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On a generalization of the Selberg trace formula

by

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1. Introduction. The Selberg trace formula (the original paper is [Se]; for a nice account see [I]) is obtained (e.g. for a co-compact Fuchsian group Γ with fundamental domain F in H, where H is the upper half-plane) by computing in two different ways (geometrically and spectrally) the integral

$$\operatorname{Tr} K = \int_F K(z,z)\,d\mu_z,$$

where $K(z,w)$ is an automorphic kernel function. We take here instead of $\operatorname{Tr} K$ an integral of the form

$$\operatorname{Tr}_u K = \int_F K(z,z)u(z)\,d\mu_z,$$

where u is an automorphic eigenfunction of the Laplace operator, so we write $u(z)$ in place of the identically 1 function.

On the geometric side of our formula we get integrals of u on certain closed geodesics of the Riemann surface $\Gamma\backslash H$. On the spectral side integrals of the form

$$\int_F |u_j(z)|^2 u(z)\,d\mu_z$$

appear (the u_j run over an orthonormal basis of automorphic Laplace-eigenforms), so our formula (Theorem 1) is a duality between such integrals and certain geodesic integrals of u. New integral transformations are involved depending on the Laplace-eigenvalue of u. We invert these integral transformations in Section 5, Theorem 2.

We develop the formula for finite volume Fuchsian groups, so (as in the case of the Selberg trace formula) $\int_F K(z,z)u(z)\,d\mu_z$ will not be convergent, and we take instead

$$\operatorname{Tr}_u^Y K = \int_{F(Y)} K(z,z)u(z)\,d\mu_z,$$

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where $F(Y)$ is obtained from F by cutting off the cuspidal zones at height Y. We let $Y \to \infty$, and the main term (which is in our case a power of Y, while in the case of the Selberg trace formula the main term is $\log Y$) will cancel out. An interesting feature of our formula is the appearance of the Riemann zeta-function in the contribution of the parabolic conjugacy classes.

In Section 6 we prove lemmas on special functions needed in Section 5.

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2. Notations and statement of the main result. Let H be the open upper half-plane. The elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the group $\mathrm{PSL}(2,\mathbb{R})$ act on H by the rule $z \to (az+b)/(cz+d)$. The hyperbolic Laplace operator is given by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It is well known that Δ commutes with the action of $\mathrm{PSL}(2,\mathbb{R})$.

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a finite volume Fuchsian group, i.e. Γ acts discontinuously on H, and it has a fundamental domain of finite volume.

The constants in the symbols O will depend on the group Γ. For a function f we denote its jth derivative by $f^{(j)}$.

We fix a complete set A of inequivalent cusps of Γ, and we will denote the elements of A by a, b or c, so e.g. $\sum_a \sum_c$ or $\bigcup_a$ will mean that a and c run over A. We say that σ_a is a *scaling matrix* of a cusp a if $\sigma_a \infty = a$, $\sigma_a^{-1}\Gamma_a\sigma_a = B$, where Γ_a is the stability group of a in Γ, and B is the group of integer translations. The scaling matrix is determined up to composition with a translation from the right.

We also fix a complete set P of representatives of Γ-equivalence classes of the set $\{z \in H : \gamma z = z \text{ for some id} \neq \gamma \in \Gamma\}$. For a $p \in P$ let m_p be the order of the stability group of p in Γ.

Let

$$P(Y) = \{z = x + iy : 0 < x \leq 1,\ y > Y\},$$

and let Y_Γ be a constant (depending only on the group Γ) such that for any fixed $Y \geq Y_\Gamma$ the cuspidal zones $F_a(Y) = \sigma_a P(Y)$ are disjoint, and the fixed fundamental domain F of Γ (it contains exactly one point of each Γ-equivalence class of H) is partitioned into

$$F = F(Y) \cup \bigcup_a F_a(Y),$$

where $F(Y)$ is the central part,

$$F(Y) = F \setminus \bigcup_a F_a(Y),$$

and $F(Y)$ has compact closure.

Denote by $\{u_j(z) : j \geq 0\}$ a complete orthonormal system of Maass forms for Γ for the discrete spectrum ($u_0(z)$ is constant), with Laplace-eigenvalue $\lambda_j = s_j(s_j - 1)$, $\operatorname{Re} s_j \geq 1/2$, $s_j = 1/2 + it_j$ and Fourier expansion

$$u_j(\sigma_a z) = \beta_{a,j}(0)y^{1-s_j} + \sum_{n\neq 0} \beta_{a,j}(n)W_{s_j}(nz),$$

where W is the Whittaker function.

The Fourier expansion of the Eisenstein series (as in [I], (8.2)) is given by

$$E_c(\sigma_a z, 1/2 + ir) = \delta_{ac}y^{1/2+ir} + \varphi_{a,c}(1/2 + ir)y^{1/2-ir} + \sum_{n\neq 0} \varphi_{a,c}(n, 1/2 + ir)W_{1/2+ir}(nz).$$

Let $\{s_l : l \in L\}$ be the set of the poles of the Eisenstein series for Γ. Then $1/2 < s_l \leq 1$ for every $l \in L$, and L is a finite set. We have $\beta_{a,j}(0) = 0$ if $j > 0$, and $u_j(z)$ is not a linear combination of residues of Eisenstein series, so if $j > 0$ is such that $\beta_{a,j}(0) \neq 0$ for some a, then $s_j = s_l$ for some $l \in L$. The functions $\varphi_{a,a}(s)$ may have poles only at the points s_l. Let us denote the residue of $\varphi_{a,a}(s)$ at $s = s_l$ by R_{a,s_l}, when $l \in L$.

Let $1/2 \leq \operatorname{Re} s < 1$, and let $u(z)$ be a fixed Γ-automorphic eigenfunction of the Laplace operator with eigenvalue $\lambda = s(s-1)$, and Fourier expansion at any cusp a of Γ

$$u(\sigma_a z) = \beta_a(0)y^s + \widetilde{\beta}_a(0)y^{1-s} + \sum_{n\neq 0} \beta_a(n)W_s(nz).$$

For simplicity we assume that $s \neq 2s_l - 1$ for $l \in L$.

We introduce the notations

$$B_u = \sum_a \beta_a(0), \quad \widetilde{B}_u = \sum_a \widetilde{\beta}_a(0),$$

$$B_u(S) = \sum_a \beta_a(0)\varphi_{a,a}\left(\frac{1+S}{2}\right), \quad \widetilde{B}_u(S) = \sum_a \widetilde{\beta}_a(0)\varphi_{a,a}\left(\frac{1+S}{2}\right).$$

Let k be a function on $[0, \infty)$, and assume that it satisfies

(A) $\quad k$ is a compactly supported continuous function on $[0, \infty)$.

As usual (see [I], (1.62)), let

$$g(a) = 2q\left(\frac{e^a + e^{-a} - 2}{4}\right), \quad \text{where} \quad q(\nu) = \int_0^\infty \frac{k(\nu + \tau)}{\sqrt{\tau}}\, d\tau,$$

and let h be the Fourier transform of g,

$$h(r) = \int_{-\infty}^\infty g(a)e^{ira}\, da.$$

We assume that

(B) $h(r)$ is even, it is holomorphic in the strip $|\operatorname{Im} r| \le 1/2+\varepsilon$, and $h(r) = O((1+|r|)^{-2-\varepsilon})$ in this strip for some $\varepsilon > 0$.

