Certain $L$-functions at $s = 1/2$

by

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**Introduction.** The vanishing orders of $L$-functions at the centers of their functional equations are interesting objects to study as one sees, for example, from the Birch–Swinnerton-Dyer conjecture on the Hasse–Weil $L$-functions associated with elliptic curves over number fields.

In this paper we study the central zeros of the following types of $L$-functions:

(i) the derivatives of the Mellin transforms of Hecke eigenforms for $\text{SL}_2(\mathbb{Z})$,

(ii) the Rankin–Selberg convolution for a pair of Hecke eigenforms for $\text{SL}_2(\mathbb{Z})$,

(iii) the Dedekind zeta functions.

The paper is organized as follows. In Section 1, the Mellin transform $L(s, f)$ of a holomorphic Hecke eigenform $f$ for $\text{SL}_2(\mathbb{Z})$ is studied. We note that every $L$-function in this paper is normalized so that it has a functional equation under the substitution $s \mapsto 1 - s$. In Section 2, we study some nonvanishing property of the Rankin–Selberg convolutions at $s = 1/2$. Section 3 contains Kurokawa’s result asserting the existence of number fields such that the vanishing order of the Dedekind zeta function at $s = 1/2$ goes to infinity.

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**Notation.** As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{C}$ the field of complex numbers. The set of positive (resp. nonnegative) integers is denoted by $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq0}$).

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For $k \in \mathbb{Z}_{>0}$, $M_k$ (resp. $S_k$) denotes the $\mathbb{C}$-vector space of holomorphic modular (resp. cusp) forms of weight $k$ for $SL_2(\mathbb{Z})$.

Let $H$ be the upper half plane, and let $f : H \to \mathbb{C}$ be a $C^\infty$-function satisfying

$$f((az+b)(cz+d)^{-1}) = (cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Such an $f$ is called a $C^\infty$-modular form of weight $k$. The Petersson inner product of $C^\infty$-modular forms $f$ and $g$ of weight $k$ is defined by

$$(f,g) := \int_{SL_2(\mathbb{Z}) \setminus H} f(z)\overline{g(z)}y^{k-2} \, dx \, dy$$

if the right-hand side is convergent. Here $z = x + iy$ with real variables $x$ and $y$ and the integral is taken over a fundamental domain of $SL_2(\mathbb{Z}) \setminus H$.

For a complex variable $s$, we put

$$e(s) := e^{2\pi i s} \quad \text{and} \quad \Gamma_C(s) := 2(2\pi)^{-s}\Gamma(s).$$

Throughout the paper, $z$ is a variable on $H$ and $s$ is a complex variable. We understand that a sum over an empty set is equal to 0.

1. Mellin transforms of modular forms. For a normalized Hecke eigenform

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_k$$

with $k \in 2\mathbb{Z}_{>0}$, the $L$-function

$$L(s, f) := \sum_{n=1}^{\infty} a(n)n^{-s-(k-1)/2} = \prod_{p \text{ prime}} (1 - a(p)p^{-s-(k-1)/2} + p^{-2s})^{-1}$$

converges absolutely and uniformly for $\text{Re}(s) \geq 1 + \delta$ for any $\delta > 0$ [De], and the function

$$A(s, f) := \Gamma_C\left(s + \frac{k-1}{2}\right)L(s, f)$$

extends to the whole $s$-plane as an entire function with functional equation

$$(1.1) \quad A(s, f) = (-1)^{k/2}A(1-s, f).$$

Hence if $k \equiv 2 \pmod{4}$, we have $L(1/2, f) = 0$. For the nonvanishing property of $L(1/2, f)$ in case $k \equiv 0 \pmod{4}$, we refer to [Ko1].

Theorem 1.1. Let $k$ be an even integer $\geq 12$ with $k \neq 14$, and let $\nu$ be a nonnegative integer with $\nu \equiv k/2 \pmod{2}$. Then there exists a normalized Hecke eigenform $f \in S_k$ such that $\Lambda^{(\nu)}(1/2, f) \neq 0$. Here the superscript $(\nu)$ denotes the $\nu$th derivative.
Remark 1.2. If $\nu \neq k/2 \pmod{2}$, then $A^{(\nu)}(1/2, f) = 0$ by (1.1).

To prove Theorem 1.1, we need

Lemma 1.3. For an even integer $k \geq 12$ with $k \neq 14$, there exists an $h \in S_k$ such that $h(it) > 0$ for all $t > 1$.

