littlewood circle method and the Bombieri–Vinogradov theorem were applied, as well as some arguments belonging to H. Mikawa. We should also mention the author's earlier paper [14] in which the same method was used.

In the present paper we apply the vector sieve, developed by Iwaniec [8] and used also by Brüdern and Fouvry in [1]. We prove the following

Theorem. *There exist infinitely many arithmetic progressions of three different primes* p_1, p_2, $p_3 = \frac{1}{2}(p_1 + p_2)$ *such that* $p_1 + 2 = \mathrm{P}_5$, $p_2 + 2 = \mathrm{P}'_5$, $p_3 + 2 = \mathrm{P}_8$.

By choosing the parameters in a different way we may obtain other similar results, for example $p_1 + 2 = \mathrm{P}_4$, $p_2 + 2 = \mathrm{P}_5$, $p_3 + 2 = \mathrm{P}_{11}$. The result would be better if it were possible to prove Lemma 12 for larger K. For example, the validity of Lemma 12 for $K = x^{1/2-\varepsilon}$, $\varepsilon > 0$ arbitrarily small, would imply the Theorem with $p_i + 2 = \mathrm{P}_5$, $i = 1, 2, 3$.

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2. Notations and some lemmas. Let x be a sufficiently large real number and let $\mathcal{L}$, α_1, α_2, α_3 be constants satisfying $\mathcal{L} \ge 1000$, $0 < \alpha_i < 1/4$, which we shall specify later. We put

$$\begin{gathered} z_i = x^{\alpha_i}, \quad i = 1, 2, 3, \quad z_0 = (\log x)^{\mathcal{L}}; \quad D_0 = \exp((\log x)^{0.6}), \\ D_1 = D_2 = x^{1/2}\exp(-2(\log x)^{0.6}), \quad D_3 = x^{1/3}\exp(-2(\log x)^{0.6}). \end{gathered} \tag{1}$$

Letters s, u, v, w, y, z, α, β, γ, ν, ε, D, M, L, K, P, H denote real numbers; m, n, d, a, q, l, k, r, h, t, δ are integers; p, p_1, $p_2, \ldots$ are prime numbers. As usual $\mu(n)$, $\varphi(n)$, $\Lambda(n)$ denote Möbius' function, Euler's function and von Mangoldt's function, respectively; $\tau_k(n)$ denotes the number of solutions of the equation $m_1 \ldots m_k = n$ in integers $m_1, \ldots, m_k$; $\tau(n) = \tau_2(n)$. We denote by $(m_1, \ldots, m_k)$ and $[m_1, \ldots, m_k]$ the greatest common divisor and the least common multiple of $m_1, \ldots, m_k$, respectively. For real y, z, however, (y, z) denotes the open interval on the real line with endpoints y and z. The meaning is always clear from the context. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n\,(k)$. As usual, $[y]$ denotes the integer part of y, $\|y\|$ the distance from y to the nearest integer, $e(y) = \exp(2\pi i y)$. For positive A and B we write $A \asymp B$ instead of $A \ll B \ll A$. The letter c denotes some positive real number, not the same in all appearances. This convention allows us to write

$$(\log y)e^{-c\sqrt{\log y}} \ll e^{-c\sqrt{\log y}},$$

for example.

We put

$$Q = (\log x)^{10\mathcal{L}}, \qquad \tau = xQ^{-1}, \tag{2}$$

$$E_1 = \bigcup_{q \le Q} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left(\frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right), \qquad E_2 = \left(-\frac{1}{\tau}, 1 - \frac{1}{\tau} \right) \setminus E_1, \tag{3}$$

$$S_k(\alpha) = \sum_{\substack{x < p \le 2x \\ p+2 \equiv 0\,(k)}} (\log p) e(\alpha p), \qquad M(\alpha) = \sum_{x < m \le 2x} e(\alpha m), \tag{4}$$

$$I_{k_1,k_2,k_3}(x) = \sum_{\substack{x < p_1, p_2, p_3 \le 2x \\ p_i + 2 \equiv 0\,(k_i),\, i=1,2,3 \\ p_1 + p_2 = 2p_3}} \log p_1 \log p_2 \log p_3. \tag{5}$$

Clearly

$$I_{k_1,k_2,k_3}(x) = \int_0^1 S_{k_1}(\alpha) S_{k_2}(\alpha) S_{k_3}(-2\alpha)\, d\alpha = I^{(1)}_{k_1,k_2,k_3}(x) + I^{(2)}_{k_1,k_2,k_3}(x), \tag{6}$$

where

$$I^{(i)}_{k_1,k_2,k_3}(x) = \int_{E_i} S_{k_1}(\alpha) S_{k_2}(\alpha) S_{k_3}(-2\alpha)\, d\alpha, \qquad i = 1, 2. \tag{7}$$

If D is a positive number we consider Rosser's weights $\lambda^{\pm}(d)$ of order D (see Iwaniec [9], [10]). Define $\lambda^{\pm}(1) = 1$, $\lambda^{\pm}(d) = 0$ if d is not squarefree. If $d = p_1 \ldots p_r$ with $p_1 > \ldots > p_r$ we put

$$\lambda^{+}(d) = \begin{cases} (-1)^r & \text{if } p_1 \ldots p_{2l} p_{2l+1}^3 < D \text{ for all } 0 \le l \le (r-1)/2, \\ 0 & \text{otherwise;} \end{cases}$$

$$\lambda^{-}(d) = \begin{cases} (-1)^r & \text{if } p_1 \ldots p_{2l-1} p_{2l}^3 < D \text{ for all } 1 \le l \le r/2, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\lambda_i^{\pm}(d)$ Rosser's weights of order D_i, $0 \le i \le 3$. In particular, we have

$$|\lambda_i^{\pm}(d)| \le 1, \qquad \lambda_i^{\pm}(d) = 0 \text{ for } d \ge D_i, \qquad 0 \le i \le 3. \tag{8}$$

Let $f(s)$ and $F(s)$ denote the functions of the linear sieve. They are continuous and satisfy

$$\begin{aligned} sF(s) &= 2e^\gamma && \text{if } 0 < s \le 3, \\ sf(s) &= 0 && \text{if } 0 < s \le 2, \\ (sF(s))' &= f(s-1) && \text{if } s > 3, \\ (sf(s))' &= F(s-1) && \text{if } s > 2, \end{aligned}$$

where $\gamma = 0.577\ldots$ is the Euler constant.

Let $\mathcal{P}$ denote a set of primes. We put

$$P(w) = \prod_{\substack{p<w \\ p\in\mathcal{P}}} p, \qquad P(w_1, w_2) = \frac{P(w_2)}{P(w_1)}, \qquad 2 \le w_1 \le w_2.$$

The following lemma is one of the main results in sieve theory. For the proof see [9], [10].

Lemma 1. *Suppose that $\mathcal{P}$ is any set of primes and ω is a multiplicative function satisfying*

$$0 < \omega(p) < p \quad \text{if } p \in \mathcal{P}, \qquad \omega(p) = 0 \quad \text{if } p \notin \mathcal{P},$$
$$\prod_{w_1 \le p < w_2} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \le \frac{\log w_2}{\log w_1}\left(1 + \frac{\mathcal{K}}{\log w_1}\right)$$

for some $\mathcal{K} > 0$ and for all $2 \le w_1 \le w_2$. Assume that $\lambda^{\pm}(d)$ are Rosser's weights of order D and let $s = (\log D)/(\log w)$. We have

$$\begin{aligned} \prod_{p<w}\left(1 - \frac{\omega(p)}{p}\right) &\le \sum_{d \mid P(w)} \lambda^+(d)\frac{\omega(d)}{d} \\ &\le \prod_{p<w}\left(1 - \frac{\omega(p)}{p}\right)(F(s) + \mathcal{O}(e^{\sqrt{\mathcal{K}}-s}(\log D)^{-1/3})), \end{aligned}$$

provided that $2 \le w \le D$, and

$$\begin{aligned} \prod_{p<w}\left(1 - \frac{\omega(p)}{p}\right) &\ge \sum_{d \mid P(w)} \lambda^-(d)\frac{\omega(d)}{d} \\ &\ge \prod_{p<w}\left(1 - \frac{\omega(p)}{p}\right)(f(s) + \mathcal{O}(e^{\sqrt{\mathcal{K}}-s}(\log D)^{-1/3})), \end{aligned}$$

provided that $2 \le w \le D^{1/2}$. Moreover, for any integer n we have

$$\sum_{d \mid (n, P(w_1,w_2))} \lambda^-(d) \le \sum_{d \mid (n, P(w_1,w_2))} \mu(d) \le \sum_{d \mid (n, P(w_1,w_2))} \lambda^+(d).$$

The next statement is Lemma 11 of [1], written in a slightly different form.

Lemma 2. *On the hypotheses of Lemma* 1 *let* $\delta \mid P(w)$ *and* $s \geq 2$. *We have*

$$\sum_{\substack{d|P(w)\\ d\equiv 0\,(\delta)}} \lambda^{\pm}(d)\frac{\omega(d)}{d} = \sum_{\substack{d|P(w)\\ d\equiv 0\,(\delta)}} \mu(d)\frac{\omega(d)}{d} + \mathcal{O}(\tau(\delta)(s^{-s} + e^{\sqrt{\mathcal{K}}-s}(\log D)^{-1/3})).$$

The next statement is the analog of Lemma 13 of [1]. The proof is almost the same.

Lemma 3. *Suppose that* $\Lambda_i, \Lambda_i^{\pm}$, $1 \leq i \leq 6$, *are numbers satisfying* $\Lambda_i = 0$ *or* 1, $\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+$, $1 \leq i \leq 6$. *Then*

$$\begin{aligned}\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6 \geq{}& \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+\\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+\Lambda_5^+\Lambda_6^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^-\Lambda_5^+\Lambda_6^+\\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^-\Lambda_6^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^-\\ &- 5\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+.\end{aligned}$$

The next lemma is Heath-Brown's decomposition of the sum

$$\sum_{P<n\leq P_1} \Lambda(n)G(n) \tag{9}$$

into sums of two types.

Type I sums are

$$\sum_{\substack{M<m\leq M_1\\ L<l\leq L_1\\ P<ml\leq P_1}}\sum a_m G(ml) \quad \text{and} \quad \sum_{\substack{M<m\leq M_1\\ L<l\leq L_1\\ P<ml\leq P_1}}\sum a_m (\log l)G(ml),$$

where $M_1 \leq 2M$, $L_1 \leq 2L$, $|a_m| \ll \tau_5(m)\log P$.

Type II sums are

$$\sum_{\substack{M<m\leq M_1\\ L<l\leq L_1\\ P<ml\leq P_1}}\sum a_m b_l G(ml),$$

where $M_1 \leq 2M$, $L_1 \leq 2L$, $|a_m| \ll \tau_5(m)\log P$, $|b_l| \ll \tau_5(l)\log P$.

The following lemma comes from [7].

