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A uniform version of Jarník's theorem

by

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1. Introduction. In his original paper [4], Jarník proved that any strictly convex curve Γ satisfies

$$N(\Gamma) \ll l(\Gamma)^{2/3}.$$

For any curve C, here and in the sequel, $N(C)$ and $l(C)$ denote respectively its number of integer points and its length. On the other hand, Jarník constructed a family of strictly convex curves Γ_0, with $l(\Gamma_0)$ tending to infinity, such that

$$N(\Gamma_0) \asymp l(\Gamma_0)^{2/3}. \tag{1}$$

Later on in [2], Grekos refined the previous results, in some cases, by introducing the infimum of the radii of curvature $r(\Gamma)$ of the curve. He succeeded in showing an upper bound of the shape

$$N(\Gamma) \ll l(\Gamma) r(\Gamma)^{-1/3}$$

and conversely constructed a family of curves Γ_0 with

$$N(\Gamma_0) \asymp l(\Gamma_0) r(\Gamma_0)^{-1/3}.$$

Naturally, Grekos' results suppose that the curve has at least C^2-regularity. In fact, this is the maximal regularity one can have for a "uniform" bound as good as (1), because of Swinnerton-Dyer's results in [8] showing that if Γ is C^3, one has

$$N(t\Gamma) \ll_{\Gamma,\varepsilon} t^{3/5+\varepsilon}. \tag{2}$$

where $t\Gamma$ denotes the homothetic dilatation (with respect to the origin) of Γ by a factor $t \geq 1$.

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Then, Schmidt [7] removed the dependence on Γ in formula (2) provided that the third derivative does not vanish, and gave generalizations in higher dimensions. Later, Bombieri and Pila in [1] improved the value $3/5+\varepsilon$ of the exponent to $1/2+\varepsilon_d$ for curves with a continuous d-derivative satisfying a condition similar to that of Schmidt on the third derivative and where ε_d tends to zero as d tends to infinity. This was a first step (the C^∞ case) in the direction of Schmidt's conjecture stating that

$$N(\Gamma) \ll_\varepsilon l(\Gamma)^{1/2+\varepsilon} \tag{3}$$

if Γ is C^3 and its third derivative does not vanish (this would naturally be the best possible, except for the ε, in view of the parabola case). Finally, Pila [6] made a next step toward Schmidt's conjecture by succeeding in showing that C^{105} curves (and not only C^∞ curves) have to satisfy (3), provided once again that a specific condition is fulfilled.

For connections with probability theory and partitions, we refer to Vershik's paper [10] on the statistics of convex lattice polygons: interesting related results have been shown.

Now, let us return to Jarník's second result (1). It can be restated as follows: for every integer q_0, there exists a strictly convex curve C_0 in $[0,1]^2$ and an integer $q \ge q_0$ such that

$$\left| C_0 \cap \left(\frac{1}{q}\mathbb{Z}\right)^2 \right| \asymp q^{2/3}.$$

In this paper we partially answer the question about the possibility of partial uniformization (in a sense to be specified) of this result by proving

THEOREM 1. *Let χ be any function tending to infinity. Then there exist a strictly convex curve $\mathcal{C}$ and a strictly increasing sequence of integers $(q_n)_{n\ge0}$ such that for each n, one has*

$$\left| \mathcal{C} \cap \left(\frac{1}{q_n}\mathbb{Z}\right)^2 \right| \gg \frac{q_n^{2/3}}{\chi(q_n)}.$$

Notice that the curve $\mathcal{C}$, as any convex curve, is continuous but unfortunately, I could not construct a C^1 curve with the required properties (although $\mathcal{C}$ will clearly be almost everywhere C^∞).

In fact, we prove the following more precise result which clearly implies Theorem 1. Indeed, it suffices to take the r_n's of Theorem 2 large enough to ensure that $\chi(x) \ge K^n$ for all $x \ge r_n$.

THEOREM 2. *Let $(r_n)_{n\ge0}$ be any sequence of real numbers. There exist a strictly convex curve $\mathcal{C}$ and a strictly increasing sequence of integers $(q_n)_{n\ge0}$*

such that for each n, *one has* $q_n \ge r_n$ *and*

$$\left|\mathcal{C} \cap \left(\frac{1}{q_n}\mathbb{Z}\right)^2\right| \gg \frac{q_n^{2/3}}{K^n},$$

where $K = \frac{(16\pi)^{2/3}\sqrt{2}}{3(\sqrt{2}-1)} = 15.50\ldots$

Following Swinnerton-Dyer's notation define, for a strictly convex curve S in $[0,1]^2$,

$$\sigma(S) = \limsup_{q\to\infty}\left(\frac{\log\left|S \cap \left(\frac{1}{q}\mathbb{Z}\right)^2\right|}{\log q}\right).$$

A simple application of Theorem 1 with for instance $\chi(x) = \log x$ implies that there exists a curve S with $\sigma(S) = 2/3$. That is optimal in view of Jarník's theorem.