The point-pair invariant determined by k is

$$k(z,w) = k\left(\frac{|z-w|^2}{4\operatorname{Im} z \operatorname{Im} w}\right)$$

for $z, w \in H$. The automorphic kernel function $K(z,w)$ is given by

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w).$$

Define

$$\operatorname{Tr}_u^Y K = \int_{F(Y)} K(z,z)u(z)\,d\mu_z.$$

We will determine the asymptotic behaviour of $\operatorname{Tr}_u^Y K$ as $Y \to \infty$ in two different ways. Firstly, by partitioning Γ into conjugacy classes, and secondly, using the spectral theorem (which is applicable by our conditions on k and h), since introducing the notations

$$I_u^Y(r) = \sum_c \int_{F(Y)} |E_c(z,1/2+ir)|^2 u(z)\,d\mu_z, \quad I_u^Y(u_j) = \int_{F(Y)} |u_j(z)|^2 u(z)\,d\mu_z,$$

we have by the spectral theorem

$$\operatorname{Tr}_u^Y K = \sum_j h(t_j) I_u^Y(u_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r) I_u^Y(r)\,dr.$$

We give here the statement of our Lemma 4 (which will be proved in Section 4 below), because to state Theorem 1 we need the quantities $I_u(u_j)$ and $I_u(r)$ defined in that lemma.

Lemma 4. *Define*

$$\psi_a^Y(r,s) = \frac{Y^s}{s} + \varphi_{a,a}(1/2+ir)\frac{Y^{s-2ir}}{s-2ir},$$
$$\widetilde{I}_u^Y(r) = I_u^Y(r) - \Big(\sum_a \beta_a(0)(\psi_a^Y(r,s)+\psi_a^Y(-r,s))$$
$$+ \widetilde{\beta}_a(0)(\psi_a^Y(r,1-s)+\psi_a^Y(-r,1-s))\Big),$$

and

$$\widetilde{I}_u^Y(u_j) = I_u^Y(u_j) - \sum_a |\beta_{a,j}(0)|^2\left(\beta_a(0)\frac{Y^{1+s-2s_j}}{1+s-2s_j} + \widetilde{\beta}_a(0)\frac{Y^{2-s-2s_j}}{2-s-2s_j}\right).$$

Then

$$\sum_{|t_j|\le R} |\widetilde{I}_u^Y(u_j)| + \int_{-R}^{R} |\widetilde{I}_u^Y(r)|\,dr = O(R^2),$$

uniformly in Y. *The limits*

$$I_u(r) = \lim_{Y\to\infty} \widetilde{I}_u^Y(r), \quad I_u(u_j) = \lim_{Y\to\infty} \widetilde{I}_u^Y(u_j)$$

obviously exist, *and then*, *of course*,

$$\sum_{|t_j|\le R} |I_u(u_j)| + \int_{-R}^{R} |I_u(r)|\,dr = O(R^2).$$

If $u_j(z)$ *is not a linear combination of residues of Eisenstein series*, *then*

$$I_u(u_j) = \int_F |u_j(z)|^2 u(z)\,d\mu_z.$$

In particular $I_u(u_0) = 0$.

With the above notations and assumptions our formula is the following:

THEOREM 1. *Assume that* k *satisfies condition* (A) *and* h *satisfies condition* (B). *Let*

$$\Sigma_{\mathrm{hyp}} = \sum_{\substack{[\gamma] \\ \gamma\ hyperbolic}} \Big(\int_{C_\gamma} u\,dS\Big) \int_{-\pi/2}^{\pi/2} k\bigg(\frac{N(\gamma)+N(\gamma)^{-1}-2}{4\cos^2\vartheta}\bigg) f_\lambda(\vartheta)\,\frac{d\vartheta}{\cos^2\vartheta},$$

where the summation is over the hyperbolic conjugacy classes of Γ, $N(\gamma)$ *is the norm of* (*the conjugacy class of*) γ, C_γ *is the closed geodesic obtained by factorizing the noneuclidean line connecting the fixed points of* γ *by the action of the centralizer of* γ *in* Γ, $dS = |dz|/y$ *is the hyperbolic arc length*, *and* $f_\lambda(\vartheta)$ *is the solution of the differential equation*

$$f^{(2)}(\vartheta) = \frac{\lambda}{\cos^2\vartheta} f(\vartheta), \quad \vartheta \in (-\pi/2, \pi/2),$$

with $f_\lambda(0) = 1$, $f_\lambda^{(1)}(0) = 0$. *Let*

$$\Sigma_{\mathrm{ell}} = \sum_{p\in P} \frac{2\pi}{m_p} u(p) \sum_{l=1}^{m_p-1} \int_0^\infty k\bigg(\sin^2\frac{l\pi}{m_p}\sinh^2 r\bigg) g_\lambda(r)\sinh r\,dr,$$

where $g_\lambda(r)$ $(r\in[0,\infty))$ *is the unique solution of*

$$g^{(2)}(r) + \frac{\cosh r}{\sinh r} g^{(1)}(r) = \lambda g(r)$$

with $g_\lambda(0)=1$. *Let*

$$\Sigma_{\text{par}} = B_u 2^{1-s}\zeta(1-s)\int_0^\infty k(\nu)\nu^{-(1+s)/2}\,d\nu + \widetilde{B}_u 2^s\zeta(s)\int_0^\infty k(\nu)\nu^{(s-2)/2}\,d\nu,$$

where ζ *is the Riemann zeta-function. Then the equality*

$$\Sigma_{\text{hyp}}+\Sigma_{\text{ell}}+\Sigma_{\text{par}} = \frac{1}{2}h\left(i\frac{s}{2}\right)B_u(s)+\frac{1}{2}h\left(i\frac{1-s}{2}\right)\widetilde{B}_u(1-s) + \sum_{j>0}h(t_j)I_u(u_j)+\frac{1}{4\pi}\int_{-\infty}^{\infty}h(r)I_u(r)\,dr$$

holds, *where* $I_u(u_j)$ *and* $I_u(r)$ *is given in Lemma* 4.

