ACTA ARITHMETICA
LXXXVIII.1 (1999)

Certain L-functions at $s = 1/2$

by

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Introduction. The vanishing orders of L-functions at the centers of their functional equations are interesting objects to study as one sees, for example, from the Birch–Swinnerton-Dyer conjecture on the Hasse–Weil L-functions associated with elliptic curves over number fields.

In this paper we study the central zeros of the following types of L-functions:

(i) the derivatives of the Mellin transforms of Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$,
(ii) the Rankin–Selberg convolution for a pair of Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$,
(iii) the Dedekind zeta functions.

The paper is organized as follows. In Section 1, the Mellin transform $L(s, f)$ of a holomorphic Hecke eigenform f for $\mathrm{SL}_2(\mathbb{Z})$ is studied. We note that every L-function in this paper is normalized so that it has a functional equation under the substitution $s \mapsto 1 - s$. In Section 2, we study some nonvanishing property of the Rankin–Selberg convolutions at $s = 1/2$. Section 3 contains Kurokawa's result asserting the existence of number fields such that the vanishing order of the Dedekind zeta function at $s = 1/2$ goes to infinity.

Acknowledgements. The author would like to thank Professor Kurokawa for his permission to include his result (Theorem 3.1 below) in this paper.

Notation. As usual, $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{C}$ the field of complex numbers. The set of positive (resp. nonnegative) integers is denoted by $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$).

1991 *Mathematics Subject Classification*: 11F11, 11F66, 11R42.

For $k \in \mathbb{Z}_{>0}$, M_k (resp. S_k) denotes the $\mathbb{C}$-vector space of holomorphic modular (resp. cusp) forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Let H be the upper half plane, and let $f : H \to \mathbb{C}$ be a C^∞-function satisfying

$$f((az+b)(cz+d)^{-1}) = (cz+d)^k f(z) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Such an f is called a *C^∞-modular form of weight k*. The *Petersson inner product* of C^∞-modular forms f and g of weight k is defined by

$$(f,g) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash H} f(z)\overline{g(z)} y^{k-2}\, dx\, dy$$

if the right-hand side is convergent. Here $z = x + iy$ with real variables x and y and the integral is taken over a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash H$.

For a complex variable s, we put

$$\mathbf{e}(s) := e^{2\pi i s} \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s).$$

Throughout the paper, z is a variable on H and s is a complex variable.

We understand that a sum over an empty set is equal to 0.

1. Mellin transforms of modular forms. For a normalized Hecke eigenform

$$f(z) = \sum_{n=1}^{\infty} a(n)\mathbf{e}(nz) \in S_k$$

with $k \in 2\mathbb{Z}_{>0}$, the L-function

$$L(s,f) := \sum_{n=1}^{\infty} a(n) n^{-s-(k-1)/2} = \prod_{p \text{ prime}} (1 - a(p)p^{-s-(k-1)/2} + p^{-2s})^{-1}$$

converges absolutely and uniformly for $\mathrm{Re}(s) \ge 1 + \delta$ for any $\delta > 0$ [De], and the function

$$\Lambda(s,f) := \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right) L(s,f)$$

extends to the whole s-plane as an entire function with functional equation

$$\Lambda(s,f) = (-1)^{k/2}\Lambda(1-s,f). \tag{1.1}$$

Hence if $k \equiv 2 \pmod 4$, we have $L(1/2,f) = 0$. For the nonvanishing property of $L(1/2,f)$ in case $k \equiv 0 \pmod 4$, we refer to [Ko1].

Theorem 1.1. *Let k be an even integer ≥ 12 with $k \ne 14$, and let ν be a nonnegative integer with $\nu \equiv k/2 \pmod 2$. Then there exists a normalized Hecke eigenform $f \in S_k$ such that $\Lambda^{(\nu)}(1/2,f) \ne 0$. Here the superscript (ν) denotes the νth derivative.*

REMARK 1.2. If $\nu \not\equiv k/2 \pmod 2$, then $\Lambda^{(\nu)}(1/2, f) = 0$ by (1.1).

To prove Theorem 1.1, we need

LEMMA 1.3. *For an even integer $k \geq 12$ with $k \neq 14$, there exists an $h \in S_k$ such that $h(it) > 0$ for all $t > 1$.*

P r o o f. The space S_{12} is spanned by

$$\Delta(z) = \mathbf{e}(z) \prod_{n=1}^{\infty} (1 - \mathbf{e}(nz))^{24}.$$

This infinite-product expression implies in particular:

$$\Delta(it) > 0 \quad \text{for all } t > 1.$$

Let $E_l(z) \in M_l$ be the Eisenstein series of weight l for $\mathrm{SL}_2(\mathbb{Z})$ with $l = 4$ or 6 such that $\lim_{t\to\infty} E_l(it) = 1$. Put

$$\sigma_s(n) := \sum_{\substack{d|n \\ d>0}} d^s. \tag{1.2}$$

Then

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\mathbf{e}(nz)$$

implies $E_4(it) > 0$ for $t > 1$, and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\mathbf{e}(nz)$$

implies $E_6(it) > E_6(i) = 0$ for $t > 1$.

There exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $4a + 6b = k - 12$. Then

$$\Delta(z)E_4(z)^a E_6(z)^b \in S_k$$

satisfies the asserted condition. ■

Proof of Theorem 1.1. For $h(z) = \sum_{n=1}^{\infty} c(n)\mathbf{e}(nz) \in S_k$, we put

$$D(s, h) := \sum_{n=1}^{\infty} c(n) n^{-s-(k-1)/2}$$

for $\mathrm{Re}(s) > 1$. Then

$$\begin{aligned} \Lambda(s, h) &:= \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right) D(s, h) \\ &= 2 \int_1^{\infty} h(it)\{t^{s+(k-3)/2} + (-1)^{k/2} t^{(k-1)/2-s}\}\, dt, \end{aligned} \tag{1.3}$$

which gives analytic continuation of $\Lambda(s,h)$ to the whole s-plane. Hence

$$\Lambda^{(\nu)}(1/2,h)=2\{1+(-1)^{\nu+k/2}\}\int_1^\infty h(it)t^{k/2-1}(\log t)^\nu\,dt$$

for every $\nu\in\mathbb{Z}_{\geq 0}$. Thus Lemma 1.3 implies $\Lambda^{(\nu)}(1/2,h)>0$ for some $h\in S_k$ under the condition $\nu\equiv k/2 \pmod 2$. Writing h as a $\mathbb{C}$-linear combination of Hecke eigenforms in S_k, we obtain the asserted result. ∎

Applying Lemma 1.3 to (1.3), we have

Corollary 1.4 (to the proof). *Under the same assumption as in Theorem* 1.1, *let* $\sigma\in\mathbb{R}$ *with* $0<\sigma<1$. *Suppose* $\sigma\neq 1/2$ *if* $k\equiv 2 \pmod 4$. *Then there exists a normalized Hecke eigenform* $f\in S_k$ *such that* $L(\sigma,f)\neq 0$.

Remark 1.5. (1) Corollary 1.4 should be compared with the following result of [Ko2]: Let $f_{k,1},\dots,f_{k,d_k}$ be the basis of normalized Hecke eigenforms of S_k with $k\in 2\mathbb{Z}_{>0}$. Let $t_0\in\mathbb{R}$ and $\varepsilon>0$. Then there exists a constant $C(t_0,\varepsilon)>0$ depending only on t_0 and ε such that for $k>C(t_0,\varepsilon)$ the function

$$\sum_{\nu=1}^{d_k}\frac{1}{(f_{k,\nu},f_{k,\nu})}\Lambda(s,f_{k,\nu})$$

does not vanish at any point $s=\sigma+it$ with $t=t_0$, $0<\sigma<1/2-\varepsilon$, $1/2+\varepsilon<\sigma<1$.

(2) The nonvanishing property of $L(s,f)$ in the interval $(0,1)$ is important in the study of holomorphy of the third symmetric power L-function attached to f (cf. [Sha]).

2. Rankin–Selberg convolutions. Let

$$f(z)=\sum_{n=1}^\infty a(n)\mathbf{e}(nz)\in S_k$$

be a normalized Hecke eigenform with $k\in 2\mathbb{Z}_{>0}$. Let 1_ν be the identity matrix of size $\nu\in\mathbb{Z}_{>0}$. For each prime number p, we take $M_p(f)\in\mathrm{GL}_2(\mathbb{C})$ such that

$$1-a(p)p^{-(k-1)/2}T+T^2=\det(1_2-M_p(f)T), \tag{2.1}$$

where T is an indeterminate. Each $M_p(f)$ is determined up to conjugacy.

For normalized Hecke eigenforms $f\in S_k$ and $g\in S_l$ with $k,l\in 2\mathbb{Z}_{>0}$, we put

$$L(s,f\times g):=\prod_{p\text{ prime}}(1_4-p^{-s}M_p(f)\otimes M_p(g))^{-1},$$

where $\otimes$ stands for the Kronecker product of matrices. The right-hand side converges absolutely and uniformly for $\mathrm{Re}(s)\geq 1+\delta$ for any $\delta>0$ [De]. By

[Ran], [Se], [Sh],

$$\Gamma_{\mathbb{C}}\left(s-1+\frac{k+l}{2}\right)\Gamma_{\mathbb{C}}\left(s+\frac{|k-l|}{2}\right)L(s,f\times g)$$

extends to the whole s-plane as a meromorphic function which is invariant under the substitution $s \mapsto 1-s$; it is holomorphic expect for possible simple poles at $s=0$ and 1.