Lemma 4. *Let* $G(n)$ *be a complex-valued function. Let* P, P_1, u, v, z *be positive numbers satisfying* $P > 2, P_1 \leq 2P$, $2 \leq u < v \leq z \leq P$, $u^2 \leq z$, $128uz^2 \leq P_1$, $2^{18}P_1 \leq v^3$. *Then the sum* (9) *may be decomposed into* $\mathcal{O}((\log P)^6)$ *sums*, *each of which is either of type I with* $L \geq z$ *or of type II with* $u \leq L \leq v$.

The next lemma is Bombieri–Vinogradov's theorem (see [3], Chapter 28).

Lemma 5. *Define*

$$\Delta(y,h) = \max_{z\le y}\max_{(l,h)=1}\Bigg|\sum_{\substack{p\le z\\ p\equiv l\,(h)}}\log p - \frac{z}{\varphi(h)}\Bigg|. \tag{10}$$

For any $A>0$ we have

$$\sum_{k\le\sqrt{y}/(\log y)^{A+5}} \Delta(y,k) \ll \frac{y}{(\log y)^A}.$$

For the proofs of the next two lemmas, see [11], Chapter 6, and [16], Chapter 2.

Lemma 6. *If $X\ge 1$ then*

$$\Big|\sum_{n\le X} e(\alpha n)\Big| \le \min\bigg(X, \frac{1}{2\|\alpha\|}\bigg).$$

Lemma 7. *Suppose that $X,Y\ge 1$, $|\alpha - a/q|\le 1/q^2$, $(a,q)=1$, $q\ge 1$. Then*

(i) $$\sum_{n\le X}\min\bigg(Y, \frac{1}{\|\alpha n\|}\bigg) \le 6\bigg(\frac{X}{q}+1\bigg)(Y+q\log q),$$

(ii) $$\sum_{n\le X}\min\bigg(\frac{XY}{n}, \frac{1}{\|\alpha n\|}\bigg) \ll XY\bigg(\frac{1}{q}+\frac{1}{Y}+\frac{q}{XY}\bigg)\log(2Xq).$$

Finally, in the next lemma we summarize some well-known properties of the functions $\tau_k(n)$ and $\varphi(n)$.

Lemma 8. *Let $X\ge 2$, $k\ge 2$, $\varepsilon>0$. We have*

(i) $$\sum_{n\le X}\tau_k^2(n) \ll X(\log X)^{k^2-1},$$ (ii) $$\sum_{n\le X}\tau^k(n) \ll X(\log X)^{2^k-1},$$

(iii) $$\sum_{n\le X}\frac{\tau^k(n)}{n} \ll (\log X)^{2^k},$$ (iv) $$\tau_k(n) \ll n^{\varepsilon},$$

(v) $$\frac{n}{\varphi(n)} \ll \log\log(10n).$$

3. Outline of the proof. A reasonable approach to proving the theorem would be to establish a Bombieri–Vinogradov type result for the sum $I_{k_1,k_2,k_3}(x)$, defined by (5). More precisely, it would be interesting to prove that for each $A>0$ there exists $B=B(A)>0$ such that

$$\sum\sum\sum_{\substack{k_1,k_2,k_3\le\sqrt{x}/(\log x)^B\\ (k_1k_2k_3,2)=1}} |I_{k_1,k_2,k_3}(x) - (\text{expected main term})| \ll \frac{x^2}{(\log x)^A}. \tag{11}$$

This estimate (or the estimate for the sum over squarefree k_i only) would imply the solvability of $p_1 + p_2 = 2p_3$ in different primes such that $p_i + 2$, $i = 1, 2, 3$, are almost-primes.

Using (6) we see that (11) is a consequence of the estimates

$$\sum\sum\sum_{\substack{k_1,k_2,k_3 \le \sqrt{x}/(\log x)^B \\ (k_1k_2k_3,2)=1}} |I^{(1)}_{k_1,k_2,k_3}(x) - (\text{expected main term})| \ll \frac{x^2}{(\log x)^A} \tag{12}$$

and

$$\sum\sum\sum_{\substack{k_1,k_2,k_3 \le \sqrt{x}/(\log x)^B \\ (k_1k_2k_3,2)=1}} |I^{(2)}_{k_1,k_2,k_3}(x)| \ll \frac{x^2}{(\log x)^A}. \tag{13}$$

Proceeding as in [13] we may prove (12) provided that B and $\mathcal{L}$ are large in terms of A (see the proof of Lemma 11). However, we are not able to adapt the method of [13] in order to establish (13) and that is the reason we cannot prove (11) at present.

It was noticed by Professor D. R. Heath-Brown that there exists some $\nu > 0$ such that if β_k are any numbers satisfying $|\beta_k| \le 1$ and if $\mathcal{L}$ is large in terms of A then

$$\max_{\alpha \in E_2} \Big| \sum_{k \le x^\nu} \beta_k S_k(\alpha) \Big| \ll \frac{x}{(\log x)^A}. \tag{14}$$

This observation enables us to find that

$$\Big| \sum\sum\sum_{\substack{k_1,k_2 \le \sqrt{x}/(\log x)^B,\ k_3 \le x^\nu \\ (k_1k_2k_3,2)=1}} \beta_{k_1}\beta_{k_2}\beta_{k_3} I^{(2)}_{k_1,k_2,k_3}(x) \Big| \ll \frac{x^2}{(\log x)^A}.$$

The last estimate may serve as an analog of (13).

We are able to prove (14) for any $\nu < 1/3$. A slightly different sum is estimated in Lemma 12. Working in this way we are not able to apply standard sieve results, as was done in [13]. In the present paper we use the vector sieve of Iwaniec [8] and Brüdern–Fouvry [1].

Suppose that $\mathcal{P}$ is the set of odd primes and consider the sum

$$\Gamma = \sum_{\substack{x < p_1, p_2, p_3 \le 2x \\ (p_i + 2, P(z_i)) = 1,\ i = 1,2,3 \\ p_1 + p_2 = 2p_3}} \log p_1 \log p_2 \log p_3.$$

Any non-trivial estimate from below of Γ implies the solvability of $p_1 + p_2 = 2p_3$ in primes such that $p_i + 2 = \mathrm{P}_{h_i}$, $h_i = [\alpha_i^{-1}]$, $i = 1, 2, 3$. For technical reasons we sieve by small primes separately. We have

$$\Gamma = \sum_{\substack{x<p_1,p_2,p_3\le 2x \\ p_1+p_2=2p_3}} (\log p_1 \log p_2 \log p_3) \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \Lambda_5 \Lambda_6,$$

where

$$\Lambda_i = \begin{cases} \sum_{d|(p_i+2, P(z_0,z_i))} \mu(d) & \text{for } i=1,2,3, \\ \sum_{d|(p_{i-3}+2, P(z_0))} \mu(d) & \text{for } i=4,5,6. \end{cases}$$

Set

$$\Lambda_i^{\pm} = \begin{cases} \sum_{d|(p_i+2, P(z_0,z_i))} \lambda_i^{\pm}(d) & \text{for } i=1,2,3, \\ \sum_{d|(p_{i-3}+2, P(z_0))} \lambda_0^{\pm}(d) & \text{for } i=4,5,6. \end{cases} \tag{15}$$

By Lemma 1 we have $\Lambda_i^- \le \Lambda_i \le \Lambda_i^+$, $1 \le i \le 6$; consequently, we may apply Lemma 3 to get

$$\begin{aligned} \Gamma \ge \Gamma_0 = \sum_{\substack{x<p_1,p_2,p_3\le 2x \\ p_1+p_2=2p_3}} & (\log p_1 \log p_2 \log p_3)(\Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+ \\ & + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+ + \ldots + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^- \\ & - 5\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \Lambda_5^+ \Lambda_6^+). \end{aligned} \tag{16}$$

We use (5), (15) and change the order of summation to obtain

$$\Gamma_0 = \sum_{\substack{d_i|P(z_0,z_i),\, i=1,2,3 \\ \delta_i|P(z_0),\, i=1,2,3}} \kappa(d_1,d_2,d_3,\delta_1,\delta_2,\delta_3) I_{d_1\delta_1,d_2\delta_2,d_3\delta_3}(x),$$

where

$$\begin{aligned} \kappa(d_1,d_2,d_3,\delta_1,\delta_2,\delta_3) = {} & \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^-(\delta_3) \\ & - 5\lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3). \end{aligned} \tag{17}$$

Hence by (6) we get

$$\Gamma_0 = \Gamma_1 + \Gamma_2, \tag{18}$$

where

$$\Gamma_j = \sum_{\substack{d_i|P(z_0,z_i),\, i=1,2,3 \\ \delta_i|P(z_0),\, i=1,2,3}} \kappa(d_1,d_2,d_3,\delta_1,\delta_2,\delta_3) I^{(j)}_{d_1\delta_1,d_2\delta_2,d_3\delta_3}(x), \qquad j=1,2. \tag{19}$$

In Section 4, Lemma 10, we study $I^{(1)}_{k_1,k_2,k_3}(x)$ for squarefree odd $k_1, k_2, k_3 \le \sqrt{x}$ and we find

$$I^{(1)}_{k_1,k_2,k_3}(x) = \sigma_0 x^2 \Omega(k_1, k_2, k_3) + \mathcal{O}(\Xi(x; k_1, k_2, k_3)),$$

where the quantities on the right-hand side are defined by (30)–(32). Therefore

$$\Gamma_1 = \sigma_0 x^2 W + \mathcal{O}(\Gamma_3), \tag{20}$$

where

$$W = \sum_{\substack{d_i \mid P(z_0,z_i),\, i=1,2,3\\ \delta_i \mid P(z_0),\, i=1,2,3}} \kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3)\Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3), \tag{21}$$

$$\Gamma_3 = \sum_{\substack{d_i \mid P(z_0,z_i),\, i=1,2,3\\ \delta_i \mid P(z_0),\, i=1,2,3}} |\kappa(d_1, d_2, d_3, \delta_1, \delta_2, \delta_3)|\Xi(x; d_1\delta_1, d_2\delta_2, d_3\delta_3). \tag{22}$$

In Section 5 we consider Γ_3 by the method of [13] and [14]. We do not know much about the quantity $\Xi(x; k_1, k_2, k_3)$ for individual large k_1, k_2, k_3 (unless we use some hypotheses which have not been proved so far). However, in order to estimate Γ_3 we need an estimate for $\Xi(x; k_1, k_2, k_3)$ "on average", so we may refer to Bombieri–Vinogradov's theorem.

In Section 6 we treat Γ_2 following the approach proposed by Heath-Brown.

In Section 7 we estimate W from below using the method of Brüdern and Fouvry [1]. Suppose that the integers d_1, d_2, d_3, δ_1, δ_2, δ_3 satisfy the conditions imposed in (21). From the explicit formula (31) we get

$$\Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3) = \Omega(d_1, d_2, d_3)\Omega(\delta_1, \delta_2, \delta_3).$$

Hence, by (17), (21) we obtain

$$W = \sum_{i=1}^{6} L_i H_i - 5L_7 H_7,$$

where $L_i, H_i, 1 \le i \le 7$, are defined by (75).