3. The geometric trace. For the first computation of $\mathrm{Tr}_u^Y K$ we partition Γ into conjugacy classes $[\gamma]=\{\tau^{-1}\gamma\tau : \tau\in\Gamma\}$. Let $\mathrm{id}\neq\gamma\in\Gamma$, and

$$T_\gamma^Y = \sum_{\delta\in[\gamma]}\int_{F(Y)} k(z,\delta z)u(z)\,d\mu_z.$$

We have $\tau_1^{-1}\gamma\tau_1=\tau_2^{-1}\gamma\tau_2$ if and only if $\tau_2\tau_1^{-1}\in C(\gamma)$, where $C(\gamma)$ is the centralizer of γ in Γ. So

$$T_\gamma^Y = \sum_{\tau\in C(\gamma)\backslash\Gamma}\int_{F(Y)} k(z,\tau^{-1}\gamma\tau z)u(z)\,d\mu_z.$$

Since $k(z,\tau^{-1}\gamma\tau z)=k(\tau z,\gamma\tau z)$ and $u(z)=u(\tau z)$, we obtain

$$T_\gamma^Y = \int_{C(\gamma)\backslash H(Y)} k(z,\gamma z)u(z)\,d\mu_z,$$

where $H(Y)=\bigcup_{\gamma\in\Gamma}\gamma F(Y)$. Let $h\in\mathrm{SL}(2,\mathbb{R})$. Then

$$T_\gamma^Y = \int_{h^{-1}(C(\gamma)\backslash H(Y))} k(hz,\gamma hz)u(hz)\,d\mu_z = \int_{(h^{-1}C(\gamma)h)\backslash(h^{-1}H(Y))} k(z,h^{-1}\gamma hz)u(hz)\,d\mu_z. \tag{1}$$

So far this is valid for every $\mathrm{id}\neq\gamma\in\Gamma$. We now examine separately the case of hyperbolic, elliptic or parabolic transformations.

If γ is hyperbolic or elliptic, then $T_\gamma=\lim_{Y\to\infty}T_\gamma^Y$ exists, and by (1) we get

$$T_\gamma = \int_{(h^{-1}C(\gamma)h)\backslash H} k(z,h^{-1}\gamma hz)u(hz)\,d\mu_z. \tag{2}$$

If γ is hyperbolic, then we choose $h\in\mathrm{SL}(2,\mathbb{R})$ so that $h^{-1}\gamma h$ is a dilation, i.e. $h^{-1}\gamma hz=N(\gamma)z$ for $z\in H$, where $N(\gamma)>1$ ($N(\gamma)$ is the "norm" of γ). Then, if the fixed points of γ are z_1 and z_2, then $C(\gamma) = \{\sigma \in \Gamma : \sigma z_1 = z_1,\ \sigma z_2 = z_2\}$. This is an infinite cyclic group. Let γ_0 be the generator of $C(\gamma)$ with the property that $\gamma = \gamma_0^l$ with a positive integer l. Then $h^{-1}C(\gamma)h$ is the group generated by the dilation $z \to N(\gamma_0)z$, and a fundamental domain of this group in H is $\{z \in H : 1 \le |z| < N(\gamma_0)\}$, so by the substitution $z = re^{i(\pi/2+\vartheta)}$ ($r \in (1, N(\gamma_0))$, $\vartheta \in (-\pi/2, \pi/2)$) we deduce (since $d\mu_z = \frac{dxdy}{y^2} = \frac{rdrd\vartheta}{r^2\cos^2\vartheta}$) by (2) that

$$T_\gamma = \int_{-\pi/2}^{\pi/2} \int_1^{N(\gamma_0)} k\left(\frac{N(\gamma) + N(\gamma)^{-1} - 2}{4\cos^2\vartheta}\right) u(h(re^{i(\pi/2+\vartheta)}))\,\frac{dr\,d\vartheta}{r\cos^2\vartheta}.$$

Introduce the notation

$$F(z) = \int_1^{N(\gamma_0)} u(h(rz))\,\frac{dr}{r} \quad (z \in H).$$

Then

$$T_\gamma = \int_{-\pi/2}^{\pi/2} k\left(\frac{N(\gamma) + N(\gamma)^{-1} - 2}{4\cos^2\vartheta}\right) F(e^{i(\pi/2+\vartheta)})\,\frac{d\vartheta}{\cos^2\vartheta}.$$

Now, F is constant on euclidean lines through the origin, i.e. $F(z) = F(rz)$ for all $r > 0$, because $u(h(z))$ is automorphic with respect to $h^{-1}\Gamma h$. In particular,

$$u(h(N(\gamma_0)z)) = u(h(z)) \quad \text{for } z \in H.$$

So F depends only on ϑ (if $z = re^{i(\pi/2+\vartheta)}$), i.e. $F(z) = F(\vartheta)$, where F is a function on $(-\pi/2, \pi/2)$.

On the other hand, since u is an eigenfunction of Δ with eigenvalue λ, so is $F(z)$ (because Δ commutes with the group action). Using the form of the Laplace operator in polar coordinates ($\Delta = (r\cos\vartheta)^2(\partial^2/\partial r^2 + r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial\vartheta^2)$), we find that $F(\vartheta)$ satisfies a second order ordinary differential equation, which depends only on λ:

$$F^{(2)}(\vartheta) = \frac{\lambda}{\cos^2\vartheta} F(\vartheta) \quad (\vartheta \in (-\pi/2, \pi/2)).$$

Let $f_\lambda(\vartheta)$ be the solution of this differential equation with $f_\lambda(0) = 1$, $f_\lambda^{(1)}(0) = 0$, and $\widetilde{f}_\lambda(\vartheta)$ the one with $\widetilde{f}_\lambda(0) = 0$, $\widetilde{f}_\lambda^{(1)}(0) = 1$. Then $F(\vartheta) = F(0)f_\lambda(\vartheta) + F^{(1)}(0)\widetilde{f}_\lambda(\vartheta)$, and $\widetilde{f}_\lambda(\vartheta)$ is an odd function, so it gives 0 in T_γ, i.e.

$$T_\gamma = F(0) \int_{-\pi/2}^{\pi/2} k\left(\frac{N(\gamma) + N(\gamma)^{-1} - 2}{4\cos^2\vartheta}\right) f_\lambda(\vartheta)\,\frac{d\vartheta}{\cos^2\vartheta}.$$

Here $F(0) = \int_1^{N(\gamma_0)} u(h(ri))\frac{dr}{r} = \int_{C_\gamma} u\,dS$, where $dS = |dz|/y$ is the hyperbolic arc length, and C_γ is the closed geodesic $C_\gamma = C(\gamma)\setminus l_\gamma$, where l_γ is the noneuclidean line connecting the fixed points (z_1 and z_2) of γ, and we factorize it by the action of $C(\gamma)$ (so we can take for C_γ any segment of length $\log N(\gamma_0)$ on l_γ). Hence

$$T_\gamma = \Big(\int_{C_\gamma} u\,dS\Big)\int_{-\pi/2}^{\pi/2} k\Big(\frac{N(\gamma)+N(\gamma)^{-1}-2}{4\cos^2\vartheta}\Big) f_\lambda(\vartheta)\,\frac{d\vartheta}{\cos^2\vartheta}, \tag{3}$$

when γ is hyperbolic.