THEOREM 2.1. *Let $f \in S_k$ be a normalized Hecke eigenform and let l be an even integer satisfying $l \geq k$ and $l \neq 14$. Then there exists a normalized Hecke eigenform $g \in S_l$ such that $L(1/2, f \times g) \neq 0$.*

REMARK 2.2. Some results have been known concerning the nonvanishing at $s = 1/2$ of automorphic L-functions for GL(2) twisted by characters on GL(1) ([F-H], [Ko-Za], [W1], [W2]). The above theorem may be seen as a result on such L-functions twisted, in contrast, by automorphic forms on GL(2).

The rest of this section is devoted to the proof of Theorem 2.1.

We fix k and l as in the assumption of Theorem 2.1 and put

$$\lambda := (l-k)/2 \quad \text{and} \quad \mu := (l+k)/2. \tag{2.2}$$

For normalized Hecke eigenforms

$$f(z)=\sum_{n=1}^{\infty} a(n)\mathbf{e}(nz) \in S_k, \quad g(z)=\sum_{n=1}^{\infty} b(n)\mathbf{e}(nz) \in S_l,$$

[Sh, Lemma 1] gives

$$L(s,f\times g)=\zeta(2s)\sum_{n=1}^{\infty} a(n)b(n)n^{1-\mu-s}$$

for $\mathrm{Re}(s) > 1$. For $\nu \in 2\mathbb{Z}_{\geq 0}$, the Eisenstein series

$$E_\nu(z,s) := \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} (mz+n)^{-\nu}|mz+n|^{-2s}$$

has a meromorphic continuation to the whole s-plane. By [Sh], we have

$$\begin{aligned} 2(4\pi)^{-s-\mu+1}\Gamma(s+\mu-1)L(s,f\times g) \\ = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash H} f(z)\overline{g(z)}E_{2\lambda}(z,s-\lambda)y^{s+\mu-2}\,dx\,dy, \end{aligned} \tag{2.3}$$

where $z = x+iy$. This gives analytic continuation of $L(s, f\times g)$ to the whole s-plane. We study (2.3) at $s = 1/2$.

LEMMA 2.3. *Put*

$$C_\lambda(z) := \tfrac{1}{2}y^{1/2-\lambda}E_{2\lambda}(z,1/2-\lambda) \tag{2.4}$$

for $\lambda \in \mathbb{Z}_{\geq 0}$ *and* $z \in H$. *Let*

$$C_\lambda(z) = \sum_{n\in\mathbb{Z}} c_\lambda(n,y)\mathbf{e}(nx) \tag{2.5}$$

be the Fourier expansion, *where* $z = x+iy$. *Then*

$$c_\lambda(n,y) = \begin{cases} y^{1/2-\lambda}\Big\{\Big(\gamma - \log 4\pi + 2\sum_{r=1}^{\lambda}\dfrac{1}{2r-1}\Big) + \log y\Big\} & \text{if } n = 0, \\ (-1)^\lambda \sqrt{\pi}\, y^{-\lambda}\dfrac{\sigma_0(|n|)}{\sqrt{|n|}} \cdot \dfrac{W_{\mathrm{sgn}(n)\lambda,0}(4\pi|n|y)}{\Gamma(\mathrm{sgn}(n)\lambda+1/2)} & \text{if } n \neq 0. \end{cases} \tag{2.6}$$

Here σ_0 *is defined as in* (1.2) *with* $s = 0$, $\mathrm{sgn}(n) := n/|n|$ *for* $n \neq 0$, γ *is the Euler constant*

$$\gamma = 0.57721\ldots, \tag{2.7}$$

and $W_{a,b}(y)$ *is Whittaker's function which is a solution of the differential equation*

$$\Big(4y^2\frac{d^2}{dy^2} + 1 - 4b^2 + 4ay - y^2\Big)W(y) = 0$$

(see, e.g., [E-M-O-T1, p. 264]).