First we study the sums $H_i, 1 \le i \le 7$. The quantity D_0, defined by (1), is large enough with respect to z_0, so Rosser's weights $\lambda_0^{\pm}(\delta_i)$ behave like the Möbius function (see Lemma 2). Hence we may approximate $H_i, 1 \le i \le 7$, by

$$\begin{aligned}\mathcal{D}(z_0) &= \sum_{\delta_i \mid P(z_0),\, i=1,2,3} \mu(\delta_1)\mu(\delta_2)\mu(\delta_3)\Omega(\delta_1, \delta_2, \delta_3)\\ &= \prod_{2<p<z_0} \left(1 - \frac{3p-8}{(p-1)(p-2)}\right).\end{aligned}$$

Therefore W is close to the product $\mathcal{D}(z_0)W^*$, where

$$W^* = \sum_{i=1}^{6} L_i - 5L_7 = \sum_{i=1}^{3} L_i - 2L_4$$
$$= \sum_{d_i \mid P(z_0,z_i),\, i=1,2,3} \xi(d_1,d_2,d_3)\Omega(d_1,d_2,d_3)$$

and where $\xi(d_1,d_2,d_3)$ is defined by (89). The summation in the last sum is taken over integers with no small prime factors. This enables us to approximate W^* with the sum

$$\sum_{d_i \mid P(z_0,z_i),\, i=1,2,3} \frac{\xi(d_1,d_2,d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)},$$

which we may estimate from below using Lemma 1.

Let us notice that the sixfold nature of the vector sieve is merely a technical device to treat small primes separately; in essence a three-dimensional vector sieve is being used.

In Section 8 we summarize the estimates from the previous sections and choose the constants $\mathcal{L}, \alpha_1, \alpha_2, \alpha_3$ in a suitable way in order to prove that

$$\Gamma \gg x^2/(\log x)^3.$$

The last estimate implies the proof of the Theorem.

4. Asymptotic formula for $I^{(1)}_{k_1,k_2,k_3}(x)$. The main result of this section is Lemma 10 in which an asymptotic formula for $I^{(1)}_{k_1,k_2,k_3}(x)$ is found.

Using (3) and (7) we get

$$I^{(1)}_{k_1,k_2,k_3}(x) = \sum_{q \le Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} H(a,q), \tag{23}$$

where

$$H(a,q) = \int_{-1/(q\tau)}^{1/(q\tau)} S_{k_1}\Big(\frac{a}{q}+\alpha\Big) S_{k_2}\Big(\frac{a}{q}+\alpha\Big) S_{k_3}\Big(-2\frac{a}{q}-2\alpha\Big)\, d\alpha. \tag{24}$$

First we study the sums S_{k_i} from the last expression, assuming that

$$|\alpha| \le 1/(q\tau), \quad q \le Q, \quad (a,q)=1. \tag{25}$$

Let $M(\alpha)$ and $\Delta(y,h)$ be defined by (4) and (10) and put

$$c_k(a,q) = \sum_{\substack{m=1 \\ (m,q)=1 \\ m \equiv -2\,((k,q))}}^{q} e\Big(\frac{am}{q}\Big), \quad c_k^*(a,q) = \sum_{\substack{m=1 \\ (m,q)=1 \\ m \equiv -2\,((k,q))}}^{q} e\Big(\frac{-2am}{q}\Big). \tag{26}$$

We have the following

LEMMA 9. *Suppose that $k \le \sqrt{x}$ is an odd integer and that* (25) *holds. Then*

$$S_k\left(\frac{a}{q}+\alpha\right) = \frac{c_k(a,q)}{\varphi([k,q])}M(\alpha) + \mathcal{O}(Q(\log x)\varDelta(2x,[k,q])), \tag{27}$$

$$S_k\left(-2\frac{a}{q}-2\alpha\right) = \frac{c_k^*(a,q)}{\varphi([k,q])}M(-2\alpha) + \mathcal{O}(Q(\log x)\varDelta(2x,[k,q])). \tag{28}$$

We also have

$$|c_k(a,q)| \le 1, \qquad |c_k^*(a,q)| \le 2. \tag{29}$$

The proof of (27) may be found in [13], the proof of (28) is similar. The first of the inequalities (29) is proved in [12], p. 218, where an explicit formula for $c_k(a,q)$ is found. The second of the inequalities (29) may be established similarly.

Suppose that k_1, k_2, k_3 are odd squarefree integers and define

$$\varphi_2(n) = n\prod_{p\mid n}\left(1-\frac{2}{p}\right), \qquad \sigma_0 = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right), \tag{30}$$

$$\begin{aligned}&\varOmega(k_1,k_2,k_3)\\ &= \frac{\varphi_2^2((k_1,k_2,k_3))\varphi((k_1,k_2))\varphi((k_1,k_3))\varphi((k_2,k_3))}{\varphi((k_1,k_2,k_3))\varphi_2((k_1,k_2))\varphi_2((k_1,k_3))\varphi_2((k_2,k_3))\varphi(k_1)\varphi(k_2)\varphi(k_3)},\end{aligned} \tag{31}$$

$$\begin{aligned}\varXi(x;k_1,k_2,k_3) &= \frac{x^2\log x}{k_1k_2k_3}\sum_{q>Q}\frac{(k_1,q)(k_2,q)(k_3,q)\log q}{q^2}\\ &\quad + \frac{\tau^2\log x}{k_1k_2k_3}\sum_{q\le Q}(k_1,q)(k_2,q)(k_3,q)\\ &\quad + xQ^2(\log x)^3\sum_{q\le Q}\bigg(\frac{\varDelta(2x,[k_1,q])}{k_2k_3}\\ &\qquad\qquad + \frac{\varDelta(2x,[k_2,q])}{k_1k_3} + \frac{\varDelta(2x,[k_3,q])}{k_1k_2}\bigg).\end{aligned} \tag{32}$$

We have

LEMMA 10. *For any squarefree odd integers $k_1,k_2,k_3 \le \sqrt{x}$ the following asymptotic formula holds*:

$$I^{(1)}_{k_1,k_2,k_3}(x) = \sigma_0x^2\varOmega(k_1,k_2,k_3) + \mathcal{O}(\varXi(x;k_1,k_2,k_3)).$$

P r o o f. Suppose that a,q,α satisfy (25). We use the trivial estimates

$$\left|S_k\left(\frac{a}{q}+\alpha\right)\right| \ll \frac{x\log x}{k}, \qquad |M(\alpha)| \ll x,$$

Lemma 8(v), Lemma 9 and (29) to obtain

$$
\begin{aligned}
(33)\qquad & S_{k_1}\Big(\frac{a}{q}+\alpha\Big) S_{k_2}\Big(\frac{a}{q}+\alpha\Big) S_{k_3}\Big(-2\frac{a}{q}-2\alpha\Big) \\
&= \frac{c_{k_1}(a,q)c_{k_2}(a,q)c^*_{k_3}(a,q)}{\varphi([k_1,q])\varphi([k_2,q])\varphi([k_3,q])} M^2(\alpha)M(-2\alpha) \\
&\quad + \mathcal{O}\bigg(x^2Q(\log x)^3\bigg(\frac{\Delta(2x,[k_1,q])}{k_2k_3} + \frac{\Delta(2x,[k_2,q])}{k_1k_3} + \frac{\Delta(2x,[k_3,q])}{k_1k_2}\bigg)\bigg).
\end{aligned}
$$

Using (23)–(25) and (32) we see that the contribution to $I^{(1)}_{k_1,k_2,k_3}(x)$ arising from the error term in (33) is $\mathcal{O}(\Xi(x;k_1,k_2,k_3))$. Hence by (23), (24) and (33) we obtain

$$
\begin{aligned}
(34)\quad I^{(1)}_{k_1,k_2,k_3}(x) = \sum_{q\le Q} & \frac{b_{k_1,k_2,k_3}(q)}{\varphi([k_1,q])\varphi([k_2,q])\varphi([k_3,q])} \\
& \times \int_{-1/(q\tau)}^{1/(q\tau)} M^2(\alpha)M(-2\alpha)\,d\alpha + \mathcal{O}(\Xi(x;k_1,k_2,k_3)),
\end{aligned}
$$

where

$$
(35)\qquad b_{k_1,k_2,k_3}(q) = \sum_{\substack{a=0\\(a,q)=1}}^{q-1} c_{k_1}(a,q)c_{k_2}(a,q)c^*_{k_3}(a,q).
$$

We know that

$$
\int_{-1/(q\tau)}^{1/(q\tau)} M^2(\alpha)M(-2\alpha)\,d\alpha = \tfrac{1}{2}x^2 + \mathcal{O}(q^2\tau^2)
$$

(see the proof of Theorem 3.3 from [16]). Therefore by (29), (32), (34), (35) and Lemma 8(v) we get

$$
(36)\qquad I^{(1)}_{k_1,k_2,k_3}(x) = \tfrac{1}{2}x^2\mathcal{B} + \mathcal{O}(\Xi(x;k_1,k_2,k_3)),
$$

where

$$
(37)\qquad \mathcal{B} = \sum_{q\le Q} \frac{b_{k_1,k_2,k_3}(q)}{\varphi([k_1,q])\varphi([k_2,q])\varphi([k_3,q])}.
$$

Define

$$
(38)\qquad h_{k_1,k_2,k_3}(q) = \frac{b_{k_1,k_2,k_3}(q)\varphi((k_1,q))\varphi((k_2,q))\varphi((k_3,q))}{\varphi^3(q)},
$$

$$
(39)\qquad \eta_{k_1,k_2,k_3} = \sum_{q=1}^{\infty} h_{k_1,k_2,k_3}(q).
$$

We apply (29), (35), (37)–(39), Lemma 8(v) and the identity

$$\varphi([k,q])\varphi((k,q))=\varphi(k)\varphi(q)$$

to get

$$\mathcal{B}=\frac{\eta_{k_1,k_2,k_3}}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}+\mathcal{O}\bigg(\frac{\log x}{k_1k_2k_3}\sum_{q>Q}\frac{(k_1,q)(k_2,q)(k_3,q)\log q}{q^2}\bigg). \tag{40}$$

It remains to compute η_{k_1,k_2,k_3}. It is easy to see that the function $h_{k_1,k_2,k_3}(q)$ is multiplicative with respect to q. We use (26), (35), (38) and after some calculations we get

$$h_{k_1,k_2,k_3}(p^m)=0 \quad \text{for } m\ge 2.$$

Obviously $h_{k_1,k_2,k_3}(2)=1$. It is not difficult to find that for a prime $p>2$ we have: $h_{k_1,k_2,k_3}(p)=-1/(p-1)^2$ if p divides no more than one of the numbers k_1,k_2,k_3; $h_{k_1,k_2,k_3}(p)=1/(p-1)$ if p divides exactly two of k_1,k_2,k_3; finally $h_{k_1,k_2,k_3}(p)=p-1$ if $p\mid k_1$, $p\mid k_2$, $p\mid k_3$. We apply Euler's identity (see [5], Theorem 286) and after some calculations we obtain

$$\begin{aligned}\eta_{k_1,k_2,k_3}&=2\prod_{p>2}\bigg(1-\frac{1}{(p-1)^2}\bigg)\prod_{p\mid(k_1,k_2,k_3)}\frac{(p-2)^2}{p-1}\\&\quad\times\prod_{p\mid(k_1,k_2)}\frac{p-1}{p-2}\prod_{p\mid(k_1,k_3)}\frac{p-1}{p-2}\prod_{p\mid(k_2,k_3)}\frac{p-1}{p-2}.\end{aligned}\tag{41}$$

The proof of the lemma follows from (30)–(32), (36), (40) and (41).