Proof. The space $S_{12}$ is spanned by

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}. $$

This infinite-product expression implies in particular:

$$\Delta(it) > 0 \quad \text{for all} \quad t > 1.$$

Let $E_l(z) \in M_l$ be the Eisenstein series of weight $l$ for $\text{SL}_2(\mathbb{Z})$ with $l = 4$ or $6$ such that $\lim_{t \to \infty} E_l(it) = 1$. Put

(1.2) $$\sigma_s(n) := \sum_{\substack{d | n \\ d > 0}} d^s.$$  

Then

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e(nz)$$ implies $E_4(it) > 0$ for $t > 1$, and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e(nz)$$ implies $E_6(it) > E_6(i) = 0$ for $t > 1$.

There exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $4a + 6b = k - 12$. Then

$$\Delta(z) E_4(z)^a E_6(z)^b \in S_k$$ satisfies the asserted condition. \quad \blacksquare

Proof of Theorem 1.1. For $h(z) = \sum_{n=1}^{\infty} c(n) e(nz) \in S_k$, we put

$$D(s, h) := \sum_{n=1}^{\infty} c(n) n^{-s - (k-1)/2}$$

for $\text{Re}(s) > 1$. Then

(1.3) $$A(s, h) := \Gamma_C \left( s + \frac{k-1}{2} \right) D(s, h)$$

$$= 2 \int h(it) \left\{ t^{s+(k-3)/2} + (-1)^{k/2} t^{(k-1)/2-s} \right\} dt,$$
which gives analytic continuation of $\Lambda(s, h)$ to the whole $s$-plane. Hence

$$
\Lambda^{(\nu)}(1/2, h) = 2\{1 + (-1)^{\nu+k/2}\} \int_1^\infty h(it) t^{k/2 - 1} (\log t)^\nu \, dt
$$

for every $\nu \in \mathbb{Z}_{\geq 0}$. Thus Lemma 1.3 implies $\Lambda^{(\nu)}(1/2, h) > 0$ for some $h \in S_k$ under the condition $\nu \equiv k/2 \pmod{2}$. Writing $h$ as a $\mathbb{C}$-linear combination of Hecke eigenforms in $S_k$, we obtain the asserted result.

Applying Lemma 1.3 to (1.3), we have

Corollary 1.4 (to the proof). Under the same assumption as in Theorem 1.1, let $\sigma \in \mathbb{R}$ with $0 < \sigma < 1$. Suppose $\sigma \neq 1/2$ if $k \equiv 2 \pmod{4}$. Then there exists a normalized Hecke eigenform $f \in S_k$ such that $L(\sigma, f) \neq 0$.

Remark 1.5. (1) Corollary 1.4 should be compared with the following result of [Ko2]: Let $f_{k,1}, \ldots, f_{k,d_k}$ be the basis of normalized Hecke eigenforms of $S_k$ with $k \in 2\mathbb{Z}_{>0}$. Let $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then there exists a constant $C(t_0, \varepsilon) > 0$ depending only on $t_0$ and $\varepsilon$ such that for $k > C(t_0, \varepsilon)$ the function

$$
\sum_{\nu=1}^{d_k} \frac{1}{(f_{k,\nu}, f_{k,\nu})} \Lambda(s, f_{k,\nu})
$$

does not vanish at any point $s = \sigma + it$ with $t = t_0$, $0 < \sigma < 1/2 - \varepsilon$, $1/2 + \varepsilon < \sigma < 1$.

(2) The nonvanishing property of $L(s, f)$ in the interval $(0, 1)$ is important in the study of holomorphy of the third symmetric power $L$-function attached to $f$ (cf. [Sha]).

2. Rankin–Selberg convolutions. Let

$$
f(z) = \sum_{n=1}^\infty a(n)e(nz) \in S_k
$$

be a normalized Hecke eigenform with $k \in 2\mathbb{Z}_{>0}$. Let $1_\nu$ be the identity matrix of size $\nu \in \mathbb{Z}_{>0}$. For each prime number $p$, we take $M_p(f) \in \text{GL}_2(\mathbb{C})$ such that

$$
1 - a(p)p^{-(k-1)/2}T + T^2 = \det(1_2 - M_p(f)T),
$$

where $T$ is an indeterminate. Each $M_p(f)$ is determined up to conjugacy.

For normalized Hecke eigenforms $f \in S_k$ and $g \in S_l$ with $k, l \in 2\mathbb{Z}_{>0}$, we put

$$
L(s, f \times g) := \prod_{p \text{ prime}} (1_4 - p^{-s}M_p(f) \otimes M_p(g))^{-1},
$$

where $\otimes$ stands for the Kronecker product of matrices. The right-hand side converges absolutely and uniformly for $\text{Re}(s) \geq 1 + \delta$ for any $\delta > 0$ [De]. By
\[
\Gamma_C \left( s - 1 + \frac{k + l}{2} \right) \Gamma_C \left( s + \frac{|k - l|}{2} \right) L(s, f \times g)
\]

extends to the whole \( s \)-plane as a meromorphic function which is invariant under the substitution \( s \mapsto 1 - s \); it is holomorphic except for possible simple poles at \( s = 0 \) and \( 1 \).