Proof. By [Ma, p. 210],

$$E_{2\lambda}(z,1/2-\lambda) = \lim_{s\to1}\varphi_\lambda(y,s) + (-1)^\lambda \cdot 2\sqrt{\frac{\pi}{y}}\sum_{\substack{n\in\mathbb{Z}\\ n\neq0}}\frac{\sigma_0(|n|)}{\sqrt{|n|}}\cdot\frac{W_{\mathrm{sgn}(n)\lambda,0}(4\pi|n|y)}{\Gamma(\mathrm{sgn}(n)\lambda+1/2)}\mathbf{e}(nx)$$

with

$$\varphi_\lambda(y,s) := 2\zeta(s) + (-1)^\lambda 2^{3-s}\pi y^{1-s}\frac{\Gamma(s-1)\zeta(s-1)}{\Gamma(s/2+\lambda)\Gamma(s/2-\lambda)}.$$

The di-gamma function

$$\psi(s) := \Gamma'(s)/\Gamma(s) \tag{2.8}$$

assumes the values

$$\psi(m+1/2) = 2\sum_{r=1}^{m}\frac{1}{2r-1} - \log 4 - \gamma \tag{2.9}$$

for $m \in \mathbb{Z}_{\geq 0}$, hence by a direct computation we have

$$\lim_{s\to1}\varphi_\lambda(y,s) = 2\Big(\gamma - \log 4\pi + 2\sum_{r=1}^{\lambda}\frac{1}{2r-1}\Big) + 2\log y. \ \blacksquare$$

We have

$$(4\pi)^{1/2-\mu}\Gamma(\mu-1/2)L(1/2, f\times g) = (f\cdot C_\lambda, g) \tag{2.10}$$

from (2.3). Here

$$f(z)C_\lambda(z) = \sum_{\nu\in\mathbb{Z}}\Big\{\sum_{m+n=\nu} a(m)c(n,y)e^{-2\pi my}\Big\}\mathbf{e}(\nu x) \tag{2.11}$$

with $a(m) := 0$ for $m \le 0$.

Now we recall the holomorphic projection operator of [St] (specialized to our case): Let $\varphi : H \to \mathbb{C}$ be a C^∞-modular form (see the Notation above) of weight $w \in \mathbb{Z}$ for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$\varphi(z) = \sum_{\nu\in\mathbb{Z}} r(\nu, y)\mathbf{e}(\nu x).$$

Suppose that $w \ge 6$ and that φ satisfies the growth condition

$$\int_0^1 dx \int_0^\infty |\varphi(z)|y^{w-2}e^{-\varepsilon y}\,dy < \infty \tag{2.12}$$

for every $\varepsilon > 0$. Put

$$r(\nu) := \frac{(4\pi\nu)^{w-1}}{\Gamma(w-1)}\int_0^\infty r(\nu,y)e^{-2\pi\nu y}y^{w-2}\,dy \tag{2.13}$$

and

$$\pi_{\mathrm{hol}}(\varphi)(z) := \sum_{\nu=1}^\infty r(\nu)\mathbf{e}(\nu z).$$

Then $\pi_{\mathrm{hol}}(\varphi) \in S_w$ and $(\varphi, h) = (\pi_{\mathrm{hol}}(\varphi), h)$ for all $h \in S_w$. This $\pi_{\mathrm{hol}}(\varphi)$ is called the *holomorphic projection* of φ.

We apply π_{hol} to $\varphi = f\cdot C_\lambda$ in (2.10). The validity of the condition (2.12) in this case (with $w = l$) follows from the first assertion of

LEMMA 2.4. (1) *For any $\varepsilon > 0$ there exists a positive constant M such that*

$$|f(z)C_\lambda(z)| < My^{-l/2}(y^{1/2+\varepsilon} + y^{-1/2-\varepsilon}) \quad \textit{for all } z \in H.$$

(2) *We have*

$$\sum_{m+n=\nu}\int_0^\infty |a(m)c_\lambda(n,y)|e^{-2\pi(m+\nu)}y^{l-2}\,dy < \infty \quad \textit{for every } \nu\in\mathbb{Z}.$$

P r o o f. We denote by $c_1, c_2, \ldots$ some positive constants independent of n and y.

(1) Suppose $y \geq \delta$ for some $\delta > 0$. By (2.6) we have

$$\begin{aligned} |C_\lambda(z)| &\leq \sum_{n\in\mathbb{Z}} |c_\lambda(n,y)| \\ &\leq c_1 y^{1/2-\lambda} + c_2 y^{1/2-\lambda}|\log y| \\ &\quad + c_3 y^{-\lambda} \sum_{n\neq 0} |W_{\mathrm{sgn}(n)\lambda,0}(4\pi|n|y)|. \end{aligned} \tag{2.14}$$

By [E-M-O-T1, p. 264, (5)],

$$W_{\kappa,0}(y) = e^{-y/2} y^\kappa \, {}_2F_0(1/2-\kappa, 1/2-\kappa; -y^{-1})$$

for $\kappa \in \mathbb{C}$ with the hypergeometric function ${}_2F_0$, hence

$$|W_{\mathrm{sgn}(n)\lambda,0}(4\pi|n|y)| \leq c_4 e^{-2\pi|n|y}(|n|y)^{\mathrm{sgn}(n)\lambda} \tag{2.15}$$

for $n \neq 0$. It follows from (2.14) that $|C_\lambda(z)| \leq c_5 y^{1/2-\lambda}|\log y|$, hence

$$|f(z)C_\lambda(z)| \leq c_6 y^{1/2-\lambda-k/2}|\log y| \quad \text{for } y \geq \delta. \tag{2.16}$$

Since $y^{l/2}|f(z)C_\lambda(z)|$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant, the assertion (1) follows from (2.16) and [St, Proposition 2].