5. The estimate of Γ_3. The main result of this section is the following

Lemma 11. *For the sum Γ_3, defined by* (22), *we have*

$$\Gamma_3\ll x^2(\log x)^{100-5\mathcal{L}}.$$

Proof. Using (1), (8), (17) and (22) we get

$$\Gamma_3\ll\sum_{k_1,k_2,k_3\le H}\tau(k_1)\tau(k_2)\tau(k_3)\Xi(x;k_1,k_2,k_3), \tag{42}$$

where

$$H=x^{1/2}(\log x)^{-60\mathcal{L}}. \tag{43}$$

We find by (32) and (42) that

$$\Gamma_3\ll x^2(\log x)\Sigma_1+\tau^2(\log x)\Sigma_2+xQ^2(\log x)^3\Sigma_3, \tag{44}$$

where

$$\Sigma_1 = \sum_{k_1,k_2,k_3\le H} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3} \sum_{q>Q} \frac{(k_1,q)(k_2,q)(k_3,q)\log q}{q^2},$$

$$\Sigma_2 = \sum_{k_1,k_2,k_3\le H} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3} \sum_{q\le Q} (k_1,q)(k_2,q)(k_3,q),$$

$$\Sigma_3 = \sum_{k_1,k_2,k_3\le H} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_2k_3} \sum_{q\le Q} \Delta(2x,[k_1,q]).$$

Let us consider Σ_1. We have

$$\Sigma_1 = \Sigma_1' + \Sigma_1'', \tag{45}$$

where

$$\Sigma_1' = \sum_{\substack{d_1,d_2,d_3\le H\\ [d_1,d_2,d_3]>Q}} d_1d_2d_3 \sum_{\substack{k_1,k_2,k_3\le H\\ (k_i,q)=d_i,\, i=1,2,3}} \sum_{q>Q} \frac{\tau(k_1)\tau(k_2)\tau(k_3)\log q}{k_1k_2k_3q^2},$$

$$\Sigma_1'' = \sum_{[d_1,d_2,d_3]\le Q} d_1d_2d_3 \sum_{\substack{k_1,k_2,k_3\le H\\ (k_i,q)=d_i,\, i=1,2,3}} \sum_{q>Q} \frac{\tau(k_1)\tau(k_2)\tau(k_3)\log q}{k_1k_2k_3q^2}.$$

First we estimate Σ_1'. We use (2) and Lemma 8(iii), (iv) to get

$$\begin{aligned}
\Sigma_1' &\ll \sum_{\substack{d_1,d_2,d_3\le H\\ [d_1,d_2,d_3]>Q}} d_1d_2d_3 \\
&\quad\times \sum_{\substack{k_1,k_2,k_3\le H\\ k_i\equiv 0\,(d_i),\, i=1,2,3}} \sum_{\substack{q>Q\\ q\equiv 0\,([d_1,d_2,d_3])}} \frac{\tau(k_1)\tau(k_2)\tau(k_3)\log q}{k_1k_2k_3q^2} \\
&\ll (\log x) \sum_{\substack{d_1,d_2,d_3\le H\\ [d_1,d_2,d_3]>Q}} \frac{\tau(d_1)\tau(d_2)\tau(d_3)}{[d_1,d_2,d_3]^2} \\
&\quad\times \sum_{k_i\le H/d_i,\, i=1,2,3} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3} \sum_{q=1}^{\infty} \frac{1+\log q}{q^2} \\
&\ll (\log x)^7 \sum_{h>Q} \frac{1}{h^2} \sum_{[d_1,d_2,d_3]=h} \tau(d_1)\tau(d_2)\tau(d_3) \\
&\ll (\log x)^7 \sum_{h>Q} \frac{\tau^6(h)}{h^2} \ll (\log x)^{7-5\mathcal{L}}.
\end{aligned} \tag{46}$$

For the sum Σ_1'' we get by (2) and Lemma 8(iii)

$$(47)\qquad \Sigma_1'' \ll \sum_{[d_1,d_2,d_3]\le Q} d_1d_2d_3 \times \sum_{\substack{k_1,k_2,k_3\le H\\ k_i\equiv 0\,(d_i),\, i=1,2,3}} \sum_{\substack{q>Q\\ q\equiv 0\,([d_1,d_2,d_3])}} \frac{\tau(k_1)\tau(k_2)\tau(k_3)\log q}{k_1k_2k_3q^2}$$

$$\ll (\log x) \sum_{[d_1,d_2,d_3]\le Q} \frac{\tau(d_1)\tau(d_2)\tau(d_3)}{[d_1,d_2,d_3]^2} \times \sum_{k_i\le H/d_i,\, i=1,2,3} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3} \sum_{q>Q/[d_1,d_2,d_3]} \frac{\log q}{q^2}$$

$$\ll (\log x)^7 \sum_{[d_1,d_2,d_3]\le Q} \frac{\tau(d_1)\tau(d_2)\tau(d_3)}{[d_1,d_2,d_3]^2} \cdot \frac{\log Q}{Q/[d_1,d_2,d_3]}$$

$$\ll \frac{(\log x)^7 \log Q}{Q} \sum_{h\le Q} \frac{\tau^6(h)}{h} \ll (\log x)^{8-10\mathcal{L}}.$$

We shall now treat Σ_2. We use again (2) and Lemma 8(iii) to find

$$(48)\qquad \Sigma_2 = \sum_{d_1,d_2,d_3\le Q} d_1d_2d_3 \sum_{\substack{k_1,k_2,k_3\le H\\ (k_i,q)=d_i,\, i=1,2,3}} \sum_{q\le Q} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3}$$

$$\ll \sum_{d_1,d_2,d_3\le Q} \tau(d_1)\tau(d_2)\tau(d_3) \times \sum_{k_i\le H/d_i,\, i=1,2,3} \frac{\tau(k_1)\tau(k_2)\tau(k_3)}{k_1k_2k_3} \sum_{q\le Q/[d_1,d_2,d_3]} 1$$

$$\ll Q(\log x)^6 \sum_{d_1,d_2,d_3\le Q} \frac{\tau(d_1)\tau(d_2)\tau(d_3)}{[d_1,d_2,d_3]}$$

$$\ll Q(\log x)^6 \sum_{h\le Q^3} \frac{\tau^6(h)}{h} \ll Q(\log x)^7.$$

Finally, we estimate Σ_3. By (2), (43), Lemma 5 and Lemma 8(iii) we get

$$(49)\qquad \Sigma_3 \ll (\log x)^4 \sum_{k\le H} \sum_{q\le Q} \tau(k)\Delta(2x,[k,q])$$

$$\ll (\log x)^4 \sum_{h\le HQ} \tau^3(h)\Delta(2x,h)$$

$$\ll (\log x)^4 \Big(\sum_{h\le HQ} \tau^6(h)\Delta(2x,h)\Big)^{1/2} \Big(\sum_{h\le HQ} \Delta(2x,h)\Big)^{1/2}$$

$$\ll (\log x)^4 \Big(x(\log x) \sum_{h \le HQ} \frac{\tau^6(h)}{h} \Big)^{1/2} \Big(\sum_{h \le HQ} \Delta(2x, h) \Big)^{1/2}$$
$$\ll x(\log x)^{50 - 25\mathcal{L}}.$$

The assertion of the lemma follows from (44)–(49).

6. The estimate of Γ_2. In this section we estimate the sum Γ_2 defined by (19). Define

$$W(K, \alpha) = \sum_{k \le K} \gamma_k S_k(2\alpha),$$

where γ_k are any numbers such that

(50) $$|\gamma_k| \le \tau(k) \quad \text{and} \quad \gamma_k = 0 \quad \text{for} \quad 2 \mid k.$$

In the next lemma we estimate $W(K, \alpha)$ uniformly for $\alpha \in E_2$, assuming that

(51) $$K \le x^{1/3} (\log x)^{-5\mathcal{L}}.$$

Lemma 12. *Suppose that conditions* (50) *and* (51) *hold. We have*

$$\max_{\alpha \in E_2} |W(K, \alpha)| \ll x(\log x)^{350 - 2\mathcal{L}}.$$

P r o o f. We use the definition of $S_k(\alpha)$ and Lemma 8(iv) to get

(52) $$W(K, \alpha) = W^*(K, \alpha) + \mathcal{O}(x^{2/3}),$$

where

$$W^*(K, \alpha) = \sum_{x < n \le 2x} \Lambda(n) e(2\alpha n) \sum_{\substack{k \le K \\ k \mid n+2}} \gamma_k.$$

We apply Lemma 4 with $P = x$, $P_1 = 2x$, $u = x^{0.001}$, $v = 2^{30} x^{1/3}$, $z = x^{0.498}$ to decompose $W^*(K, \alpha)$ into $\mathcal{O}((\log x)^6)$ sums of two types.

Type I sums are

$$W_1 = \sum_{\substack{M < m \le M_1 \\ x < ml \le 2x}} \sum_{L < l \le L_1} a_m e(2\alpha m l) \sum_{\substack{k \le K \\ k \mid ml+2}} \gamma_k$$

and

$$W_1' = \sum_{\substack{M < m \le M_1 \\ x < ml \le 2x}} \sum_{L < l \le L_1} a_m (\log l) e(2\alpha m l) \sum_{\substack{k \le K \\ k \mid ml+2}} \gamma_k,$$

where

(53) $$M_1 \le 2M, \qquad L_1 \le 2L, \qquad ML \asymp x,$$
$$L \ge x^{0.498}, \qquad |a_m| \ll \tau_5(m) \log x.$$

Type II sums are

$$W_2 = \sum_{\substack{M<m\le M_1\\ x<ml\le 2x}} \sum_{L<l\le L_1} a_m b_l e(2\alpha ml) \sum_{\substack{k\le K\\ k\mid ml+2}} \gamma_k,$$

where

(54) $$M_1 \le 2M, \quad L_1 \le 2L, \quad ML \asymp x, \quad x^{0.001} \le L \le 2^{30}x^{1/3}, \\ |a_m| \ll \tau_5(m)\log x, \quad |b_l| \ll \tau_5(l)\log x.$$

Let us consider type II sums. We have

$$|W_2| \ll (\log x) \sum_{M<m\le M_1} \tau_5(m) \Big| \sum_{\substack{L<l\le L_1\\ x<ml\le 2x\\ ml+2\equiv 0\,(k)}} \sum_{k\le K} b_l\gamma_k e(2\alpha ml)\Big|.$$

Using Cauchy's inequality and Lemma 8(i) we get

$$|W_2|^2 \ll M(\log x)^{26} \sum_{M<m\le M_1} \Big| \sum_{\substack{L<l\le L_1\\ x<ml\le 2x\\ ml+2\equiv 0\,(k)}} \sum_{k\le K} b_l\gamma_k e(2\alpha ml)\Big|^2$$
$$= M(\log x)^{26} \times \sum_{M<m\le M_1} \sum_{\substack{L<l_1,l_2\le L_1\\ x<l_1m,l_2m\le 2x\\ l_im+2\equiv 0\,(k_i),\, i=1,2}} \sum_{k_1,k_2\le K} b_{l_1}\overline{b}_{l_2}\gamma_{k_1}\overline{\gamma}_{k_2} e(2\alpha m(l_1-l_2)).$$