**Theorem 2.1.** Let \( f \in S_k \) be a normalized Hecke eigenform and let \( l \) be an even integer satisfying \( l \geq k \) and \( l \neq 14 \). Then there exists a normalized Hecke eigenform \( g \in S_l \) such that \( L(1/2, f \times g) \neq 0 \).

**Remark 2.2.** Some results have been known concerning the nonvanishing at \( s = 1/2 \) of automorphic \( L \)-functions for \( GL(2) \) twisted by characters on \( GL(1) \) ([F-H], [Ko-Za], [W1], [W2]). The above theorem may be seen as a result on such \( L \)-functions twisted, in contrast, by automorphic forms on \( GL(2) \).

The rest of this section is devoted to the proof of Theorem 2.1.

We fix \( k \) and \( l \) as in the assumption of Theorem 2.1 and put

\[
\lambda := \frac{l - k}{2} \quad \text{and} \quad \mu := \frac{l + k}{2}.
\]

For normalized Hecke eigenforms

\[
f(z) = \sum_{n=1}^{\infty} a(n) e(nz) \in S_k, \quad g(z) = \sum_{n=1}^{\infty} b(n) e(nz) \in S_l,
\]

[Sh, Lemma 1] gives

\[
L(s, f \times g) = \zeta(2s) \sum_{n=1}^{\infty} a(n) b(n) n^{1-\mu-s}
\]

for \( \Re(s) > 1 \). For \( \nu \in 2\mathbb{Z}_{\geq 0} \), the Eisenstein series

\[
E_{\nu}(z, s) := \sum_{(m,n) \in \mathbb{Z}^2-(0,0)} (mz + n)^{-\nu} |mz + n|^{-2s}
\]

has a meromorphic continuation to the whole \( s \)-plane. By [Sh], we have

\[
(2) \quad 2(4\pi)^{-s-\mu+1} \Gamma(s + \mu - 1) L(s, f \times g) = \int_{SL_2(\mathbb{Z}) \backslash H} f(z) \overline{g(z)} E_{2\lambda}(z, s - \lambda) y^{s+\mu-2} \, dx \, dy,
\]

where \( z = x + iy \). This gives analytic continuation of \( L(s, f \times g) \) to the whole \( s \)-plane. We study (2.3) at \( s = 1/2 \).

**Lemma 2.3.** Put

\[
C_\lambda(z) := \frac{1}{2} y^{1/2-\lambda} E_{2\lambda}(z, 1/2 - \lambda)
\]
for \( \lambda \in \mathbb{Z}_{\geq 0} \) and \( z \in H \). Let
\[
C_\lambda(z) = \sum_{n \in \mathbb{Z}} c_\lambda(n, y) e(nx)
\]
be the Fourier expansion, where \( z = x + iy \). Then
\[
c_\lambda(n, y) = \begin{cases} 
  y^{1/2 - \lambda} \left\{ \gamma - \log 4\pi + 2 \sum_{r=1}^{\lambda} \frac{1}{2r - 1} \right\} + \log y & \text{if } n = 0, \\
  (-1)^\lambda \sqrt{\pi} y^{-\lambda} \sigma_0(|n|) \cdot \frac{W_{\sgn(n)\lambda,0}(4\pi|n|y)}{\Gamma(\sgn(n)\lambda + 1/2)} & \text{if } n \neq 0.
\end{cases}
\]
Here \( \sigma_0 \) is defined as in (1.2) with \( s = 0 \), \( \sgn(n) := n/|n| \) for \( n \neq 0 \), \( \gamma \) is the Euler constant
\[
\gamma = 0.57721 \ldots,
\]
and \( W_{a,b}(y) \) is Whittaker’s function which is a solution of the differential equation
\[
\left( 4y^2 \frac{d^2}{dy^2} + 1 - 4b^2 + 4ay - y^2 \right) W(y) = 0
\]
(see, e.g., [E-M-O-T1, p. 264]).