(2) First we show the finiteness of the integral

$$I(m,n) := \int_0^\infty |a(m)c_\lambda(n,y)| e^{-2\pi(m+\nu)y} y^{l-2}\, dy$$

for $\nu \in \mathbb{Z}_{>0}$, $0 \leq n \leq \nu - 1$, and $m = \nu - n$. If $n = 0$, we have

$$I(\nu, 0) \leq c_7 \int_0^\infty (y^{1/2-\lambda} + y^{1/2-\lambda}|\log y|) e^{-4\pi\nu y} y^{l-2}\, dy < \infty$$

since $l - \lambda = \mu \geq 12$. Next we use the estimate

$$|W_{\lambda,0}(x)| \leq c_8 x^{1/2}|\log x| \quad \text{as } x \to +0, \tag{2.17}$$

which follows from [E-M-O-T1, p. 264, (2), p. 262, (5),(10)]. If $1 \leq n \leq \nu - 1$, we have

$$I(n-\nu, n) \leq c_9 \int_0^\infty |W_{\lambda,0}(4\pi n y)| y^{l-\lambda-2}\, dy < \infty$$

by (2.15) and (2.17). Hence it remains to show that

$$\sum_{n<0} I(\nu - n, n) < \infty.$$

From $|a(m)| = O(m^{k/2})$ it follows that

$$\begin{aligned}\sum_{n<0} I(\nu-n,n) &\le c_{10}\sum_{n=1}^{\infty} n^{k/2}\int_0^{\infty} e^{-2\pi(2\nu+n)y}y^{l-2}|c_\lambda(-n,y)|\,dy\\ &\le c_{11}\sum_{n=1}^{\infty} n^{k/2}\Big(n\int_0^{1/n} e^{-2\pi ny}y^{l-1}dy + n^{-\lambda}\int_{1/n}^{\infty} e^{-4\pi ny}y^{l-\lambda-2}\,dy\Big)\end{aligned}$$

by (2.15) and (2.17). Hence

$$\sum_{n<0} I(\nu-n,n) \le c_{12}\zeta(l-k/2-1) < \infty. \ \blacksquare$$

PROPOSITION 2.5. *The notation being as above, let*

$$\pi_{\mathrm{hol}}(f\cdot C_\lambda)(z) = \sum_{\nu=1}^{\infty}\alpha(\nu)\mathbf{e}(\nu z) \in S_l.$$

Then

$$\begin{aligned}\alpha(\nu) = {} & (4\pi\nu)^{\lambda-1/2}\frac{\Gamma(\mu-1/2)}{\Gamma(l-1)}a(\nu)\\ &\times\Big\{2\sum_{r=1}^{\lambda}\frac{1}{2r-1} + 2\sum_{r=1}^{\mu-1}\frac{1}{2r-1} - \log(64\pi^2\nu)\Big\}\\ &+ (-1)^\lambda 2^{l-1}(2\pi)^{\lambda+1/2}\frac{\Gamma(\mu-1/2)^2}{\Gamma(l-1)}\\ &\times\sum_{\substack{m+n=\nu\\ m,n\in\mathbb{Z}\\ n\neq 0}}\frac{a(m)\sigma_0(|n|)}{\Gamma(\mathrm{sgn}(n)\lambda+1/2)\Gamma(\mu-\mathrm{sgn}(n)\lambda)}\cdot\frac{\nu^{l-1}}{(m+\nu+|n|)^{\mu-1/2}}\\ &\times F\Big(\mu-1/2, 1/2-\mathrm{sgn}(n)\lambda; \mu-\mathrm{sgn}(n)\lambda; \frac{m+\nu-|n|}{m+\nu+|n|}\Big).\end{aligned}$$

Here $F = {}_2F_1$ is the hypergeometric function. The above series converges absolutely for every $\nu\in\mathbb{Z}_{>0}$.