Therefore, by (50) and (54),

(55) $$|W_2|^2 \ll M(\log x)^{28} \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=(l_1,k_1)=(l_2,k_2)=1}} \sum_{L<l_1,l_2\le L_1} \tau(k_1)\tau(k_2)\tau_5(l_1)\tau_5(l_2)|V|,$$

where

$$V = \sum_{\substack{M'<m\le M_1'\\ l_im+2\equiv 0\,(k_i),\, i=1,2}} e(2\alpha m(l_1-l_2)),$$
$$M' = \max\Big(\frac{x}{l_1}, \frac{x}{l_2}, M\Big), \quad M_1' = \min\Big(\frac{2x}{l_1}, \frac{2x}{l_2}, M_1\Big).$$

If the system of congruences $l_im+2\equiv 0\,(k_i)$, $i=1,2$, is not solvable, then $V=0$. If it is solvable, then there exists some $f = f(l_1,l_2,k_1,k_2)$ such that the system $l_im+2\equiv 0\,(k_i)$, $i=1,2$, is equivalent to $m\equiv f\,([k_1,k_2])$. In this case we have

$$V = \sum_{\substack{M' < m \le M_1' \\ m \equiv f\,([k_1,k_2])}} e(2\alpha m(l_1 - l_2))$$

$$= \sum_{(M'-f)/[k_1,k_2] < r \le (M_1'-f)/[k_1,k_2]} e(2\alpha(f + r[k_1,k_2])(l_1 - l_2)).$$

Obviously

$$|V| \ll M/[k_1,k_2] \quad \text{for } l_1 = l_2.$$

If $l_1 \neq l_2$ then by Lemma 6 we get

$$|V| \ll \min\left(\frac{M}{[k_1,k_2]}, \frac{1}{\|2\alpha(l_1 - l_2)[k_1,k_2]\|}\right).$$

We substitute these estimates for $|V|$ in (55) and use Lemma 8(i) to find

$$|W_2|^2 \ll M^2 L V_1 (\log x)^{52} + M V_2 (\log x)^{28}, \tag{56}$$

where

$$V_1 = \sum_{k_1,k_2 \le K} \frac{\tau(k_1)\tau(k_2)}{[k_1,k_2]},$$

$$V_2 = \sum_{k_1,k_2 \le K} \tau(k_1)\tau(k_2) \times \sum_{\substack{L < l_1,l_2 \le L_1 \\ l_1 \neq l_2}} \tau_5(l_1)\tau_5(l_2) \min\left(\frac{M}{[k_1,k_2]}, \frac{1}{\|2\alpha(l_1 - l_2)[k_1,k_2]\|}\right).$$

Obviously

$$\sum_{[k_1,k_2]=h} \tau(k_1)\tau(k_2) \le \tau^4(h), \tag{57}$$

hence using Lemma 8(iii) we get

$$V_1 \ll \sum_{h \le K^2} \frac{\tau^4(h)}{h} \ll (\log x)^{16}. \tag{58}$$

Consider V_2. We have

$$V_2 \ll \sum_{h \le K^2} \Big(\sum_{[k_1,k_2]=h} \tau(k_1)\tau(k_2) \Big) \times \sum_{0 < |r| \le L} \Big(\sum_{\substack{L < l_1,l_2 \le L_1 \\ l_1 - l_2 = r}} \tau_5(l_1)\tau_5(l_2) \Big) \min\left(\frac{M}{h}, \frac{1}{\|2\alpha r h\|}\right). \tag{59}$$

Using Cauchy's inequality and Lemma 8(i) we get

$$\sum_{\substack{L<l_1,l_2\le L_1\\ l_1-l_2=r}} \tau_5(l_1)\tau_5(l_2) = \sum_{L<l,l+r\le L_1} \tau_5(l+r)\tau_5(l)$$
$$\le \Big(\sum_{L<l,l+r\le L_1} \tau_5^2(l+r)\Big)^{1/2}\Big(\sum_{L<l,l+r\le L_1} \tau_5^2(l)\Big)^{1/2}$$
$$\ll L(\log x)^{24}.$$

The last estimate and (57), (59) imply

$$V_2 \ll L(\log x)^{24} \sum_{h\le K^2} \tau^4(h) \sum_{1\le r\le L} \min\left(\frac{M}{h}, \frac{1}{\|2\alpha rh\|}\right) \tag{60}$$
$$\ll L(\log x)^{25} \max_{H\le K^2} V_3,$$

where

$$V_3 = V_3(H) = \sum_{h\le H} \tau^4(h) \sum_{1\le r\le L} \min\left(\frac{M}{H}, \frac{1}{\|2\alpha rh\|}\right).$$

We have

$$V_3 = \sum_{m\le 2HL} \Big(\sum_{\substack{h\le H\\ 2rh=m}}\sum_{1\le r\le L} \tau^4(h)\Big) \min\left(\frac{M}{H}, \frac{1}{\|\alpha m\|}\right)$$
$$\ll \sum_{m\le 2HL} \tau^5(m) \min\left(\frac{M}{H}, \frac{1}{\|\alpha m\|}\right).$$

Therefore by Cauchy's inequality and Lemma 8(ii) we get

$$V_3 \ll \left(\sum_{m\le 2HL} \tau^{10}(m)\frac{M}{H}\right)^{1/2} V_4^{1/2} \ll M^{1/2}L^{1/2}V_4^{1/2}(\log x)^{550}, \tag{61}$$

where

$$V_4 = \sum_{m\le 2HL} \min\left(\frac{M}{H}, \frac{1}{\|\alpha m\|}\right).$$

If $\alpha \in E_2$ then there exist a and q such that

$$Q < q \le \tau, \quad (a,q) = 1, \quad |\alpha - a/q| \le 1/q^2. \tag{62}$$

We apply Lemma 7(i) and (2), (51), (54), (60) to get

$$V_4 \ll ML\left(\frac{1}{q} + \frac{q}{ML} + \frac{H}{M} + \frac{1}{HL}\right)\log x$$
$$\ll x\left(\frac{1}{Q} + \frac{K^2}{M}\right)\log x \ll x(\log x)^{1-10\mathcal{L}}.$$

The last inequality and (54), (56), (58), (60), (61) imply

$$|W_2| \ll x(\log x)^{310-2\mathcal{L}}. \tag{63}$$

Consider now the type I sum W_1. By (50) and (53) we find

$$|W_1| \ll (\log x) \sum_{\substack{k\le K\\ (k,2)=1}} \tau(k) \sum_{\substack{M<m\le M_1\\ (m,k)=1}} \tau_5(m)|U|, \tag{64}$$

where

$$U = \sum_{\substack{L'<l\le L_1'\\ ml+2\equiv 0\,(k)}} e(2\alpha ml),$$

$$L' = \max(L, x/m), \qquad L_1' = \min(L_1, 2x/m).$$

Define $\overline{m}$ by $m\overline{m} \equiv 1\,(k)$. We have

$$U = \sum_{\substack{L'<l\le L_1'\\ l\equiv -2\overline{m}\,(k)}} e(2\alpha ml) = \sum_{(L'+2\overline{m})/k<r\le(L_1'+2\overline{m})/k} e(2\alpha m(-2\overline{m}+rk)).$$

By Lemma 6 and (53), (64),

$$|U| \ll \min\left(\frac{x}{mk}, \frac{1}{\|2\alpha mk\|}\right).$$

We substitute the last estimate for $|U|$ in (64), we apply Cauchy's inequality, Lemma 7(ii), Lemma 8(iii) and also (2), (51), (53), (62) to get

$$\begin{aligned}
|W_1| &\ll (\log x) \sum_{k\le K} \tau(k) \sum_{M<m\le M_1} \tau_5(m) \min\left(\frac{x}{mk}, \frac{1}{\|2\alpha mk\|}\right) \\
&\ll (\log x) \sum_{n\le 4MK} \Big(\sum_{\substack{k\le K\ \ M<m\le M_1\\ 2mk=n}} \sum \tau(k)\tau_5(m)\Big) \min\left(\frac{x}{n}, \frac{1}{\|\alpha n\|}\right) \\
&\ll (\log x) \sum_{n\le 4MK} \tau^6(n) \min\left(\frac{x}{n}, \frac{1}{\|\alpha n\|}\right) \\
&\ll (\log x) \bigg(\sum_{n\le 4MK} \frac{\tau^{12}(n)}{n} x\bigg)^{1/2} \bigg(\sum_{n\le 4MK} \min\left(\frac{x}{n}, \frac{1}{\|\alpha n\|}\right)\bigg)^{1/2} \\
&\ll x^{1/2} (\log x)^{2049} \left(x\left(\frac{1}{q} + \frac{q}{x} + \frac{MK}{x}\right)\log x\right)^{1/2} \\
&\ll x(\log x)^{2050-5\mathcal{L}}.
\end{aligned} \tag{65}$$

To estimate W_1' we apply Abel's formula and proceed in the same way to find

$$|W_1'| \ll x(\log x)^{2050-5\mathcal{L}}. \tag{66}$$

The assertion of the lemma follows from the inequality $\mathcal{L} \ge 1000$ and from (52), (63), (65) and (66).