If γ is elliptic, then by conjugation in Γ we may assume that its fixed point is a $p\in P$. Then $C(\gamma)=\Gamma_p=\{\sigma\in\Gamma:\sigma p=p\}$; this is a finite set, $|\Gamma_p|=m_p$. We choose $h\in \mathrm{SL}(2,\mathbb{R})$ such that $h(i)=p$, then $h^{-1}\gamma h = R(l\vartheta_p)$ for some integer $0<l<m_p$, where $\vartheta_p=\pi/m_p$, and

$$R(\varphi)=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}.$$

Then, by (2),

$$T_\gamma = \frac{1}{m_p}\int_H k(z, R(l\vartheta_p)z)u(hz)\,d\mu_z.$$

We use the substitution $z=R(\varphi)e^{-r}i$, i.e. we use geodesic polar coordinates (see [I], Sections 1.3 and 10.6), where $r\in(0,\infty)$, $\varphi\in(0,\pi)$, getting

$$T_\gamma = \frac{1}{m_p}\int_0^\infty k(\sin^2 l\vartheta_p \sinh^2 r)\Big(\int_0^\pi u(h(R(\varphi)e^{-r}i))\,d\varphi\Big)(2\sinh r)\,dr,$$

because $R(\varphi)$ commutes with $R(l\vartheta_p)$, and with $z=e^{-r}i$ we have

$$\frac{|z-R(l\vartheta_p)z|^2}{4\,\mathrm{Im}\,z\,\mathrm{Im}\,R(l\vartheta_p)z} = \sin^2 l\vartheta_p\sinh^2 r,$$

and furthermore $d\mu_z=(2\sinh r)\,dr\,d\varphi$. Define

$$G(z)=\frac{1}{\pi}\int_0^\pi u(h(R(\varphi)z))\,d\varphi.$$

One obtains $G(z)$ by averaging the function $u(h(z))$ over the stability group of i in $\mathrm{SL}(2,\mathbb{R})$ (or what amounts to the same, by averaging over noneuclidean circles around i), so $G(z)$ is radial at i, i.e. it depends only on the noneuclidean distance of z and i (see [I], Lemma 1.10). On the other hand, since u is an eigenfunction of Δ with eigenvalue λ, so is $G(z)$ (because Δ commutes with the group action). A radial (at i) eigenfunction of Δ of eigenvalue λ is determined up to a constant factor ([I], Lemma 1.12), so using the form of the Laplace operator in geodesic polar coordinates ($\Delta=\partial^2/\partial r^2+(\cosh r/\sinh r)\partial/\partial r+(2\sinh r)^{-2}\partial^2/\partial\varphi^2$, see [I], (1.20)), we find that if $g_\lambda(r)$ ($r \in [0,\infty)$) is the unique solution of

$$g^{(2)}(r) + \frac{\cosh r}{\sinh r} g^{(1)}(r) = \lambda g(r)$$

with $g_\lambda(0) = 1$, then $G(z) = u(p)g_\lambda(r)$ for $z = R(\varphi)e^{-r}i$, since $h(i) = p$. This shows

$$T_\gamma = \frac{2\pi}{m_p} u(p) \int_0^\infty k(\sin^2 l\vartheta_p \sinh^2 r) g_\lambda(r) \sinh r \, dr, \tag{4}$$

when γ is elliptic.

If γ is parabolic, then by conjugation in Γ we may assume that its fixed point is an $a \in A$. Then $C(\gamma) = \Gamma_a = \{\sigma \in \Gamma : \sigma a = a\}$. Let γ_a be a generator of Γ_a. Then $\gamma = \gamma_a^l$ for some $l \neq 0$. In this case we choose $h = \sigma_a$, the scaling matrix, i.e. $\sigma_a \infty = a$, $\sigma_a^{-1}\gamma_a\sigma_a = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then by (1) we have

$$T_\gamma^Y = \int_{B\backslash(\sigma_a^{-1}H(Y))} k(z, z+l) u(\sigma_a z) \, d\mu_z, \tag{5}$$

where B is the set of integer translations.

Lemma 1. *There is a constant c_Γ such that*

$$\{z \in H : c_\Gamma/Y \le \operatorname{Im} z \le Y\} \subseteq \sigma_a^{-1}H(Y) \subseteq \{z \in H : \operatorname{Im} z \le Y\}.$$

Proof. If $z \in (\sigma_a^{-1}H(Y)) \cap P(Y)$, then $\gamma\sigma_a z \in F(Y) \subseteq F$ for some $\gamma \in \Gamma$ and $\sigma_a z \in F_a(Y) \subseteq F$, so $\gamma\sigma_a z = \sigma_a z \in F(Y) \cap F_a(Y) = \emptyset$, a contradiction. This proves one half of the lemma, because $\sigma_a^{-1}H(Y)$ is invariant under $B = \sigma_a^{-1}\Gamma_a\sigma_a$.

If $z \notin \sigma_a^{-1}H(Y)$, then $\operatorname{Im} \sigma_b^{-1}\gamma\sigma_a z > Y$ for some $\gamma \in \Gamma$ and $b \in A$. Let $\sigma_b^{-1}\gamma\sigma_a = \left(\begin{smallmatrix} \cdot & \cdot \\ C & D \end{smallmatrix}\right)$. Then either $C = 0$, and in this case $\sigma_b^{-1}\gamma\sigma_a\infty = \infty$, so $a = b$, $\sigma_a^{-1}\gamma\sigma_a \in \sigma_a^{-1}\Gamma_a\sigma_a = B$, and $\operatorname{Im} z > Y$, or $C \neq 0$, when $\operatorname{Im} \sigma_b^{-1}\gamma\sigma_a z = \operatorname{Im} z/|Cz+D|^2 \le 1/(C^2 \operatorname{Im} z)$, which means $\operatorname{Im} z < 1/(C^2 Y)$. This proves the lemma, because $\min\left\{C > 0 : \left(\begin{smallmatrix} \cdot & \cdot \\ C & \cdot \end{smallmatrix}\right) \in \sigma_b^{-1}\Gamma\sigma_a\right\}$ exists (see [I], pp. 53–54), $a, b \in A$, A is finite.

This shows that for γ parabolic and Y large enough we can integrate in (5) over $\{z = x + iy : 0 \le y \le Y,\ 0 \le x \le 1\}$, because $k(z, z+l) = k(l^2/(4y^2))$, and this is 0 for y small, since $l \neq 0$, and k has compact support. So

$$T_\gamma^Y = \int_0^Y k\left(\frac{l^2}{4y^2}\right)(\beta_a(0)y^s + \widetilde{\beta}_a(0)y^{1-s}) \frac{dy}{y^2},$$

and with the substitution $\nu = l^2/(4y^2)$ this is

$$|l|^{-1} \int_{l^2/(4Y^2)}^\infty k(\nu)(\beta_a(0)(|l|/(2\nu^{1/2}))^s + \widetilde{\beta}_a(0)(|l|/(2\nu^{1/2}))^{1-s})\nu^{-1/2} \, d\nu.$$

Lemma 2. *If* $0 < \mathrm{Re}\, S < 1$, *then*, *as* $Y \to \infty$,

$$\sum_{l\neq 0} |l|^{S-1} \int_{l^2/(4Y^2)}^{\infty} k(\nu)(4\nu)^{-S/2}\nu^{-1/2}\, d\nu$$

$$= g(0)\frac{Y^S}{S} + 2^{1-S}\zeta(1-S)\int_0^{\infty} k(\nu)\nu^{-(1+S)/2}\, d\nu + O(Y^{\mathrm{Re}\, S-1}\log Y),$$

where ζ *is the Riemann zeta-function.*

Proof. Summing the left-hand side over l gives

$$2\int_{1/(4Y^2)}^{\infty} k(\nu)(4\nu)^{-S/2}\nu^{-1/2}\Big(\sum_{1\le l\le 2Y\sqrt{\nu}} l^{S-1}\Big)\, d\nu.$$

Since

$$\sum_{1\le l\le 2Y\sqrt{\nu}} l^{S-1} = (2Y\sqrt{\nu})^S/S + \zeta(1-S) + O((Y\sqrt{\nu})^{\mathrm{Re}\, S-1}),$$

the lemma follows, because k has compact support, $k(0)$ is finite, k is continuous at 0, and $2\int_0^\infty k(\nu)\nu^{-1/2}\, d\nu = g(0)$.