Proof. By Lemma 2.4(1), $\varphi = f\cdot C_\lambda$ satisfies the growth condition (2.12). So from (2.13) we have

$$\alpha(\nu) = \frac{(4\pi\nu)^{l-1}}{\Gamma(l-1)}\int_0^{\infty}\Big(\sum_{m+n=\nu} a(m)c(n,y)e^{-2\pi my}\Big)e^{-2\pi\nu y}y^{l-2}\,dy.$$

By Lemma 2.4(2),

$$\alpha(\nu) = \frac{(4\pi\nu)^{l-1}}{\Gamma(l-1)} \sum_{m+n=\nu} a(m) I_1(n) \tag{2.18}$$

with

$$I_1(n) := \int_0^\infty e^{-2\pi(m+\nu)y} y^{l-2} c(n,y)\, dy,$$

and the right-hand side of (2.18) is absolutely convergent; here we fix ν and put $m = \nu - n$. From (2.6) and (2.9),

$$I_1(0) = (4\pi\nu)^{-\mu+1/2}\Gamma(\mu - 1/2)\Big\{2\sum_{r=1}^{\lambda}\frac{1}{2r-1} + 2\sum_{r=1}^{\mu-1}\frac{1}{2r-1} - \log(64\pi^2\nu)\Big\}.$$

If $n \neq 0$, (2.6) gives

$$I_1(n) = (-1)^\lambda \sqrt{\pi} \cdot \frac{\sigma_0(|n|)}{\sqrt{|n|}} \cdot \Gamma(\mathrm{sgn}(n)\lambda + 1/2)^{-1} I_2(n)$$

with

$$I_2(n) := \int_0^\infty e^{-2\pi(m+\nu)y} y^{\mu-2} W_{\mathrm{sgn}(n)\lambda,0}(4\pi|n|y)\, dy.$$

By [E-M-O-T2, p. 216, (16)],

$$\begin{aligned} I_2(n) &= 2\sqrt{\pi|n|} \cdot \frac{\Gamma(\mu-1/2)^2}{\Gamma(\mu - \mathrm{sgn}(n)\lambda)} \cdot \{2\pi(m+\nu+|n|)\}^{-\mu+1/2} \\ &\quad \times F\Big(\mu - 1/2, 1/2 - \mathrm{sgn}(n)\lambda; \mu - \mathrm{sgn}(n)\lambda; \frac{m+\nu-|n|}{m+\nu+|n|}\Big). \end{aligned}$$

Thus from (2.18) the result follows. ∎

By Proposition 2.5 we have

$$(4\pi)^{1/2-\lambda}\frac{\Gamma(l-1)}{\Gamma(\mu-1/2)}\alpha(1) = A(\lambda,\mu) + R(\lambda,\mu) \tag{2.19}$$

with

$$A(\lambda,\mu) := 2\sum_{r=1}^{\lambda}\frac{1}{2r-1} + 2\sum_{r=1}^{\mu-1}\frac{1}{2r-1} - \log(64\pi^2), \tag{2.20}$$

$$\begin{aligned} R(\lambda,\mu) &:= \frac{(-1)^\lambda 2\pi\Gamma(\mu-1/2)}{\Gamma(1/2-\lambda)\Gamma(l)} \sum_{m=2}^\infty a(m)\sigma_0(m-1)m^{-\mu+1/2} \\ &\quad \times F(\mu-1/2, \lambda+1/2; l; 1/m). \end{aligned} \tag{2.21}$$

LEMMA 2.6. *For all λ and μ we have*

$$|R(\lambda,\mu)| \le R^*(\lambda,\mu)$$

with

$$R^*(\lambda,\mu) := \frac{4\Gamma(\mu-1/2)\Gamma(\lambda+1/2)}{\Gamma(l)}\cdot\left\{\zeta\left(\frac{k-1}{2}\right)^2-1\right\}.$$

Proof. Euler's integral representation [E-M-O-T1, p. 59, (10)] gives

$$F(\mu-1/2,\lambda+1/2;l;1/m) = \frac{\Gamma(l)}{\Gamma(\mu-1/2)\Gamma(\lambda+1/2)}\int_0^1 t^{\mu-3/2}(1-t)^{\lambda-1/2}(1-t/m)^{-\lambda-1/2}\,dt.$$

Hence

$$F(\mu-1/2,\lambda+1/2;l;1/m) \le \frac{\Gamma(l)}{\Gamma(\mu-1/2)\Gamma(\lambda+1/2)}(1-1/m)^{-\lambda-1/2}\int_0^1 t^{\mu-3/2}(1-t)^{\lambda-1/2}\,dt \le \left(1+\frac{1}{m-1}\right)^{\lambda+1/2}. \tag{2.22}$$

Similarly we have

$$F(\mu-1/2,\lambda+1/2;l;1/m) > 1. \tag{2.23}$$

Using Deligne's bound [De]

$$|a(m)| \le \sigma_0(m)m^{(k-1)/2},$$

from (2.22) and (2.23) we obtain

$$\Big|\sum_{m=2}^{\infty} a(m)\sigma_0(m-1)m^{-\mu+1/2}\cdot F(\mu-1/2,\lambda+1/2;l;1/m)\Big| \le \sum_{m=2}^{\infty}\sigma_0(m)\sigma_0(m-1)m^{-l/2}\left(1+\frac{1}{m-1}\right)^{\lambda+1/2}. \tag{2.24}$$