Proof. By [Ma, p. 210],
\[
E_{2\lambda}(z, 1/2 - \lambda) = \lim_{s \to 1} \varphi_\lambda(y, s)
\]
\[
+ (-1)^\lambda \cdot 2 \sqrt{\pi} y \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\sigma_0(|n|)}{\sqrt{|n|}} \cdot \frac{W_{\sgn(n)\lambda,0}(4\pi|n|y)}{\Gamma(\sgn(n)\lambda + 1/2)} e(nx)
\]
with
\[
\varphi_\lambda(y, s) := 2\zeta(s) + (-1)^\lambda 2^{3-s} \pi y^{1-s} \frac{\Gamma(s-1)\zeta(s-1)}{\Gamma(s/2 + \lambda)\Gamma(s/2 - \lambda)}.
\]
The di-gamma function
\[
\psi(s) := \Gamma'(s)/\Gamma(s)
\]
assumes the values
\[
\psi(m + 1/2) = 2 \sum_{r=1}^{m} \frac{1}{2r - 1} - \log 4 - \gamma
\]
for \( m \in \mathbb{Z}_{\geq 0} \), hence by a direct computation we have
\[
\lim_{s \to 1} \varphi_\lambda(y, s) = 2 \left( \gamma - \log 4\pi + 2 \sum_{r=1}^{\lambda} \frac{1}{2r - 1} \right) + 2 \log y.
\]
We have

\[(2.10) \quad (4\pi)^{1/2-\mu} \Gamma(\mu - 1/2) L(1/2, f \times g) = (f \cdot C_\lambda, g)\]

from (2.3). Here

\[(2.11) \quad f(z)C_\lambda(z) = \sum_{\nu \in \mathbb{Z}} \left\{ \sum_{m+n=\nu} a(m)c(n, y)e^{-2\pi my} \right\} e(\nu x)\]

with \(a(m) := 0\) for \(m \leq 0\).

Now we recall the holomorphic projection operator of [St] (specialized to our case): Let \(\varphi : H \to \mathbb{C}\) be a \(C^\infty\)-modular form (see the Notation above) of weight \(w \in \mathbb{Z}\) for \(SL_2(\mathbb{Z})\) with Fourier expansion

\[\varphi(z) = \sum_{\nu \in \mathbb{Z}} r(\nu, y)e(\nu x).\]

Suppose that \(w \geq 6\) and that \(\varphi\) satisfies the growth condition

\[(2.12) \quad \int_0^1 dx \int_0^\infty |\varphi(z)|y^{w-2}e^{-\varepsilon y} dy < \infty\]

for every \(\varepsilon > 0\). Put

\[(2.13) \quad r(\nu) := \frac{(4\pi \nu)^{w-1}}{\Gamma(w-1)} \int_0^\infty r(\nu, y)e^{-2\pi \nu y}y^{w-2} dy\]

and

\[\pi_{\text{hol}}(\varphi)(z) := \sum_{\nu=1}^\infty r(\nu)e(\nu z).\]

Then \(\pi_{\text{hol}}(\varphi) \in S_w\) and \((\varphi, h) = (\pi_{\text{hol}}(\varphi), h)\) for all \(h \in S_w\). This \(\pi_{\text{hol}}(\varphi)\) is called the holomorphic projection of \(\varphi\).

We apply \(\pi_{\text{hol}}\) to \(\varphi = f \cdot C_\lambda\) in (2.10). The validity of the condition (2.12) in this case (with \(w = l\)) follows from the first assertion of

**Lemma 2.4.** (1) For any \(\varepsilon > 0\) there exists a positive constant \(M\) such that

\[|f(z)C_\lambda(z)| < My^{-l/2}(y^{1/2+\varepsilon} + y^{-1/2-\varepsilon}) \quad \text{for all } z \in H.\]

(2) We have

\[\sum_{m+n=\nu} \int_0^\infty |a(m)c(n, y)|e^{-2\pi(m+n)}y^{l-2} dy < \infty \quad \text{for every } \nu \in \mathbb{Z}.\]

**Proof.** We denote by \(c_1, c_2, \ldots\) some positive constants independent of \(n\) and \(y\).
(1) Suppose $y \geq \delta$ for some $\delta > 0$. By (2.6) we have

\begin{equation}
|C_\lambda(z)| \leq \sum_{n \in \mathbb{Z}} |c_\lambda(n, y)|
\leq c_1 y^{1/2-\lambda} + c_2 y^{1/2-\lambda} |\log y|
+ c_3 y^{-\lambda} \sum_{n \neq 0} |W_{\text{sgn}(n)\lambda,0}(4\pi|n|y)|.
\end{equation}

By [E-M-O-T1, p. 264, (5)],

$$W_{\kappa,0}(y) = e^{-y/2}y^{\kappa} 2F_0(1/2 - \kappa, 1/2 - \kappa; -y^{-1})$$

for $\kappa \in \mathbb{C}$ with the hypergeometric function $2F_0$, hence

\begin{equation}
|W_{\text{sgn}(n)\lambda,0}(4\pi|n|y)| \leq c_4 e^{-2\pi|n|y}(|n|y)^{\text{sgn}(n)\lambda}
\end{equation}

for $n \neq 0$. It follows from (2.14) that $|C_\lambda(z)| \leq c_5 y^{1/2-\lambda} |\log y|$, hence

\begin{equation}
|f(z)C_\lambda(z)| \leq c_6 y^{1/2-\lambda-k/2} |\log y|
\quad \text{for } y \geq \delta.
\end{equation}

Since $y^{l/2}|f(z)C_\lambda(z)|$ is SL$_2(\mathbb{Z})$-invariant, the assertion (1) follows from (2.16) and [St, Proposition 2].

