ACTA ARITHMETICA
LXXXVII.3 (1999)

Errata to "The Dirichlet series of $\zeta(s)\zeta^{\alpha}(s+1)f(s+1)$: On an error term associated with its coefficients"

(Acta Arith. 75 (1996), 39–69)

by

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1. Hypotheses in Theorem 2. We recall the statement of our Theorem 2 in Section 3 (page 50).

Theorem 2. *Let $v_n = v(n)$ be a real multiplicative arithmetical function satisfying, for some real numbers $\alpha > 0$ and $\beta \ge 0$,*

$$\sum_{n\le x} |v_n| = O((\log x)^{\alpha}); \tag{h1}$$

$$\sum_{n\le x} (nv_n)^2 = O(x(\log x)^{\beta}); \tag{h2}$$

(h3) $p^k v(p^k)$ *is an ultimately monotonic function of p for $k = 1$ and $k = 2$, and is bounded for every $k \ge 1$.*

Then, if we set $y := x\exp(-(\log x)^b)$ for some positive number b, $t := \log x$, and $u := \log t = \log\log x$, we have

$$\sum_{n\le y} v_n \psi(x/n) = O(t^{2\alpha/3}u^{4\alpha/3}).$$

Hypothesis (h1) is insufficient to ensure the validity of the proof sketched in the paper (the last argument in the proof of Lemma 3.3 on pages 52–53 is not valid under that too weak hypothesis). It can however be replaced by

$$\sum_{n\le x} n|v_n| = x\sum_{r=0}^{\lambda+[\alpha]} F_r(\log x)^{\alpha-r} + O(x(\log x)^{-\lambda}) \tag{h1$*$}$$

for every positive natural number λ (and where the F_r are real constants). A complete proof of Theorem 2 under hypothesis (h1$*$) (and under a weakened hypothesis (h3$*$)) is given in [2]. And as noted in the latter (Section 4), the stronger hypothesis (h1$*$) is still satisfied by all the examples we treat in our paper, and in particular by $a(n) = (1 * v)(n) = (\sigma(n)/n)^r$ or $(\phi(n)/n)^r$ or $(\sigma(n)/\phi(n))^r$.

2. A mistake in Saltykov's paper. For the second time, we thank Professor A. Schinzel for twisting our arm to make us read [3]. The first time we did so was in the note "added in proof" of our paper, in which we claimed that A. I. Saltykov's result

$$H(x) = O((\log x)^{2/3}(\log\log x)^{1+\varepsilon}) \tag{1}$$

for the error term

$$H(x) = \sum_{n\le x} \phi(n)/n - 6x/\pi^2,$$

which is a better estimate than Walfisz'

$$H(x) = O((\log x)^{2/3}(\log\log x)^{4/3}) \tag{2}$$

(proved in [4]), is correct. We then realised that the reason why Saltykov's paper was considered suspect, is because it makes use of a theorem proved in [1], in which Korobov also makes an unproved claim on the zeros of the Riemann zeta function. (Korobov's paper, together with a paper of Vinogradov, in which the same unverified claim is made, is the starting point of a famous controversy which was never really completely settled.) But we also realised that the theorem Saltykov uses is perfectly sound, and we concluded in our note that his estimate on $H(x)$ was undisputable and the best to date. We even announced an improvement of our Theorem 2 based on Saltykov's ideas.

Our reading of [3] was however not careful enough (was Professor Schinzel's twist not strong enough?), and Pétermann [2] discovered later a mistake in Saltykov's argument. More exactly, the proof of his Lemma 2.6, which states an estimate involving two parameters γ_1 and γ_2, is not valid for $\gamma_1 = 0$. This value $\gamma_1 = 0$, however, is the value needed in order to deduce (1) from Saltykov's result. And it is clear that the modification needed in Lemma 2.6 in order to treat the case $\gamma_1 = 0$ "only" yields Walfisz' estimate (2), which thus cannot be improved with the method. More details on this can be found in [2] (Section 5).

References

[1] M. N. Korobov, *Estimates of trigonometrical sums and their applications*, Uspekhi Mat. Nauk 13 (1958), no. 4, 185–192 (in Russian).

[2] Y.-F. S. Pétermann, *On an estimate of Walfisz and Saltykov for an error term related to the Euler function*, J. Théor. Nombres Bordeaux, to appear.

[3] A. I. Saltykov, *On Euler's function*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1960, no. 6, 34–50 (in Russian).

[4] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Deutscher Verlag Wiss., Berlin, 1963.

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Received on 12.8.1998 (3436)