Since $\sigma_0(m-1)\le 2\sqrt{m-1}$, the sum (2.24) is majorized by

$$2\sum_{m=2}^{\infty}\sigma_0(m)m^{(1-k)/2} = 2\left\{\zeta\left(\frac{k-1}{2}\right)^2-1\right\}. \blacksquare$$

LEMMA 2.7. (1) *If* $2 \le m \in \mathbb{Z}$, *then*

$$2\sum_{r=1}^{m}\frac{1}{2r-1} = \gamma + \log 2 + \log(2m-1) + \frac{1}{2m-1} - \frac{1}{3(2m-1)^2} + \frac{\theta_m}{120(m-1/2)^4}$$

with $0 < \theta_m < 1$, *where* γ *is the Euler constant as in* (2.7).

(2) *If* $12 \le k \in \mathbb{Z}$, *then*

$$\zeta\left(\frac{k-1}{2}\right)^2 \le 1 + 2^{1-\frac{21}{44}(k-1)} + 2^{-\frac{21}{22}(k-1)}.$$

(3) *If* $1 < x \in \mathbb{R}$, *then*

$$\frac{\Gamma(x-1/2)}{\Gamma(x)} \le \frac{1}{\sqrt{x-1/2}}\cdot\exp\left(\frac{1}{4x-2} + \frac{1}{(4x-2)^2}\right).$$

Proof. (1) is immediate from the Euler–MacLaurin formula (see, e.g., [Rad]).

(2) Suppose $1 < \sigma \in \mathbb{R}$ and $2 \le N \in \mathbb{Z}$. The Euler–MacLaurin formula gives

$$\zeta(\sigma) = \sum_{n=1}^{N} n^{-\sigma} + \frac{N^{1-\sigma}}{\sigma-1} - \frac{N^{-\sigma}}{2} + \frac{\sigma}{12}N^{-\sigma-1}\theta \tag{2.25}$$

with $0 < \theta < 1$. If we put $N = 2$ and use the inequality

$$\frac{1}{2} + \frac{2}{\sigma-1} + \frac{\sigma}{24} \le 2^{\sigma/22} \quad \text{for } \sigma \ge \frac{11}{2},$$

the result follows.

(3) The di-gamma function $\psi(s)$ defined by (2.8) is increasing for $0 < s \in \mathbb{R}$. Hence from the mean value theorem it follows that

$$\frac{\Gamma(x-1/2)}{\Gamma(x)} \le \exp\left(-\frac{1}{2}\psi\left(x-\frac{1}{2}\right)\right) \quad \text{for } 1 < x \in \mathbb{R}. \tag{2.26}$$

By [Rad, p. 37],

$$\log\Gamma(s) = \frac{1}{2}\log(2\pi) + \left(s-\frac{1}{2}\right)\log s - s - \frac{1}{2}\int_0^\infty \frac{B_2(x-[x]) - B_2}{(x+s)^2}\,dx \quad \text{for } 0 < s \in \mathbb{R}$$

with $B_2(x)$ being the second Bernoulli polynomial. Differentiating, we have

$$\psi(s) = \log s - \frac{1}{2s} + \int_0^\infty \frac{B_2(x-[x]) - B_2}{(x+s)^3}\,dx.$$

Here

$$\left|\int_0^\infty \frac{B_2(x-[x])-B_2}{(x+s)^3}\,dx\right| \le \max_{0\le x\le 1}|B_2(x)-B_2|\cdot\int_0^\infty \frac{dx}{(x+s)^3} = \frac{1}{8s^2}$$

since $B_2(x)-B_2 = x^2 - x$. Hence (3) follows from (2.26). ■

LEMMA 2.8. *In the notation of* (2.20) *and Lemma* 2.6, *we have*

$$|A(\lambda,\mu)| > R^*(\lambda,\mu)$$

for all λ,μ *defined by* (2.2).

Proof. (i) *Estimate for* $A(\lambda,\mu)$. First note that

$$A(\lambda,\mu) \le A(\lambda',\mu') \quad \text{if} \quad \lambda\le\lambda' \text{ and } \mu\le\mu'\,.$$