Now we are in a position to estimate the sum Γ_2, defined by (19). The following lemma holds:

LEMMA 13. *We have*

$$|\Gamma_2| \ll x^2 (\log x)^{370-2\mathcal{L}}.$$

Proof. By (17), (19) we get

$$\Gamma_2 = F_1 + \ldots + F_6 - 5F_7, \tag{67}$$

where

$$F_1 = \sum_{\substack{d_i | P(z_0, z_i),\, i=1,2,3 \\ \delta_i | P(z_0),\, i=1,2,3}} \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) \times I^{(2)}_{d_1\delta_1, d_2\delta_2, d_3\delta_3}(x);$$

the meaning of other F_i is clear. Let us estimate F_1. Using (1) and (8) we find

$$F_1 = \sum_{k_1, k_2 \le \sqrt{x}} \sum_{k_3 \le x^{1/3}/(\log x)^{5\mathcal{L}}} a_1(k_1) a_2(k_2) a_3(k_3) I^{(2)}_{k_1,k_2,k_3}(x), \tag{68}$$

where

$$a_1(k) = \sum_{\substack{d | P(z_0, z_1) \\ \delta | P(z_0) \\ d\delta = k}} \lambda_1^-(d)\lambda_0^+(\delta),$$

$$a_i(k) = \sum_{\substack{d | P(z_0, z_i) \\ \delta | P(z_0) \\ d\delta = k}} \lambda_i^+(d)\lambda_0^+(\delta), \qquad i = 2, 3.$$

Obviously

$$|a_i(k)| \le \tau(k), \quad i = 1, 2, 3; \qquad a_i(k) = 0 \quad \text{if } 2 \,|\, k \text{ or } \mu(k) = 0. \tag{69}$$

We use (7), (68) and change the order of summation and integration to get

$$F_1 = \int_{E_2} \mathcal{H}_1(\alpha)\mathcal{H}_2(\alpha)\mathcal{H}_3(\alpha)\, d\alpha,$$

where

$$\begin{aligned} \mathcal{H}_i(\alpha) &= \sum_{k \le \sqrt{x}} a_i(k) S_k(\alpha), \qquad i = 1, 2, \\ \mathcal{H}_3(\alpha) &= \sum_{k \le x^{1/3}/(\log x)^{5\mathcal{L}}} a_3(k) S_k(-2\alpha). \end{aligned} \tag{70}$$

Hence

$$(71)\qquad |F_1| \ll \max_{\alpha\in E_2} |\mathcal{H}_3(\alpha)| \cdot \int_0^1 |\mathcal{H}_1(\alpha)\mathcal{H}_2(\alpha)|\, d\alpha$$
$$\ll \max_{\alpha\in E_2} |\mathcal{H}_3(\alpha)| \cdot \Big(\int_0^1 |\mathcal{H}_1(\alpha)|^2\, d\alpha\Big)^{1/2}\Big(\int_0^1 |\mathcal{H}_2(\alpha)|^2\, d\alpha\Big)^{1/2}.$$

By Lemma 12 and (69), (70) we get

$$(72)\qquad \max_{\alpha\in E_2} |\mathcal{H}_3(\alpha)| \ll x(\log x)^{350-2\mathcal{L}}.$$

It remains to estimate the integrals in formula (71). We use (4), (69) and (70) to obtain

$$(73)\qquad \int_0^1 |\mathcal{H}_j(\alpha)|^2\, d\alpha = \int_0^1 \sum_{k_1,k_2\le\sqrt{x}} a_j(k_1)\overline{a_j(k_2)} \times \sum_{\substack{x<p_1,p_2\le 2x\\ p_i+2\equiv 0\ (k_i),\, i=1,2}} (\log p_1)(\log p_2) e(\alpha(p_1-p_2))\, d\alpha$$
$$= \sum_{k_1,k_2\le\sqrt{x}} a_j(k_1)\overline{a_j(k_2)} \sum_{\substack{x<p\le 2x\\ p+2\equiv 0\ ([k_1,k_2])}} (\log p)^2$$
$$\ll x(\log x)^2 \sum_{k_1,k_2\le\sqrt{x}} \frac{\tau(k_1)\tau(k_2)}{[k_1,k_2]} \ll x(\log x)^{18}.$$

Hence by (71)–(73) we find

$$|F_1| \ll x^2(\log x)^{370-2\mathcal{L}}.$$

It is clear that the same estimate holds for the other F_i too. Using (67) we obtain the statement of the lemma.

7. The main term. In this section we consider the sum W defined by (21). Suppose that the integers $d_1, d_2, d_3, \delta_1, \delta_2, \delta_3$ satisfy the conditions imposed in (21). Using (31) we easily get

$$\Omega(d_1\delta_1, d_2\delta_2, d_3\delta_3) = \Omega(d_1, d_2, d_3)\Omega(\delta_1, \delta_2, \delta_3).$$

Hence, by (17) and (21) we obtain

$$(74)\qquad W = \sum_{i=1}^{6} L_i H_i - 5L_7H_7,$$

where

$$(75)\qquad L_1 = \sum_{d_i\mid P(z_0,z_i),\, i=1,2,3} \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\Omega(d_1,d_2,d_3),$$

$$L_2 = \sum_{d_i \mid P(z_0, z_i),\, i=1,2,3} \lambda_1^+(d_1)\lambda_2^-(d_2)\lambda_3^+(d_3)\Omega(d_1, d_2, d_3),$$

$$L_3 = \sum_{d_i \mid P(z_0, z_i),\, i=1,2,3} \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^-(d_3)\Omega(d_1, d_2, d_3),$$

$$L_4 = L_5 = L_6 = L_7 = \sum_{d_i \mid P(z_0, z_i),\, i=1,2,3} \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3)\Omega(d_1, d_2, d_3),$$

(75) [cont.]

$$H_1 = H_2 = H_3 = H_7 = \sum_{\delta_i \mid P(z_0),\, i=1,2,3} \lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3)\Omega(\delta_1, \delta_2, \delta_3),$$

$$H_4 = \sum_{\delta_i \mid P(z_0),\, i=1,2,3} \lambda_0^-(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3)\Omega(\delta_1, \delta_2, \delta_3),$$

$$H_5 = \sum_{\delta_i \mid P(z_0),\, i=1,2,3} \lambda_0^+(\delta_1)\lambda_0^-(\delta_2)\lambda_0^+(\delta_3)\Omega(\delta_1, \delta_2, \delta_3),$$

$$H_6 = \sum_{\delta_i \mid P(z_0),\, i=1,2,3} \lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^-(\delta_3)\Omega(\delta_1, \delta_2, \delta_3).$$

Note that the expressions for H_4, H_5, H_6 are equal because of the symmetry with respect to δ_1, δ_2, δ_3.

In the following lemma we find asymptotic formulas for the sums H_i.

LEMMA 14. *We have*

$$H_i = \mathcal{D}(z_0) + \mathcal{O}(e^{-c\sqrt{\log x}}), \qquad 1 \le i \le 7, \tag{76}$$

where

$$\mathcal{D}(z_0) = \prod_{2<p<z_0} \left(1 - \frac{3p-8}{(p-1)(p-2)}\right), \qquad \mathcal{D}(z_0) \asymp (\log z_0)^{-3}. \tag{77}$$

Proof. The estimate (77) is clear. Let us prove (76). Consider, for example, H_1. By (31) we have

$$\begin{aligned} H_1 &= \sum_{\delta \mid P(z_0)} \frac{\varphi_2^2(\delta)}{\varphi(\delta)} \sum_{\substack{\delta_1, \delta_2, \delta_3 \mid P(z_0) \\ (\delta_1, \delta_2, \delta_3) = \delta}} \lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) \\ &\quad \times \frac{\varphi((\delta_1, \delta_2))\varphi((\delta_1, \delta_3))\varphi((\delta_2, \delta_3))}{\varphi(\delta_1)\varphi(\delta_2)\varphi(\delta_3)\varphi_2((\delta_1, \delta_2))\varphi_2((\delta_1, \delta_3))\varphi_2((\delta_2, \delta_3))} \\ &= \sum_{\delta \mid P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{\substack{\delta_1, \delta_2, \delta_3 \mid P(z_0)/\delta \\ (\delta_1, \delta_2, \delta_3) = 1}} \lambda_0^+(\delta_1\delta)\lambda_0^+(\delta_2\delta)\lambda_0^+(\delta_3\delta) \end{aligned} \tag{78}$$

$$
\begin{aligned}
&\times \frac{\varphi((\delta_1,\delta_2))\varphi((\delta_1,\delta_3))\varphi((\delta_2,\delta_3))}{\varphi(\delta_1\delta)\varphi(\delta_2\delta)\varphi(\delta_3\delta)\varphi_2((\delta_1,\delta_2))\varphi_2((\delta_1,\delta_3))\varphi_2((\delta_2,\delta_3))} \\
&= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{\delta_1,\delta_2,\delta_3|P(z_0)/\delta} \lambda_0^+(\delta_1\delta)\lambda_0^+(\delta_2\delta)\lambda_0^+(\delta_3\delta) \\
&\quad \times \Big(\sum_{t|(\delta_1,\delta_2,\delta_3)} \mu(t) \Big) \\
&\quad \times \frac{\varphi((\delta_1,\delta_2))\varphi((\delta_1,\delta_3))\varphi((\delta_2,\delta_3))}{\varphi(\delta_1\delta)\varphi(\delta_2\delta)\varphi(\delta_3\delta)\varphi_2((\delta_1,\delta_2))\varphi_2((\delta_1,\delta_3))\varphi_2((\delta_2,\delta_3))} \\
&= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \mu(t) \sum_{\substack{\delta_1,\delta_2,\delta_3|P(z_0)/\delta \\ \delta_i\equiv 0\,(t),\, i=1,2,3}} \lambda_0^+(\delta_1\delta)\lambda_0^+(\delta_2\delta)\lambda_0^+(\delta_3\delta) \\
&\quad \times \frac{\varphi((\delta_1,\delta_2))\varphi((\delta_1,\delta_3))\varphi((\delta_2,\delta_3))}{\varphi(\delta_1\delta)\varphi(\delta_2\delta)\varphi(\delta_3\delta)\varphi_2((\delta_1,\delta_2))\varphi_2((\delta_1,\delta_3))\varphi_2((\delta_2,\delta_3))} \\
&= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \frac{\mu(t)\varphi^3(t)}{\varphi_2^3(t)} \\
&\quad \times \sum_{\delta_1,\delta_2,\delta_3|P(z_0)/(\delta t)} \lambda_0^+(\delta_1\delta t)\lambda_0^+(\delta_2\delta t)\lambda_0^+(\delta_3\delta t) \\
&\quad \times \frac{\varphi((\delta_1,\delta_2))\varphi((\delta_1,\delta_3))\varphi((\delta_2,\delta_3))}{\varphi(\delta_1\delta t)\varphi(\delta_2\delta t)\varphi(\delta_3\delta t)\varphi_2((\delta_1,\delta_2))\varphi_2((\delta_1,\delta_3))\varphi_2((\delta_2,\delta_3))} \\
&= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \frac{\mu(t)\varphi^3(t)}{\varphi_2^3(t)} \sum_{l_1,l_2,l_3|P(z_0)/(\delta t)} \frac{\varphi(l_1)\varphi(l_2)\varphi(l_3)}{\varphi_2(l_1)\varphi_2(l_2)\varphi_2(l_3)} \\
&\quad \times \sum_{\substack{\delta_1,\delta_2,\delta_3|P(z_0)/(\delta t) \\ (\delta_1,\delta_2)=l_3,\,(\delta_1,\delta_3)=l_2 \\ (\delta_2,\delta_3)=l_1}} \frac{\lambda_0^+(\delta_1\delta t)\lambda_0^+(\delta_2\delta t)\lambda_0^+(\delta_3\delta t)}{\varphi(\delta_1\delta t)\varphi(\delta_2\delta t)\varphi(\delta_3\delta t)} \\
&= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \frac{\mu(t)\varphi^3(t)}{\varphi_2^3(t)} \sum_{l_1,l_2,l_3|P(z_0)/(\delta t)} \frac{\varphi(l_1)\varphi(l_2)\varphi(l_3)}{\varphi_2(l_1)\varphi_2(l_2)\varphi_2(l_3)} \\
&\quad \times \sum_{\substack{\delta_1,\delta_2,\delta_3|P(z_0)/(\delta t) \\ \delta_1\equiv 0\,([l_2,l_3]),\,\delta_2\equiv 0\,([l_1,l_3]) \\ \delta_3\equiv 0\,([l_1,l_2])}} \frac{\lambda_0^+(\delta_1\delta t)\lambda_0^+(\delta_2\delta t)\lambda_0^+(\delta_3\delta t)}{\varphi(\delta_1\delta t)\varphi(\delta_2\delta t)\varphi(\delta_3\delta t)} \\
&\quad \times \sum_{\substack{t_1|(\delta_2/l_1,\delta_3/l_1) \\ t_2|(\delta_1/l_2,\delta_3/l_2) \\ t_3|(\delta_1/l_3,\delta_2/l_3)}} \mu(t_1)\mu(t_2)\mu(t_3)
\end{aligned}
$$

$$= \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \frac{\mu(t)\varphi^3(t)}{\varphi_2^3(t)} \sum_{l_1,l_2,l_3|P(z_0)/(\delta t)} \frac{\varphi(l_1)\varphi(l_2)\varphi(l_3)}{\varphi_2(l_1)\varphi_2(l_2)\varphi_2(l_3)}$$

$$\times \sum_{t_i|P(z_0)/(\delta t l_i),\, i=1,2,3} \mu(t_1)\mu(t_2)\mu(t_3)\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3,$$

where

$$\mathcal{U}_i = \sum_{\substack{h|P(z_0)\\ h\equiv 0\,(\varrho_i)}} \frac{\lambda_0^+(h)}{\varphi(h)}, \qquad i = 1, 2, 3,$$

and

$$\varrho_1 = [\delta t, l_2t_2, l_3t_3], \qquad \varrho_2 = [l_1t_1, \delta t, l_3t_3], \qquad \varrho_3 = [l_1t_1, l_2t_2, \delta t]. \tag{79}$$