(2) First we show the finiteness of the integral

$$I(m, n) := \int_0^\infty |a(m)c_\lambda(n, y)| e^{-2\pi(m+\nu)y} y^{l-2} \, dy$$

for $\nu \in \mathbb{Z}_{>0}$, $0 \leq n \leq \nu - 1$, and $m = \nu - n$. If $n = 0$, we have

$$I(\nu, 0) \leq c_7 \int_0^{\infty} (y^{l/2-\lambda} + y^{1/2-\lambda} |\log y|) e^{-4\pi\nu y} y^{l-2} \, dy < \infty$$

since $l - \lambda = \mu \geq 12$. Next we use the estimate

\begin{equation}
|W_{\lambda,0}(x)| \leq c_8 x^{1/2} |\log x|
\quad \text{as } x \to +0,
\end{equation}

which follows from [E-M-O-T1, p. 264, (2), p. 262, (5),(10)]. If $1 \leq n \leq \nu - 1$, we have

$$I(n - \nu, n) \leq c_9 \int_0^{\infty} |W_{\lambda,0}(4\pi ny)| y^{l-\lambda-2} \, dy < \infty$$

by (2.15) and (2.17). Hence it remains to show that

$$\sum_{n<0} I(\nu - n, n) < \infty.$$
From $|a(m)| = O(m^{k/2})$ it follows that
\[
\sum_{n<0} I(\nu - n, n) \leq c_{10} \sum_{n=1}^{\infty} n^{k/2} \int_{0}^{\infty} e^{-2\pi(2\nu+n)y} y^{l-1} |c_\lambda(-n,y)| \, dy
\]
\[
\leq c_{11} \sum_{n=1}^{\infty} n^{k/2} \left( n \int_{0}^{1/n} e^{-2\pi ny} y^{l-1} \, dy + n^{-\lambda} \int_{1/n}^{\infty} e^{-4\pi ny} y^{l-2} \, dy \right)
\]
by (2.15) and (2.17). Hence
\[
\sum_{n<0} I(\nu - n, n) \leq c_{12} \zeta(l - k/2 - 1) < \infty.
\]

**Proposition 2.5.** The notation being as above, let
\[
\pi_{\text{hol}}(f \cdot C_\lambda)(z) = \sum_{\nu=1}^{\infty} \alpha(\nu) e(\nu z) \in S_l.
\]

Then
\[
\alpha(\nu) = \frac{(4\pi\nu)^{\lambda-1/2} \Gamma(\mu-1/2)}{\Gamma(l-1)} a(\nu)
\]
\[
\times \left\{ 2 \sum_{r=1}^{\lambda} \frac{1}{2r-1} + 2 \sum_{r=1}^{\mu-1} \frac{1}{2r-1} - \log(64\pi^2\nu) \right\}
\]
\[
+ (-1)^{\lambda-1} (2\pi)^{\lambda+1/2} \frac{\Gamma(\mu-1/2)^2}{\Gamma(l-1)}
\]
\[
\times \sum_{m+n=\nu \atop m,n \in \mathbb{Z}} a(m) c_0(|n|) \frac{\Gamma(\text{sgn}(n)\lambda + 1/2) \Gamma(\mu - \text{sgn}(n)\lambda)}{(m + \nu + |n|)^{\mu-1/2}}
\]
\[
\times F\left( \mu - 1/2, 1/2 - \text{sgn}(n)\lambda; \mu - \text{sgn}(n)\lambda; \frac{m + \nu - |n|}{m + \nu + |n|} \right).
\]

Here $F = \mathbb{F}_1$ is the hypergeometric function. The above series converges absolutely for every $\nu \in \mathbb{Z}_{>0}$.