Summing over the parabolic conjugacy classes means summing over $l\neq 0$ and $a\in A$, so by the above lemma we have proved the following.

Lemma 3. *Define*

$$T^Y_{\mathrm{par}} = \sum_{\substack{\delta\in\Gamma\\ \delta\ \text{parabolic}}} \int_{F(Y)} k(z,\delta z)u(z)\, d\mu_z.$$

Then the difference of T^Y_{par} *and*

$$g(0)\Big(\frac{Y^s}{s}B_u + \frac{Y^{1-s}}{1-s}\widetilde{B}_u\Big)$$

tends to

$$B_u 2^{1-s}\zeta(1-s)\int_0^{\infty} k(\nu)\nu^{-(1+s)/2}\, d\nu + \widetilde{B}_u 2^s\zeta(s)\int_0^{\infty} k(\nu)\nu^{(s-2)/2}\, d\nu,$$

as $Y\to\infty$.

4. The spectral trace—end of the proof of Theorem 1. We now compute $\mathrm{Tr}^Y_u K$ in another way, based on the spectral theorem. Remember that

$$\mathrm{Tr}^Y_u K = \sum_j h(t_j)I^Y_u(u_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)I^Y_u(r)\, dr. \tag{6}$$

We need several lemmas. Firstly we give the proof of the lemma stated in Section 2.

Proof of Lemma 4. We get the main term in $I_u^Y(r)$ if we substitute the constant terms of $E_c(z,1/2+ir)$ and of $u(z)$ in the cuspidal zones $F_a(Y_\Gamma)$; this gives the sum of

$$\sum_a\sum_c\Big(\delta_{ac}+\Big|\varphi_{a,c}\Big(\frac12+ir\Big)\Big|^2\Big)\Big(\beta_a(0)\frac{Y^s}{s}+\widetilde\beta_a(0)\frac{Y^{1-s}}{1-s}\Big),$$

$$\sum_a\sum_c\delta_{ac}\overline{\varphi_{a,c}\Big(\frac12+ir\Big)}\Big(\beta_a(0)\frac{Y^{s+2ir}}{s+2ir}+\widetilde\beta_a(0)\frac{Y^{1-s+2ir}}{1-s+2ir}\Big)$$

and

$$\sum_a\sum_c\delta_{ac}\varphi_{a,c}\Big(\frac12+ir\Big)\Big(\beta_a(0)\frac{Y^{s-2ir}}{s-2ir}+\widetilde\beta_a(0)\frac{Y^{1-s-2ir}}{1-s-2ir}\Big).$$

Using $\sum_c|\varphi_{a,c}(1/2+ir)|^2=1$ ([I], Theorem 6.6) and $\overline{\varphi_{a,a}(1/2+ir)}=\varphi_{a,a}(1/2-ir)$, we find that this main term will be $I_u^Y(r)-\widetilde I_u^Y(r)$. Similarly, the main term of $I_u^Y(u_j)$ will be $I_u^Y(u_j)-\widetilde I_u^Y(u_j)$. Applying Lemmas 5 and 6 below we get the result.

We need the following crude bound.

Lemma 5. *For $R\ge1$ and $Y>0$ we have*

$$\int_{-R}^{R}\sum_c\int_{F(Y)}|E_c(z,1/2+ir)|^2\,d\mu_z\,dr=O(R^2(1+\log Y)).$$

Proof. This follows easily from [I], formulas (10.9), (6.24) and (10.13).

Lemma 6. *For $R\ge1$ we have*

$$\int_{-R}^{R}\Big(\int_{Y_\Gamma}^{\infty}\int_0^1|E_c(\sigma_a z,1/2+ir)-\delta_{ac}y^{1/2+ir}-\varphi_{a,c}(1/2+ir)y^{1/2-ir}|^2y^{\operatorname{Re}s}\,\frac{dx\,dy}{y^2}\Big)dR=O(R^2),$$

and

$$\sum_{|t_j|\le R}\Big(\int_{Y_\Gamma}^{\infty}\int_0^1|u_j(\sigma_a z)-\beta_{a,j}(0)y^{1-s_j}|^2y^{\operatorname{Re}s}\,\frac{dx\,dy}{y^2}\Big)=O(R^2).$$

Proof. This follows easily by Parseval's identity, Lemma 7 below, and [I], (8.27) (see also (8.4) and (8.5) there). (We use the Fourier expansions, fix an $n\ne0$ and sum over the spectrum.)

Lemma 7. *If T is real and $n \neq 0$ is an integer, then*

$$\int_{Y_\Gamma}^{\infty} |W_{1/2+iT}(iny)|^2 y^{\operatorname{Re} s} \frac{dy}{y^2}$$
$$= \begin{cases} O((|T|/|n|)^{\operatorname{Re} s-1} e^{-\pi|T|}) & \text{if } |n| = O(|T|), \\ O(e^{-c_\Gamma |n|}) & \text{if } |T|/|n| \text{ is sufficiently small (depending on } \Gamma), \end{cases}$$

where c_Γ is a positive constant depending on Γ.

Proof. By the definition of W we have

$$\int_{Y_\Gamma}^{\infty} |W_{1/2+iT}(iny)|^2 y^{\operatorname{Re} s} \frac{dy}{y^2} = O\Big(|n|^{1-\operatorname{Re} s} \int_{2\pi|n|Y_\Gamma} |K_{iT}(y)|^2 y^{\operatorname{Re} s} \frac{dy}{y}\Big),$$

and the lemma follows by [I], p. 228, the formula above (B.37), and [Le], (5.10.24).

We need one more lemma for the computation of $\operatorname{Tr}_u^Y K$.

Lemma 8. *Let $0 < \operatorname{Re} S < 1$ and $S \neq 2s_l - 1$ for $l \in L$. Then the difference of*

$$\int_{-\infty}^{\infty} h(r) \varphi_{a,a}(1/2 + ir) \frac{Y^{S-2ir}}{S - 2ir} \, dr$$

and

$$\pi \varphi_{a,a}\Big(\frac{1+S}{2}\Big) h\Big(i\frac{S}{2}\Big) - 2\pi \sum_{\substack{1/2 < s_l \le (1+\operatorname{Re} S)/2 \\ l \in L}} h\Big(i\Big(s_l - \frac{1}{2}\Big)\Big) \frac{Y^{1+S-2s_l}}{1+S-2s_l} R_{a,s_l}$$

tends to 0 *as $Y \to \infty$.*

Proof. This follows by replacing the line of integration to $\operatorname{Im} r = -\operatorname{Re} S/2 - \varepsilon$ with some $\varepsilon > 0$, passing through simple poles at $r = -iS/2$, and $r = -i(s_l - 1/2)$ lying in this strip for $l \in L$.