If $\lambda \ge 1$, then $\mu \ge 14$ and

$$A(\lambda,\mu) \ge A(1,14) = 0.0803\ldots$$

by Lemma 2.7(1). If $\lambda = 0$, the same lemma gives

$$A(0,88) = -0.018\ldots \quad \text{and} \quad A(0,90) = 0.0038\ldots$$

Hence we have

$$|A(\lambda,\mu)| > 8\times 10^{-2} \quad \text{for all } \mu \text{ if } \lambda \ge 1, \tag{2.27}$$

and

$$|A(0,\mu)| > 3.8\times 10^{-3} \quad \text{for all } \mu. \tag{2.28}$$

We also have

$$|A(0,\mu)| > 1.2 \quad \text{for } \mu \le 26 \tag{2.29}$$

since $A(0,26) = -1.26\ldots$

(ii) *Estimate for* $R^*(\lambda,\mu)$. If $\lambda\ge 1$, we have

$$\frac{R^*(\lambda,\mu)}{R^*(\lambda-1,\mu-1)} \le \frac14\cdot\frac{(l-2)^2-(k-1)^2}{(l-2)^2+(l-2)} < \frac14,$$

hence

$$R^*(\lambda,\mu) \le 2^{-2\lambda}R^*(0,k).$$

Observe that $R^*(0,k)$ is a decreasing function of k. Hence, if $\lambda\ge 1$, we have

$$R^*(\lambda,\mu) \le \max\{2^{-2}R^*(0,16), 2^{-4}R^*(0,12)\} < 7.2\times 10^{-3} \tag{2.30}$$

by Lemma 2.7(2) and (3). If $\lambda = 0$ and $\mu \ge 28$, then

$$R^*(0,\mu) \le R^*(0,28) < 3.7\times 10^{-4}. \tag{2.31}$$

If $\lambda = 0$ and $\mu \le 26$, then

$$R^*(0,\mu) \le R^*(0,12) < 0.12. \tag{2.32}$$

Comparing (2.27) with (2.30), (2.28) with (2.31), and (2.29) with (2.32), we have the assertion of Lemma 2.8. ∎

Combining Lemma 2.8 with Lemma 2.6 and (2.19), we see that $\alpha(1) \neq 0$ in the notation of Proposition 2.5. Hence $\pi_{\mathrm{hol}}(f \cdot C_\lambda) \neq 0$. Thus for some $g \in S_l$ we have

$$(f \cdot C_\lambda, g) = (\pi_{\mathrm{hol}}(f \cdot C_\lambda), g) \neq 0$$

in (2.10). This completes the proof of Theorem 2.1.

3. Dedekind zeta functions. Let K be an algebraic number field of finite degree. The functional equation of the Dedekind zeta function $\zeta_K(s)$ tells us that the vanishing order $\mathrm{ord}_{s=1/2}\,\zeta_K(s)$ is a nonnegative even integer. On this we have

THEOREM 3.1 (Kurokawa). *For every $\nu \in \mathbb{Z}_{>0}$ there exists a Galois extension K_ν over $\mathbb{Q}$ of degree $2^{3\nu}$ such that*

$$\mathrm{ord}_{s=1/2}\,\zeta_{K_\nu}(s) \ge 2\nu.$$

In particular,

$$\sup_{K/\mathbb{Q}\ \mathit{finite}} \mathrm{ord}_{s=1/2}\,\zeta_K(s) = \infty.$$

Proof. Put

$$g_K := \tfrac{1}{2}\,\mathrm{ord}_{s=1/2}\,\zeta_K(s) \in \mathbb{Z}_{\ge 0}\,. \tag{3.1}$$

Let $F/\mathbb{Q}$ be a finite extension and let L_i/F $(i = 1,2)$ be finite Galois extensions such that $L_1 \cap L_2 = F$. By [Br], the function

$$\frac{\zeta_{L_1L_2}(s)\zeta_F(s)}{\zeta_{L_1}(s)\zeta_{L_2}(s)}$$

is entire, hence

$$g_{L_1L_2} \ge g_{L_1} + g_{L_2} - g_F. \tag{3.2}$$

By [Frö], there exists a sequence $\{N_i\}_{i=1}^{\infty}$ of Galois extensions over $\mathbb{Q}$ of degree 8 such that

$$N_{i+1} \cap N_1 \ldots N_i = \mathbb{Q} \quad \text{and} \quad \zeta_{N_i}(1/2) = 0 \quad \text{for } i \ge 1.$$

Then $K_\nu := N_1 \ldots N_\nu$ satisfies $g_{K_\nu} \ge \nu$ by (3.2). ∎

REMARK 3.2. (1) We refer to [Den2] for an interpretation of the zeros of the Dedekind zeta functions at $s = 1/2$.

(2) Let g_K be as in (3.1). Theorem 3.1 leads naturally to the following problem: *Classify the algebraic number fields by g_K.* For example, one can ask whether $g_K = 0$ for every finite abelian extension $K/\mathbb{Q}$. This is equivalent to asking whether $L(1/2, \chi) \neq 0$ for every Dirichlet character χ, which is a longstanding open problem in analytic number theory.

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Received on 2.12.1997
and in revised form on 10.9.1998 (3308)