**Proof.** By Lemma 2.4(1), $\varphi = f \cdot C_\lambda$ satisfies the growth condition (2.12). So from (2.13) we have
\[
\alpha(\nu) = \frac{(4\pi\nu)^{\lambda-1} \Gamma(l-1)}{\int_{0}^{\infty} \left( \sum_{m+n=\nu} a(m) c(n,y) e^{-2\pi ny} \right) e^{-2\pi ny} y^{l-2} \, dy}.
\]
By Lemma 2.4(2),
\[\alpha(\nu) = \frac{(4\pi\nu)^{l-1}}{\Gamma(l-1)} \sum_{m+n=\nu} a(m)I_1(n)\]
with
\[I_1(n) := \int_0^\infty e^{-2\pi(m+\nu)y} y^{l-2} c(n, y) \, dy,\]
and the right-hand side of (2.18) is absolutely convergent; here we fix \(\nu\) and put \(m = \nu - n\). From (2.6) and (2.9),
\[I_1(0) = (4\pi\nu)^{\mu+1/2} \Gamma(\mu - 1/2) \left\{ 2 \sum_{r=1}^{\lambda} \frac{1}{2r-1} + 2 \sum_{r=1}^{\mu-1} \frac{1}{2r-1} - \log(64\pi^2\nu) \right\}.
\]
If \(n \neq 0\), (2.6) gives
\[I_1(n) = (-1)^\lambda \sqrt{\pi} \cdot \frac{\sigma_0(|n|)}{\sqrt{|n|}} \cdot \Gamma(\text{sgn}(n)\lambda + 1/2)^{-1} I_2(n)\]
with
\[I_2(n) := \int_0^\infty e^{-2\pi(m+\nu)y} y^{\mu-2} W_{\text{sgn}(n)\lambda,0}(4\pi|n|y) \, dy.\]
By [E-M-O-T2, p. 216, (16)],
\[I_2(n) = 2\sqrt{\pi|n|} \cdot \frac{\Gamma(\mu - 1/2)^2}{\Gamma(\mu - \text{sgn}(n)\lambda)} \cdot \left\{ 2\pi(m + \nu + |n|) \right\}^{-\mu+1/2}
\times F\left(\mu - 1/2, 1/2 - \text{sgn}(n)\lambda; \mu - \text{sgn}(n)\lambda; \frac{m + \nu - |n|}{m + \nu + |n|}\right).
\]
Thus from (2.18) the result follows. \(\blacksquare\)

By Proposition 2.5 we have
\[\frac{(4\pi)^{1/2-\lambda}}{\Gamma(\mu - 1/2)} \alpha(1) = A(\lambda, \mu) + R(\lambda, \mu)\]
with
\[A(\lambda, \mu) := 2 \sum_{r=1}^{\lambda} \frac{1}{2r-1} + 2 \sum_{r=1}^{\mu-1} \frac{1}{2r-1} - \log(64\pi^2),\]
\[R(\lambda, \mu) := \frac{(-1)^\lambda 2\pi \Gamma(\mu - 1/2)}{\Gamma(1/2 - \lambda)\Gamma(l)} \sum_{m=2}^\infty a(m)\sigma_0(m - 1)m^{-\mu+1/2}
\times F(\mu - 1/2, \lambda + 1/2; l; 1/m).\]
Lemma 2.6. For all \( \lambda \) and \( \mu \) we have

\[ |R(\lambda, \mu)| \leq R^*(\lambda, \mu) \]

with

\[ R^*(\lambda, \mu) := \frac{4\Gamma(\mu - 1/2)\Gamma(\lambda + 1/2)}{\Gamma(l)} \cdot \left\{ \zeta \left( \frac{k - 1}{2} \right)^2 - 1 \right\}. \]

Proof. Euler’s integral representation [E-M-O-T1, p. 59, (10)] gives

\[ F(\mu - 1/2, \lambda + 1/2; l; 1/m) = \frac{\Gamma(l)}{\Gamma(\mu - 1/2)\Gamma(\lambda + 1/2)} \int_0^1 t^{\mu - 3/2}(1 - t)^{-1/2}(1 - t/m)^{-\lambda - 1/2} \, dt. \]

Hence

\[ (2.22) \quad F(\mu - 1/2, \lambda + 1/2; l; 1/m) \leq \frac{\Gamma(l)}{\Gamma(\mu - 1/2)\Gamma(\lambda + 1/2)} \int_0^1 t^{\mu - 3/2}(1 - t)^{-1/2}(1 - t/m)^{-\lambda - 1/2} \, dt \]

\[ \leq \left(1 + \frac{1}{m - 1}\right)^{\lambda + 1/2}. \]

Similarly we have

\[ F(\mu - 1/2, \lambda + 1/2; l; 1/m) > 1. \]

Using Deligne’s bound [De]

\[ |a(m)| \leq \sigma_0(m)m^{(k - 1)/2}, \]

from (2.22) and (2.23) we obtain

\[ (2.24) \quad \left| \sum_{m=2}^{\infty} a(m)\sigma_0(m - 1)m^{-\mu + 1/2} \cdot F(\mu - 1/2, \lambda + 1/2; l; 1/m) \right| \]

\[ \leq \sum_{m=2}^{\infty} \sigma_0(m)\sigma_0(m - 1)m^{-l/2} \left(1 + \frac{1}{m - 1}\right)^{\lambda + 1/2}. \]