This means that if h satisfies condition (B), then with the notations

$$\Sigma_{u,h}^Y(S) = \sum_{\substack{1/2 < s_l \le (1+\operatorname{Re} S)/2 \\ l \in L}} h(t_l) \frac{Y^{1+S-2s_l}}{1+S-2s_l} \sum_a \beta_a(0) \Big(\sum_{\substack{j \\ s_j = s_l}} |\beta_{a,j}(0)|^2 - R_{a,s_l} \Big),$$

$$\widetilde{\Sigma}_{u,h}^Y(S) = \sum_{\substack{1/2 < s_l \le (1+\operatorname{Re} S)/2 \\ l \in L}} h(t_l) \frac{Y^{1+S-2s_l}}{1+S-2s_l} \sum_a \widetilde{\beta}_a(0) \Big(\sum_{\substack{j \\ s_j = s_l}} |\beta_{a,j}(0)|^2 - R_{a,s_l} \Big),$$

where $t_l = i(s_l - 1/2)$, we have proved by (6), Lemmas 4 and 8, using

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\,dr = g(0)$$

the following:

LEMMA 9. *The difference of* $\mathrm{Tr}_u^Y K$ *and*

$$g(0)\left(\frac{Y^s}{s}B_u + \frac{Y^{1-s}}{1-s}\widetilde{B}_u\right) + \frac{1}{2}h\left(i\frac{s}{2}\right)B_u(s) + \frac{1}{2}h\left(i\frac{1-s}{2}\right)\widetilde{B}_u(1-s) + \Sigma_{u,h}^Y(s) + \widetilde{\Sigma}_{u,h}^Y(1-s)$$

tends to

$$\sum_j h(t_j)I_u(u_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)I_u(r)\,dr,$$

as $Y \to \infty$.

Since k has compact support, it is obvious that

$$T_{\mathrm{hyp}}^Y = \sum_{\substack{\delta\in\Gamma\\ \delta\ \mathrm{hyperbolic}}} \int_{F(Y)} k(z,\delta z)u(z)\,d\mu_z$$

and

$$T_{\mathrm{ell}}^Y = \sum_{\substack{\delta\in\Gamma\\ \delta\ \mathrm{elliptic}}} \int_{F(Y)} k(z,\delta z)u(z)\,d\mu_z$$

have finite limits as $Y \to \infty$ (and of course $T_{\mathrm{id}}^Y = \int_{F(Y)} k(z,z)u(z)\,d\mu_z$ tends to 0 as $Y \to \infty$, because $\int_F u(z)\,d\mu_z = 0$), so by Lemmas 3 and 9 we see that $\Sigma_{u,h}^Y(s) + \widetilde{\Sigma}_{u,h}^Y(1-s)$ tends to a finite limit as $Y \to \infty$. But this sum is a finite linear combination of Y-powers with nonzero exponents, and every exponent has nonnegative real part. So the fact that this sum has a finite limit as $Y \to \infty$ implies that this sum is identically 0. (It is not hard to see that in fact $\sum_{j,\, s_j = s_l} |\beta_{a,j}(0)|^2 - R_{a,s_l} = 0$, but we do not need it.)

This last remark, together with (3), (4), Lemmas 3, 4 and 9, proves Theorem 1.

5. The inversion of the integral transformations. The transformation formulas between the functions k, g and h are well known, but we now have a new integral transformation for every $\lambda < 0$, namely

$$R(y) = \int_{-\pi/2}^{\pi/2} k\left(\frac{y}{\cos^2\vartheta}\right) f_\lambda(\vartheta)\,\frac{d\vartheta}{\cos^2\vartheta} \tag{7}$$

for $y > 0$. Our aim is now to invert this transformation, i.e. to express h (and in our way k, q and g) in terms of R.

To this end let R be a smooth, compactly supported function on $(0, \infty)$ (i.e. it is 0 in a neighbourhood of 0 as well as in a neighbourhood of ∞). Denote the Mellin transform of R by

$$\widehat{R}(s) = \int_0^\infty R(y) y^{s-1}\, dy,$$

and assume that $\widehat{R}(0) = 0$ (one needs this unsignificant restriction).

By Mellin inversion

$$R(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \widehat{R}(s)\, ds$$

for $y > 0$ and for any real σ. We see from this that (7) is satisfied with the function $k(\tau)$ $(\tau > 0)$ defined by

$$k(\tau) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{\tau^{-s}}{F_\lambda(s)} \widehat{R}(s)\, ds, \tag{8}$$

where $\sigma > 1/2$, and $F_\lambda(s) = \int_{-\pi/2}^{\pi/2} f_\lambda(\vartheta) \cos^{2s-2} \vartheta\, d\vartheta$. We know $F_\lambda(s)$ explicitly,

$$F_\lambda(s) = \sqrt{\pi}\, \frac{\Gamma(s - z_1)\Gamma(s - z_2)}{\Gamma^2(s)},$$

with $z_1 = 1/4 + it/2$, $z_2 = 1/4 - it/2$, $1/4 + t^2 = -\lambda$ (see Lemma 11).

It is easy to see from (8) that k is continuous on $[0, \infty)$, $k(0)$ is finite (replace the line of integration by $\sigma = -\varepsilon$ with some $\varepsilon > 0$, and use $\widehat{R}(0) = 0$), and k has compact support (this follows from the fact that R has compact support, letting $\sigma \to \infty$), i.e. k satisfies condition (A).

For this k by (8) we get

$$q(\nu) = \frac{1}{2\pi i} \int_0^\infty \frac{\int_{(\sigma)} (\nu + \tau)^{-s} F_\lambda^{-1}(s) \widehat{R}(s)\, ds}{\sqrt{\tau}}\, d\tau$$

for $\nu > 0$. With the substitution $\tau = \nu \sin^2 \vartheta / \cos^2 \vartheta$, $\vartheta \in (0, \pi/2)$, we have

$$\int_0^\infty \frac{(\nu + \tau)^{-s}}{\sqrt{\tau}}\, d\tau = 2\nu^{1/2-s} \int_0^{\pi/2} \cos^{2s-2} \vartheta\, d\vartheta = \nu^{1/2-s} E(s),$$

where $E(s) = \sqrt{\pi}\Gamma(s - 1/2)/\Gamma(s)$ by the Corollary to Lemma 11. So

$$q(\nu) = \frac{1}{2\pi i} \int_{(\sigma)} \nu^{1/2-s} \frac{E(s)}{F_\lambda(s)} \widehat{R}(s)\, ds \tag{9}$$

for $\nu > 0$, with

$$\frac{E(s)}{F_\lambda(s)} = \frac{\Gamma(s)\Gamma(s-1/2)}{\Gamma(s-z_1)\Gamma(s-z_2)}.$$

Since R has compact support (as a function on $(0,\infty)$), we see by (9) that q is smooth on $(0,\infty)$, $g(\nu) = 0$ for ν large enough (by letting $\sigma \to \infty$), and in a neighbourhood of 0 it has an absolutely convergent expansion of the type $q(\nu) = \sum_{n=0}^{\infty} c_n \nu^{n/2}$ with $c_1 = 0$, i.e. the coefficient of $\nu^{1/2}$ is 0 (we see this by letting $\sigma \to -\infty$, and using $\widehat{R}(0) = 0$). This implies that the function g (which is even and defined on $(-\infty,\infty)$) is smooth on $[0,\infty)$, $g(a) = 0$ for a large enough, and for small positive a it has an absolutely convergent expansion of the type $g(a) = \sum_{n=0}^{\infty} d_n a^n$ with $d_1 = 0$. These properties of g imply (after three-fold integration by parts) that h satisfies condition (B).