Define

$$\mathcal{M}_i = \sum_{\substack{h|P(z_0)\\ h\equiv 0\,(\varrho_i)}} \frac{\mu(h)}{\varphi(h)}, \qquad i = 1, 2, 3.$$

Using (1), (79) and Lemma 2 we get

$$|\mathcal{U}_i - \mathcal{M}_i| \ll \tau(\varrho_i)e^{-c\sqrt{\log x}}, \qquad i = 1, 2, 3. \tag{80}$$

It is easy to see that

$$|\mathcal{U}_i|, |\mathcal{M}_i| \le \sum_{\substack{h|P(z_0)\\ h\equiv 0\,(\varrho_i)}} \frac{\mu^2(h)}{\varphi(h)} \ll \frac{\log x}{\varrho_i}, \qquad i = 1, 2, 3. \tag{81}$$

Hence, by (80) and (81) we obtain

$$\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3 = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3 + \mathcal{O}\bigg(\bigg(\frac{\tau(\varrho_1)}{\varrho_2\varrho_3} + \frac{\tau(\varrho_2)}{\varrho_1\varrho_3} + \frac{\tau(\varrho_3)}{\varrho_1\varrho_2}\bigg)e^{-c\sqrt{\log x}}\bigg). \tag{82}$$

We substitute the last formula in (78) to get

$$H_1 = H^* + R, \tag{83}$$

where

$$H^* = \sum_{\delta|P(z_0)} \frac{\varphi^2(\delta)}{\varphi_2(\delta)} \sum_{t|P(z_0)/\delta} \frac{\mu(t)\varphi^3(t)}{\varphi_2^3(t)} \sum_{l_1,l_2,l_3|P(z_0)/(\delta t)} \frac{\varphi(l_1)\varphi(l_2)\varphi(l_3)}{\varphi_2(l_1)\varphi_2(l_2)\varphi_2(l_3)}$$

$$\times \sum_{t_i|P(z_0)/(\delta t l_i),\, i=1,2,3} \mu(t_1)\mu(t_2)\mu(t_3)\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3,$$

and where R is the contribution to (78) arising from the error term in (82). We use (1), (78), (79), (82), Lemma 8(iii), and also the estimate

$$\varphi_2(n) \gg n(\log\log 10n)^{-2} \qquad \text{for } n \not\equiv 0\,(2) \tag{84}$$

(which is an easy consequence of Lemma 8(v)) to get

$$
\begin{aligned}
(85)\qquad |R| &\ll e^{-c\sqrt{\log x}} \sum_{\delta | P(z_0)} \delta \\
&\times \sum_{t | P(z_0)/\delta} \ \sum_{l_1,l_2,l_3 | P(z_0)/(\delta t)} \ \sum_{\substack{t_i | P(z_0)/(\delta t l_i) \\ i=1,2,3}} \frac{\tau([\delta t, l_2 t_2, l_3 t_3])}{[\delta t, l_1 t_1, l_3 t_3][\delta t, l_1 t_1, l_2 t_2]} \\
&\ll e^{-c\sqrt{\log x}} \sum_{d | P(z_0)} d\tau(d) \sum_{h_1,h_2,h_3 | P(z_0)/d} \frac{\tau(h_1)\tau(h_2)\tau(h_3)\tau([d,h_2,h_3])}{[d,h_1,h_3][d,h_1,h_2]} \\
&\ll e^{-c\sqrt{\log x}} \sum_{d | P(z_0)} \frac{\tau^2(d)}{d} \\
&\times \sum_{h_1,h_2,h_3 | P(z_0)/d} \frac{\tau(h_1)\tau^2(h_2)\tau^2(h_3)(h_1,h_3)(h_1,h_2)}{h_1^2 h_2 h_3} \\
&\ll e^{-c\sqrt{\log x}} \sum_{d | P(z_0)} \frac{\tau^2(d)}{d} \\
&\times \sum_{h_2,h_3 | P(z_0)/d} \frac{\tau^2(h_2)\tau^2(h_3)}{h_2 h_3} \prod_{\substack{p | P(z_0) \\ p \nmid d}} \left(1 + \frac{2(h_2,p)(h_3,p)}{p^2}\right) \\
&\ll e^{-c\sqrt{\log x}} \sum_{d | P(z_0)} \frac{\tau^2(d)}{d} \sum_{h_2,h_3 | P(z_0)/d} \frac{\tau^2(h_2)\tau^2(h_3)\tau^2((h_2,h_3))}{h_2 h_3} \\
&\ll e^{-c\sqrt{\log x}} \sum_{d | P(z_0)} \frac{\tau^2(d)}{d} \sum_{h_2,h_3 | P(z_0)/d} \frac{\tau^3(h_2)\tau^3(h_3)}{h_2 h_3} \ll e^{-c\sqrt{\log x}}.
\end{aligned}
$$

Let us consider H^*. The calculations we did to obtain (78) are valid not only for $\lambda_0^{\pm}$ but for any functions, including Möbius' function. Therefore

$$
\begin{aligned}
H^* &= \sum_{\delta_1,\delta_2,\delta_3 | P(z_0)} \mu(\delta_1)\mu(\delta_2)\mu(\delta_3)\Omega(\delta_1,\delta_2,\delta_3) \\
&= \sum_{\delta_1,\delta_2 | P(z_0)} \frac{\mu(\delta_1)\mu(\delta_2)\varphi((\delta_1,\delta_2))}{\varphi(\delta_1)\varphi(\delta_2)\varphi_2((\delta_1,\delta_2))} \\
&\quad \times \prod_{2<p<z_0} \left(1 - \frac{\varphi_2^2((\delta_1,\delta_2,p))\varphi((\delta_1,p))\varphi((\delta_2,p))}{(p-1)\varphi((\delta_1,\delta_2,p))\varphi_2((\delta_1,p))\varphi_2((\delta_2,p))}\right) \\
&= \sum_{\substack{\delta_1,\delta_2 | P(z_0) \\ (\delta_1,\delta_2)=1}} \frac{\mu(\delta_1)\mu(\delta_2)}{\varphi(\delta_1)\varphi(\delta_2)} \prod_{\substack{2<p<z_0 \\ p \nmid \delta_1\delta_2}} \frac{p-2}{p-1} \prod_{p|\delta_1} \frac{p-3}{p-2} \prod_{p|\delta_2} \frac{p-3}{p-2}
\end{aligned}
$$

$$= \prod_{2<p<z_0} \frac{p-2}{p-1} \cdot \sum_{\substack{\delta_1,\delta_2 | P(z_0)\\ (\delta_1,\delta_2)=1}} \frac{\mu(\delta_1)\mu(\delta_2)\varphi_3(\delta_1)\varphi_3(\delta_2)}{\varphi_2^2(\delta_1)\varphi_2^2(\delta_2)},$$

where we have set

$$\varphi_3(n) = n \prod_{p|n} \left(1 - \frac{3}{p}\right).$$

Hence, using (77) we find

$$\begin{aligned} H^* &= \prod_{2<p<z_0} \frac{p-2}{p-1} \cdot \sum_{\delta_1 | P(z_0)} \frac{\mu(\delta_1)\varphi_3(\delta_1)}{\varphi_2^2(\delta_1)} \prod_{\substack{2<p<z_0\\ p\nmid \delta_1}} \left(1 - \frac{p-3}{(p-2)^2}\right) \\ &= \prod_{2<p<z_0} \frac{p-2}{p-1} \prod_{2<p<z_0} \frac{p^2-5p+7}{(p-2)^2} \\ &\quad \times \sum_{\delta_1 | P(z_0)} \mu(\delta_1)\varphi_3(\delta_1) \prod_{p|\delta_1} (p^2-5p+7)^{-1} \\ &= \mathcal{D}(z_0). \end{aligned}$$

From the last formula and (83), (85) we obtain

$$H_1 = \mathcal{D}(z_0) + \mathcal{O}(e^{-c\sqrt{\log x}}).$$

We consider the other H_i in the same way, so Lemma 14 is proved.

In the next lemma we estimate from below the quantity W defined by (21). We put

$$(86)\qquad \mathcal{F}(z_0, z_i) = \prod_{z_0 \le p < z_i} \left(1 - \frac{1}{p-1}\right), \qquad s_i = \frac{\log D_i}{\log z_i}, \qquad i = 1, 2, 3.$$

Suppose that $c^* > 0$ is an absolute constant and let θ_i, s_i, $i = 1, 2, 3$, satisfy

$$(87)\qquad \theta_1 + \theta_2 + \theta_3 = 1, \quad \theta_i > 0, \quad f(s_i) - 2\theta_i F(s_i) > c^*, \quad i = 1, 2, 3.$$

LEMMA 15. *On the hypotheses above we have*

$$W \ge \mathcal{D}(z_0) \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \Big(\sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}((\log x)^{-1/3}) \Big).$$

Proof. Using (8), (31), (75), (84) and Lemma 8(iii), (v) we see that

$$|L_i| \ll (\log x) \sum_{d_1,d_2,d_3 \le x} \frac{(d_1,d_2,d_3)}{d_1 d_2 d_3} \ll (\log x)^5, \qquad 1 \le i \le 7.$$

By (74), (75), Lemma 14 and the last estimate we obtain

$$(88)\qquad W = \mathcal{D}(z_0) W^* + \mathcal{O}(e^{-c\sqrt{\log x}}),$$

where

$$(89)\quad \begin{aligned} W^* &= \sum_{d_i \mid P(z_0,z_i),\, i=1,2,3} \xi(d_1,d_2,d_3)\Omega(d_1,d_2,d_3),\\ \xi(d_1,d_2,d_3) &= \lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3) + \lambda_1^+(d_1)\lambda_2^-(d_2)\lambda_3^+(d_3)\\ &\quad + \lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^-(d_3) - 2\lambda_1^+(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3). \end{aligned}$$