Since \( \sigma_0(m - 1) \leq 2\sqrt{m - 1} \), the sum (2.24) is majorized by

\[ 2 \sum_{m=2}^{\infty} \sigma_0(m)m^{(1-k)/2} = 2\left\{ \zeta \left( \frac{k - 1}{2} \right)^2 - 1 \right\}. \]
Lemma 2.7. (1) If \(2 \leq m \in \mathbb{Z}\), then
\[
2 \sum_{r=1}^{m} \frac{1}{2r-1} = \gamma + \log 2 + \log(2m-1) + \frac{1}{2m-1} \]
\[- \frac{1}{3(2m-1)^2} + \frac{\theta_m}{120(m-1/2)^4},
\]
with \(0 < \theta_m < 1\), where \(\gamma\) is the Euler constant as in (2.7).

(2) If \(12 \leq k \in \mathbb{Z}\), then
\[
\zeta\left(\frac{k-1}{2}\right)^2 \leq 1 + 2^1 \frac{\gamma}{2}(k-1) + 2^{2} \frac{\gamma}{2}(k-1).
\]

(3) If \(1 < x \in \mathbb{R}\), then
\[
\frac{\Gamma\left(x - \frac{1}{2}\right)}{\Gamma(x)} \leq \frac{1}{\sqrt{x - \frac{1}{2}}} \cdot \exp\left(\frac{1}{4x - 2} + \frac{1}{(4x - 2)^2}\right).
\]

Proof. (1) is immediate from the Euler–MacLaurin formula (see, e.g., [Rad]).

(2) Suppose \(1 < \sigma \in \mathbb{R}\) and \(2 \leq N \in \mathbb{Z}\). The Euler–MacLaurin formula gives
\[
(2.25) \quad \zeta(\sigma) = \sum_{n=1}^{N} n^{-\sigma} + \frac{N^{1-\sigma}}{\sigma-1} - \frac{N^{-\sigma}}{2} + \frac{\sigma}{12} N^{-\sigma-1} \theta
\]
with \(0 < \theta < 1\). If we put \(N = 2\) and use the inequality
\[
\frac{1}{2} + \frac{2}{\sigma-1} + \frac{\sigma}{24} \leq 2^{\sigma/22} \quad \text{for } \sigma \geq \frac{11}{2},
\]
the result follows.

(3) The di-gamma function \(\psi(s)\) defined by (2.8) is increasing for \(0 < s \in \mathbb{R}\). Hence from the mean value theorem it follows that
\[
(2.26) \quad \frac{\Gamma\left(x - \frac{1}{2}\right)}{\Gamma(x)} \leq \exp\left(-\frac{1}{2}\psi\left(x - \frac{1}{2}\right)\right) \quad \text{for } 1 < x \in \mathbb{R}.
\]
By [Rad, p. 37],
\[
\log \Gamma(s) = \frac{1}{2} \log(2\pi) + \left(s - \frac{1}{2}\right) \log s - s
\]
\[- \frac{1}{2} \int_{0}^{\infty} \frac{B_2(x-[x]) - B_2}{(x+s)^2} dx \quad \text{for } 0 < s \in \mathbb{R}
\]
with \(B_2(x)\) being the second Bernoulli polynomial. Differentiating, we have
\[
\psi(s) = \log s - \frac{1}{2s} + \int_{0}^{\infty} \frac{B_2(x-[x]) - B_2}{(x+s)^3} dx.
\]
Here
\[ \left| \int_0^\infty \frac{B_2(x-[x]) - B_2}{(x+s)^3} \, dx \right| \leq \max_{0 \leq x \leq 1} |B_2(x) - B_2| \cdot \int_0^\infty \frac{dx}{(x+s)^3} = \frac{1}{8s^2} \]
since \( B_2(x) - B_2 = x^2 - x \). Hence (3) follows from (2.26). \( \blacksquare \)

**Lemma 2.8.** In the notation of (2.20) and Lemma 2.6, we have
\[ |A(\lambda, \mu)| > R^*(\lambda, \mu) \]
for all \( \lambda, \mu \) defined by (2.2).