Now, let $|\operatorname{Im} r| < 1/2$. Then by (9), taking $1/2 + |\operatorname{Im} r| < \sigma < 1$ (since the double integral is absolutely convergent in this case) we have

$$h(r) = \frac{1}{2\pi i}\int_{(\sigma)} 2^{2s}\frac{E(s)}{F_\lambda(s)}\widehat{R}(s)\Big(\int_{-\infty}^{\infty}(e^a + e^{-a} - 2)^{1/2-s}e^{ira}\,da\Big)\,ds. \tag{10}$$

We have to compute the inner integral. With the notations

$$G(A,B) = \int_0^{\infty}(e^a - 1)^A e^{Ba}\,da, \quad F(r,s) = G(1-2s, -1/2+s+ir)$$

one obtains

$$\int_{-\infty}^{\infty}(e^a + e^{-a} - 2)^{1/2-s}e^{ira}\,da = F(r,s) + F(-r,s). \tag{11}$$

By Lemma 10 one has

$$\begin{aligned} F(r,s) &\\ &= \pi\frac{\Gamma(-1/2+s+ir)}{\Gamma(3/2-s+ir)\Gamma(2s-1)}\{\cot\pi(1-2s) - \cot\pi(1/2-s+ir)\}. \end{aligned} \tag{12}$$

So we have determined h, but for the application of Theorem 1 we also need

$$\int_0^{\infty} k\bigg(\sin^2\frac{l\pi}{m_p}\sinh^2 r\bigg)g_\lambda(r)\sinh r\,dr.$$

By (8) we have

$$\int_0^{\infty} k\bigg(\sin^2\frac{l\pi}{m_p}\sinh^2 r\bigg)g_\lambda(r)\sinh r\,dr = \frac{1}{2\pi i}\int_{(\sigma)}\frac{G_\lambda(s)}{F_\lambda(s)}\widehat{R}(s)\sin^{-2s}\frac{l\pi}{m_p}\,ds$$

for $1/2<\sigma<1$, where $G_\lambda(s)=\int_0^\infty g_\lambda(r)\sinh^{1-2s} r\,dr$. So by Lemma 11 we obtain

$$\int_0^\infty k\left(\sin^2\frac{l\pi}{m_p}\sinh^2 r\right)g_\lambda(r)\sinh r\,dr = \frac{1}{4i\sqrt{\pi}}\cdot\frac{1}{\Gamma(1-z_1)\Gamma(1-z_2)}\int_{(\sigma)}\frac{\widehat{R}(s)}{\sin\pi s}\sin^{-2s}\frac{l\pi}{m_p}\,ds, \tag{13}$$

where $z_1=1/4+it/2$, $z_2=1/4-it/2$, $1/4+t^2=-\lambda$.

We have proved the following.

Theorem 2. *Let R be a smooth, compactly supported function on* $(0,\infty)$ (*i.e. it is* 0 *in a neighbourhood of* 0 *as well as in a neighbourhood of* ∞). *Denote the Mellin transform of R by*

$$\widehat{R}(s)=\int_0^\infty R(y)y^{s-1}\,dy,$$

and assume that $\widehat{R}(0)=0$. *Then the function k defined by* (8) *satisfies condition* (A), *the corresponding h satisfies condition* (B), *so Theorem* 1 *is applicable for them. The function $h(r)$ for* $|\mathrm{Im}\, r|<1/2$ *is given in* (10),

$$\int_0^\infty k\left(\sin^2\frac{l\pi}{m_p}\sinh^2 r\right)g_\lambda(r)\sinh r\,dr$$

is given in (13) (*for the functions E, F_λ, and G_λ see Lemma* 11 *and its Corollary*), *and for $y>0$ we have*

$$R(y)=\int_{-\pi/2}^{\pi/2} k\left(\frac{y}{\cos^2\vartheta}\right)f_\lambda(\vartheta)\,\frac{d\vartheta}{\cos^2\vartheta}.$$

6. Two lemmas on special functions

Lemma 10. *Let*

$$G(A,B)=\int_0^\infty (e^a-1)^A e^{Ba}\,da \quad \textit{for}\ \mathrm{Re}\, A>-1,\ \mathrm{Re}(A+B)<0.$$

Then

$$G(A,B)=\pi\frac{\Gamma(B)}{\Gamma(A+B+1)\Gamma(-A)}\{\cot\pi A-\cot\pi(A+B)\},$$

where $\cot=\cos/\sin$.

Proof. We first fix $-1 < A < 0$, and consider $G(A,B)$ as a function of B. In this case we have by partial integration

$$G(A,B) = -\int\limits_0^\infty \left(\frac{(e^a-1)^{A+1}}{A+1}\right)(e^{(B-1)a}(B-1))\,da$$
$$= \frac{1-B}{1+A}(G(A,B)-G(A,B-1)),$$

and this gives

$$G(A,B) = \frac{B-1}{A+B}G(A,B-1).$$

Now let $\widetilde{G}(A,B) = \Gamma(B)/\Gamma(A+B+1)$. Then this satisfies the same functional equation as G, i.e.

$$\widetilde{G}(A,B) = \frac{B-1}{A+B}\widetilde{G}(A,B-1),$$

so $G(A,B)/\widetilde{G}(A,B)$ (as a function of B) is periodic with respect to 1, and it is meromorphic on the whole plane. For $\operatorname{Re} B < -A$ the function G is regular, so in this region the only singularities of $G/\widetilde{G}$ are the roots of $\widetilde{G}$, i.e. $B = -A-1, -A-2, \ldots$ Now, it is easy to see from the integral representation that $G(A,B)$ has a pole of order 1 with residue -1 at $B = -A$, and $\widetilde{G}(A,-A) = \Gamma(-A)$. From these considerations it follows that

$$\frac{G(A,B)}{\widetilde{G}(A,B)} + \frac{\pi}{\Gamma(-A)}\cot\pi(A+B)$$

is an entire function of B, periodic with respect to 1, it has at most polynomial growth on vertical lines, so it is a constant. Its value at $B = 0$ is $\frac{\pi}{\Gamma(-A)}\cot\pi A$. This proves the lemma for $-1 < A < 0$, and it is enough by analytic continuation.

For $\lambda < 0$ let $f_\lambda(\vartheta)$ ($\vartheta \in (-\pi/2, \pi/2)$) be the solution of the differential equation

$$f^{(2)}(\vartheta) = \frac{\lambda}{\cos^2\vartheta}f(\vartheta)$$

with $f_\lambda(0) = 1$, $f_\lambda^{(1)}(0) = 0$; and let $g_\lambda(r)$ ($r \in [0,\infty)$) be the solution of

$$g^{(2)}(r) + \frac{\cosh r}{\sinh r}g^{(1)}(r) = \lambda g(r)$$

with $g_\lambda(0) = 1$.