We have

$$(90)\qquad W^* = W_1 + W_1',$$

where

$$W_1 = \sum_{\substack{d_i \mid P(z_0,z_i),\, i=1,2,3\\ (d_1,d_2)=(d_1,d_3)=(d_2,d_3)=1}} \xi(d_1,d_2,d_3)\Omega(d_1,d_2,d_3)$$

and where W_1' is the sum over $d_i \mid P(z_0,z_i)$, $i=1,2,3$, such that $(d_i,d_j) > 1$ for some $i \neq j$. For these d_i, d_j we certainly have $(d_i,d_j) \geq z_0$. Hence, by (31), (84), Lemma 8(iv), (v) we get

$$(91)\quad \begin{aligned} |W_1'| &\ll \sum_{\substack{d_1,d_2,d_3 \leq x\\ (d_1,d_2)\geq z_0}} \Omega(d_1,d_2,d_3) \ll (\log x) \sum_{\substack{d_1,d_2,d_3 \leq x\\ (d_1,d_2)\geq z_0}} \frac{(d_1,d_2,d_3)}{d_1d_2d_3}\\ &= (\log x) \sum_{z_0 \leq t \leq x}\ \sum_{\substack{d_1,d_2,d_3 \leq x\\ (d_1,d_2)=t}} \frac{(t,d_3)}{d_1d_2d_3}\\ &\ll (\log x) \sum_{z_0 \leq t \leq x} \frac{1}{t^2} \sum_{d_1,d_2 \leq x/t} \frac{1}{d_1d_2} \sum_{d_3 \leq x} \frac{(t,d_3)}{d_3}\\ &\ll (\log x)^3 \sum_{z_0 \leq t \leq x} \frac{1}{t^2} \sum_{d \mid t} d \sum_{\substack{d_3 \leq x\\ (d_3,t)=d}} \frac{1}{d_3}\\ &\ll (\log x)^4 \sum_{z_0 \leq t} \frac{\tau(t)}{t^2} \ll \frac{(\log x)^4}{z_0^{1/2}}. \end{aligned}$$

Consider W_1. We have

$$\begin{aligned} W_1 &= \sum_{d_i \mid P(z_0,z_i),\, i=1,2,3} \frac{\xi(d_1,d_2,d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)} \sum_{\substack{l_1 \mid (d_2,d_3)\\ l_2 \mid (d_1,d_3)\\ l_3 \mid (d_1,d_2)}} \mu(l_1)\mu(l_2)\mu(l_3)\\ &= \sum_{l_1,l_2,l_3 \mid P(z_0,z^*)} \mu(l_1)\mu(l_2)\mu(l_3)\\ &\quad \times \sum_{\substack{d_i \mid P(z_0,z_i),\, i=1,2,3\\ d_1 \equiv 0\,([l_2,l_3]),\, d_2 \equiv 0\,([l_1,l_3])\\ d_3 \equiv 0\,([l_1,l_2])}} \frac{\xi(d_1,d_2,d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)}, \end{aligned}$$

where $z^* = \max(z_1, z_2, z_3)$. We have

$$W_1 = W_2 + W_2', \tag{92}$$

where

$$W_2 = \sum_{d_i \mid P(z_0, z_i),\, i=1,2,3} \frac{\xi(d_1, d_2, d_3)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)} \tag{93}$$

and where W_2' is the sum over $l_1, l_2, l_3 \mid P(z_0, z^*)$ such that $l_j > 1$ for some j. Obviously, such l_j satisfies $l_j \ge z_0$.

We use (1), (8), (89) and Lemma 8(v) to find

$$\begin{aligned}
|W_2'| &\ll (\log x) \sum_{\substack{l_1, l_2, l_3 \le x \\ z_0 \le l_1}} \mu^2(l_1)\mu^2(l_2)\mu^2(l_3) \\
&\quad \times \sum_{\substack{d_1, d_2, d_3 \le x \\ d_1 \equiv 0\,([l_2, l_3]),\, d_2 \equiv 0\,([l_1, l_3]) \\ d_3 \equiv 0\,([l_1, l_2])}} \frac{1}{d_1 d_2 d_3} \\
&\ll (\log x)^4 \sum_{\substack{z_0 \le l_1 \le x \\ l_2, l_3 \le x}} \frac{\mu^2(l_1)\mu^2(l_2)\mu^2(l_3)(l_1, l_2)(l_1, l_3)(l_2, l_3)}{l_1^2 l_2^2 l_3^2} \\
&\ll (\log x)^4 \sum_{z_0 \le l_1 \le x} \frac{\mu^2(l_1)}{l_1^2} \sum_{l_2 \le x} \frac{\mu^2(l_2)(l_1, l_2)}{l_2^2} \\
&\quad \times \prod_{p \le x} \left(1 + \frac{(l_1, p)(l_2, p)}{p^2}\right) \\
&\ll (\log x)^5 \sum_{z_0 \le l_1 \le x} \frac{\mu^2(l_1)}{l_1^2} \sum_{l_2 \le x} \frac{\mu^2(l_2)(l_1, l_2)\tau((l_1, l_2))}{l_2^2} \\
&\ll (\log x)^5 \sum_{z_0 \le l_1 \le x} \frac{\mu^2(l_1)}{l_1^2} \prod_{p \le x} \left(1 + \frac{(l_1, p)\tau((l_1, p))}{p^2}\right) \\
&\ll (\log x)^7 \sum_{z_0 \le l} \frac{\mu^2(l)}{l^2} \ll \frac{(\log x)^7}{z_0}.
\end{aligned} \tag{94}$$

Consider W_2. We find by (89) and (93) that

$$W_2 = G_1^- G_2^+ G_3^+ + G_1^+ G_2^- G_3^+ + G_1^+ G_2^+ G_3^- - 2 G_1^+ G_2^+ G_3^+,$$

where

$$G_i^{\pm} = \sum_{d \mid P(z_0, z_i)} \frac{\lambda_i^{\pm}(d)}{\varphi(d)}, \qquad i = 1, 2, 3.$$

Assume that (87) holds. We have

$$W_2 = W_2^{(1)} + W_2^{(2)} + W_2^{(3)}, \tag{95}$$

where

$$W_2^{(1)} = (G_1^- - 2\theta_1 G_1^+)G_2^+ G_3^+, \quad W_2^{(2)} = (G_2^- - 2\theta_2 G_2^+)G_1^+ G_3^+,$$
$$W_2^{(3)} = (G_3^- - 2\theta_3 G_3^+)G_1^+ G_2^+.$$

Consider, for example, $W_2^{(1)}$. Applying Lemma 1 and using (1), (86), (87) we get

$$(G_1^- - 2\theta_1 G_1^+) \geq \mathcal{F}(z_0, z_1)(f(s_1) - 2\theta_1 F(s_1) + \mathcal{O}((\log x)^{-1/3})),$$
$$G_i^+ \geq \mathcal{F}(z_0, z_i), \quad i = 2, 3.$$

Hence

$$W_2^{(1)} \geq \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \cdot (f(s_1) - 2\theta_1 F(s_1) + \mathcal{O}((\log x)^{-1/3})).$$

We find the corresponding estimates for $W_2^{(i)}$, $i = 2, 3$, similarly and we use (95) to get

$$W_2 \geq \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \cdot \Big(\sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}((\log x)^{-1/3}) \Big). \tag{96}$$

It remains to notice that

$$\mathcal{F}(z_0, z_i) \asymp \frac{\log z_0}{\log z_i} \tag{97}$$

and the conclusion of the lemma follows from (1), (77), (88), (90)–(92), (94), (96) and (97).

8. Proof of the Theorem. Consider the sum

$$\Gamma = \sum_{\substack{x < p_1, p_2, p_3 \leq 2x \\ (p_i + 2, P(z_i)) = 1,\, i = 1,2,3 \\ p_1 + p_2 = 2p_3}} \log p_1 \log p_2 \log p_3.$$

We find (see (16), (18))

$$\Gamma \geq \Gamma_1 + \Gamma_2, \tag{98}$$

where Γ_i, $i = 1, 2$, are defined by (19).

In Section 6, Lemma 13, we prove that

$$|\Gamma_2| \ll x^2 (\log x)^{370 - 2\mathcal{L}}. \tag{99}$$

For Γ_1 we have (see (20))

$$\Gamma_1 = \sigma_0 x^2 W + \mathcal{O}(\Gamma_3), \tag{100}$$

where W, Γ_3, σ_0 are defined by (21), (22), (30).

In Section 5, Lemma 11, we estimate Γ_3 to get

$$\Gamma_3 \ll x^2 (\log x)^{100 - 5\mathcal{L}}. \tag{101}$$

In Section 7, Lemma 15, we consider W. On the conditions (86) and (87) we find

$$W \geq \mathcal{D}(z_0) \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \times \Big(\sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}((\log x)^{-1/3}) \Big), \tag{102}$$

where $f(s)$ and $F(s)$ are the functions of the linear sieve. Hence, using (1), (77), (97)–(102) and assuming that $\mathcal{L} = 1000$ we obtain

$$\Gamma \geq \sigma_0 x^2 \mathcal{D}(z_0) \prod_{j=1}^{3} \mathcal{F}(z_0, z_j) \times \Big(\sum_{i=1}^{3} (f(s_i) - 2\theta_i F(s_i)) + \mathcal{O}((\log x)^{-1/3}) \Big). \tag{103}$$

For $2 \leq s \leq 3$ we have

$$f(s) = \frac{2e^\gamma \log(s-1)}{s}, \qquad F(s) = \frac{2e^\gamma}{s}$$

(γ denotes Euler's constant). We choose

$$\alpha_1 = \alpha_2 = 0.167, \qquad \alpha_3 = 0.116, \qquad \theta_1 = \theta_2 = 0.345, \qquad \theta_3 = 0.31.$$

Then, by (1) and (86),

$$s_1 = s_2 = (0.334)^{-1} + \mathcal{O}((\log x)^{-1/3}), \quad s_3 = (0.348)^{-1} + \mathcal{O}((\log x)^{-1/3}).$$

It is not difficult to compute that for sufficiently large x we have

$$f(s_i) - 2\theta_i F(s_i) > 10^{-5}, \qquad i = 1, 2, 3. \tag{104}$$

Therefore, using (1), (77), (97), (103) and (104) we get

$$\Gamma \gg x^2/(\log x)^3.$$

By the last inequality and the definition of Γ we conclude that for some constant $c_0 > 0$ there are at least $c_0 x^2 (\log x)^{-6}$ triples of primes p_1, p_2, p_3 satisfying $x < p_1, p_2, p_3 \leq 2x$, $p_1 + p_2 = 2p_3$ and such that for any prime factor p of $p_1 + 2$ or $p_2 + 2$ we have $p \geq x^{0.167}$ and for any prime factor p of $p_3 + 2$ we have $p \geq x^{0.116}$. Obviously, the number of trivial triples $p_1 = p_2 = p_3$ is $\mathcal{O}(x)$.

The proof of the Theorem is complete.

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