**Proof.** (i) **Estimate for** \( A(\lambda, \mu) \). First note that
\[ A(\lambda, \mu) \leq A(\lambda', \mu') \quad \text{if} \quad \lambda \leq \lambda' \quad \text{and} \quad \mu \leq \mu'. \]
If \( \lambda \geq 1 \), then \( \mu \geq 14 \) and
\[ A(\lambda, \mu) \geq A(1, 14) = 0.0803 \ldots \]
by Lemma 2.7(1). If \( \lambda = 0 \), the same lemma gives
\[ A(0, 88) = -0.018 \ldots \quad \text{and} \quad A(0, 90) = 0.0038 \ldots \]
Hence we have
\[ |A(\lambda, \mu)| > 8 \times 10^{-2} \quad \text{for all } \mu \text{ if } \lambda \geq 1, \]
and
\[ |A(0, \mu)| > 3.8 \times 10^{-3} \quad \text{for all } \mu. \]
We also have
\[ |A(0, \mu)| > 1.2 \quad \text{for } \mu \leq 26 \]
since \( A(0, 26) = -1.26 \ldots \)
(ii) **Estimate for** \( R^*(\lambda, \mu) \). If \( \lambda \geq 1 \), we have
\[ \frac{R^*(\lambda, \mu)}{R^*(\lambda-1, \mu-1)} \leq \frac{1}{4} \cdot \frac{(l-2)^2 - (k-1)^2}{(l-2)^2 + (l-2)} < \frac{1}{4} \]
hence
\[ R^*(\lambda, \mu) \leq 2^{-2\lambda} R^*(0, k). \]
Observe that \( R^*(0, k) \) is a decreasing function of \( k \). Hence, if \( \lambda \geq 1 \), we have
\[ R^*(\lambda, \mu) \leq \max \{ 2^{-2} R^*(0, 16), 2^{-4} R^*(0, 12) \} < 7.2 \times 10^{-3} \]
by Lemma 2.7(2) and (3). If \( \lambda = 0 \) and \( \mu \geq 28 \), then
\[ R^*(0, \mu) \leq R^*(0, 28) < 3.7 \times 10^{-4}. \]
If $\lambda = 0$ and $\mu \leq 26$, then

\[(2.32) \quad R^\ast(0, \mu) \leq R^\ast(0, 12) < 0.12.\]

Comparing (2.27) with (2.30), (2.28) with (2.31), and (2.29) with (2.32), we have the assertion of Lemma 2.8. \[\Box\]

Combining Lemma 2.8 with Lemma 2.6 and (2.19), we see that $\alpha(1) \neq 0$ in the notation of Proposition 2.5. Hence $\pi_{\text{hol}}(f \cdot C_\lambda) \neq 0$. Thus for some $g \in S_l$ we have

\[(f \cdot C_\lambda, g) = (\pi_{\text{hol}}(f \cdot C_\lambda), g) \neq 0\]

in (2.10). This completes the proof of Theorem 2.1.

3. Dedekind zeta functions. Let $K$ be an algebraic number field of finite degree. The functional equation of the Dedekind zeta function $\zeta_K(s)$ tells us that the vanishing order $\text{ord}_{s=1/2} \zeta_K(s)$ is a nonnegative even integer. On this we have

**Theorem 3.1** (Kurokawa). *For every $\nu \in \mathbb{Z}_{>0}$ there exists a Galois extension $K_\nu$ over $\mathbb{Q}$ of degree $2^{3\nu}$ such that

\[
\text{ord}_{s=1/2} \zeta_{K_\nu}(s) \geq 2\nu.
\]

In particular,

\[
\sup_{K/\mathbb{Q} \text{ finite}} \text{ord}_{s=1/2} \zeta_K(s) = \infty.
\]

**Proof.** Put

\[(3.1) \quad g_K := \frac{1}{2} \text{ord}_{s=1/2} \zeta_K(s) \in \mathbb{Z}_{\geq 0}.
\]

Let $F/\mathbb{Q}$ be a finite extension and let $L_i/F$ ($i = 1, 2$) be finite Galois extensions such that $L_1 \cap L_2 = F$. By [Br], the function

\[
\frac{\zeta_{L_1L_2}(s)\zeta_F(s)}{\zeta_{L_1}(s)\zeta_{L_2}(s)}
\]

is entire, hence

\[(3.2) \quad g_{L_1L_2} \geq g_{L_1} + g_{L_2} - g_F.
\]

By [Frö], there exists a sequence $\{N_i\}_{i=1}^\infty$ of Galois extensions over $\mathbb{Q}$ of degree 8 such that

\[N_{i+1} \cap N_1 \ldots N_i = \mathbb{Q} \quad \text{and} \quad \zeta_{N_i}(1/2) = 0 \quad \text{for } i \geq 1.
\]

Then $K_\nu := N_1 \ldots N_\nu$ satisfies $g_{K_\nu} \geq \nu$ by (3.2). \[\Box\]

**Remark 3.2.** (1) We refer to [Den2] for an interpretation of the zeros of the Dedekind zeta functions at $s = 1/2$. 

Let $g_K$ be as in (3.1). Theorem 3.1 leads naturally to the following problem: Classify the algebraic number fields by $g_K$. For example, one can ask whether $g_K = 0$ for every finite abelian extension $K/\mathbb{Q}$. This is equivalent to asking whether $L(1/2, \chi) \neq 0$ for every Dirichlet character $\chi$, which is a longstanding open problem in analytic number theory.

References


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