Lemma 11. *Let* $\lambda < 0$ *and*

$$F_\lambda(s) = \int\limits_{-\pi/2}^{\pi/2} f_\lambda(\vartheta)\cos^{2s-2}\vartheta\,d\vartheta \quad \textit{for } \operatorname{Re} s > 1/2,$$

$$G_\lambda(s) = \int_0^\infty g_\lambda(r) \sinh^{1-2s} r\, dr \quad \textit{for } 1/2 < \operatorname{Re} s < 1.$$

Then

$$F_\lambda(s) = \sqrt{\pi}\, \frac{\Gamma(s-z_1)\Gamma(s-z_2)}{\Gamma^2(s)},$$

$$G_\lambda(s) = \frac{\Gamma(s-z_1)\Gamma(s-z_2)}{\Gamma^2(s)} \cdot \frac{\pi}{2\sin \pi s} \cdot \frac{1}{\Gamma(1-z_1)\Gamma(1-z_2)},$$

where $z_1 = 1/4 + it/2$, $z_2 = 1/4 - it/2$, $1/4 + t^2 = -\lambda$.

Proof. It is easy to see by elementary considerations (using the fact that $\lambda/\cos^2\vartheta$ is negative and it is decreasing for $\vartheta \ge 0$) that $|f_\lambda(\vartheta)| \le 1$ for every ϑ, and this implies that $f_\lambda^{(1)}(\vartheta)\cos\vartheta$ is bounded for a fixed λ (since $f_\lambda^{(2)}(\vartheta)\cos^2\vartheta$ is bounded).

On the other hand, the function $g_\lambda(r)$ is also bounded for a fixed λ (for example because there are nonzero bounded eigenfunctions of the Laplace operator on H with eigenvalue λ (e.g. $f(z) = f_\lambda(\vartheta)$ for $z = re^{i(\pi/2+\vartheta)}$), and we know ([I], Cor 1.13) that averaging any eigenfunction over hyperbolic circles around any point w in H, we get a multiple of $g_\lambda(r(z,w))$, where r is the hyperbolic distance), and then $g_\lambda^{(1)}(r)$ is also bounded, because $(g_\lambda^{(1)}(r)\sinh r)^{(1)} = \lambda g_\lambda(r)\sinh r$. Observe also that $g_\lambda^{(1)}(0) = 0$. We will repeatedly use these remarks in the following calculations.

Using the differential equation for $f_\lambda(\vartheta)$ and partial integration twice we have

$$\begin{aligned}\lambda F_\lambda(s) &= \int_{-\pi/2}^{\pi/2} f_\lambda^{(2)}(\vartheta)\cos^{2s}\vartheta\, d\vartheta \\ &= \int_{-\pi/2}^{\pi/2} f_\lambda(\vartheta)[2s(2s-1)\cos^{2s-2}\vartheta\sin^2\vartheta - 2s\cos^{2s}\vartheta]\, d\vartheta,\end{aligned}$$

and this implies $\lambda F_\lambda(s) = 2s(2s-1)F_\lambda(s) - (2s)^2 F_\lambda(s+1)$, i.e. F_λ is a meromorphic function on the whole plane satisfying

$$F_\lambda(s+1) = F_\lambda(s)\frac{2s(2s-1)-\lambda}{(2s)^2}.$$

Using the differential equation for $g_\lambda(r)$ and partial integration we have

$$\lambda G_\lambda(s) = \int_0^\infty (g_\lambda^{(1)}(r)\sinh r)^{(1)} \sinh^{-2s} r\, dr = 2s\int_0^\infty g_\lambda^{(1)}(r)\sinh^{-2s} r\cosh r\, dr,$$

and a new partial integration gives, by the equality $\cosh^2 = 1 + \sinh^2$, that

$$\lambda G_\lambda(s) = 2s(2s-1)G_\lambda(s) + 2s \lim_{\varepsilon\to 0+0} \Big(2s \int_\varepsilon^\infty g_\lambda(r) \sinh^{-2s-1} r\, dr - g_\lambda(\varepsilon)\sinh^{-2s}\varepsilon \cosh\varepsilon\Big).$$

Now, $g_\lambda(\varepsilon) = 1 + O(\varepsilon^2) = \cosh\varepsilon$, from which it follows easily that this last limit is a regular function of s for $-1/2 < \operatorname{Re} s < 1$ and it equals $2sG_\lambda(s+1)$ for $-1/2 < \operatorname{Re} s < 0$. This shows that G_λ is a meromorphic function on the whole plane satisfying

$$G_\lambda(s+1) = -G_\lambda(s)\frac{2s(2s-1)-\lambda}{(2s)^2}.$$

and we also see that in $1/2 < \operatorname{Re} s < 2$ the only pole of $G_\lambda(s)$ is at $s = 1$, it is of first order and the residue is (from the integral representation) $-1/2$.

Let $X_\lambda(s) = \Gamma(s-z_1)\Gamma(s-z_2)/\Gamma^2(s)$. Then F_λ/X_λ is periodic with respect to 1, and it is regular for $\operatorname{Re} s > 1/2$, i.e. it is an entire function, and it has at most polynomial growth on vertical lines, so it is a constant. As $s \to \infty$, we see by Stirling's formula and by $f_\lambda(0) = 1$ that this constant is $\lim_{s\to\infty} \sqrt{s}\int_{-\pi/2}^{\pi/2} \cos^{2s-2}\vartheta\, d\vartheta$, so it is independent of λ. For $\lambda \to 0-0$ we have $X_\lambda(s) \to \Gamma(s-1/2)/\Gamma(s)$ and $f_\lambda(\vartheta) \to 1$ for every ϑ, so

$$\frac{F_\lambda}{X_\lambda}(1) \to \frac{\pi}{\Gamma(1/2)} = \sqrt{\pi}.$$

On the other hand, for

$$Q_\lambda(s) = \frac{G_\lambda(s)}{X_\lambda(s)} - \frac{\pi}{2\sin\pi s}\cdot\frac{1}{\Gamma(1-z_1)\Gamma(1-z_2)}$$

we have $Q_\lambda(s+1) = -Q_\lambda(s)$, and Q_λ is regular for $1/2 < \operatorname{Re} s < 2$ (including $s = 1$), so it is an entire function, it has at most polynomial growth on vertical lines, hence it is identically 0.

COROLLARY. *For* $\operatorname{Re} s > 1/2$ *let* $E(s) = \int_{-\pi/2}^{\pi/2} \cos^{2s-2}\vartheta\, d\vartheta$. *Then*

$$E(s) = \sqrt{\pi}\,\frac{\Gamma(s-1/2)}{\Gamma(s)}.$$

Proof. This follows by letting $\lambda \to 0-0$.

References

[I] H. Iwaniec, *Introduction to the Spectral Theory of Automorphic Forms*, Rev. Mat. Iberoamericana, 1995.

[Le] N. N. Lebedev, *Special Functions and their Applications*, 2nd ed., Moscow, Leningrad, 1963.

[Se] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. 20 (1956), 47–87.

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