In conclusion, I would like to thank Jean-Marc Deshouillers and Georges Grekos. I am very indebted to them for having introduced me to this problem and for their constant and various help as well as the ideas they provided me with during the elaboration of this paper.

2. Definitions, notations and basic facts. We use classical notations as $[x], \{x\}$ for the integer and fractional parts of a real x and $\gcd(m,n)$ for the greatest common divisor of two integers m and n; the Möbius function and the number of divisors function will be denoted by μ and τ respectively, the Euler function by ϕ and the Kronecker symbol of an integer k by $\delta_{0,k}$. Finally, we write λ for the Lebesgue measure.

If M is a point in $\mathbb{R}^2$, then x_M and y_M denote its coordinates and we write $M = (x_M, y_M)$; for two points A and B, we denote by $s_{A,B}$ the slope of the line joining A and B. In the following we call a point belonging to the lattice $\left(\frac{1}{q}\mathbb{Z}\right)^2$ a *q-integer point*.

Let us begin with a technical but useful definition. If q is an integer, then a *q-well weighted curve* is an object of the form

$$\mathcal{C} = (f, (S_i, \alpha_i, \beta_i)_{i\in\mathcal{E}}),$$

where $\mathcal{E}$ is a finite set of indices, f a convex function defined on $[0,1]$, the S_i's disjoint subintervals of $[0,1]$ and α_i, β_i real numbers. Moreover, we impose the following conditions. First, outside the S_i's, f is strictly convex. Second, on each $S_i = [a,b]$, f is affine and for $x < a < b < y$,

$$\frac{f(a)-f(x)}{a-x} < \alpha_i < \frac{f(b)-f(a)}{b-a} < \beta_i < \frac{f(y)-f(b)}{y-b}, \tag{4}$$

$$\left|\frac{f(b)-f(a)}{b-a} - \frac{\alpha_i+\beta_i}{2}\right| < \frac{1}{4}|\beta_i - \alpha_i|. \tag{5}$$

Finally, we require that the points on the curve corresponding to the extremities of the S_i's on the representing curve of f be q-integer points.

For such an object, we call the real number

$$\nu(\mathcal{C}) = \sum_{i\in\mathcal{E}} (\beta_i - \alpha_i)$$

the *free measure* of $\mathcal{C}$.

For $\mathcal{C}, \mathcal{C}'$ two (q- and q'- respectively) well weighted curves, we say that $\mathcal{C}'$ *refines* $\mathcal{C}$ if $\mathcal{C}'$ coincides with $\mathcal{C}$ on each interval where $\mathcal{C}$ is strictly convex (that is, outside the S_i's corresponding to $\mathcal{C}$).

We will need an easy property which is a direct consequence of the above definition.

LEMMA 1. *If $\mathcal{C}'$ refines $\mathcal{C}$ then the graph of $\mathcal{C}'$ is under that of $\mathcal{C}$.*

Finally, we recall the following fact which will be of use in the sequel and is quite clear:

LEMMA 2. *Let A and B be two points in $\mathbb{R}^2$. If $\alpha < s_{A,B} < \beta$ then there is a strictly convex curve (in C^∞) joining A to B whose derivative is strictly between α and β. We call such a curve an (α,β)-regularization of the segment $[A, B]$.*

Sometimes we will talk about the regularization of an affine function on an interval; it will simply be a convenient way of talking about the regularization of the (linear) graph of the function on this interval.

3. Some estimates related to Farey fractions. Farey fractions are well known objects. We refer to [3] for their definition and basic properties. We begin with a trivial lemma whose proof requires only calculations using $hk' - kh' = 1$ and $2T > k + k' > T$ for consecutive fractions $h'/k' < h/k$ in the set $\mathcal{F}_T$ of Farey fractions of order $T > 1$.

LEMMA 3. *Let T be an integer and $h'/k' < h/k < h''/k''$ three consecutive Farey fractions in $\mathcal{F}_T$. Then*

$$\frac{h}{k} - \frac{\alpha+\beta}{2} = \frac{k''-k'}{2k(k+k')(k+k'')}, \tag{6}$$

$$\beta - \alpha = \frac{2k+k'+k''}{k(k+k')(k+k'')}$$

$$= \frac{1}{k(k+k')} + \frac{1}{k(k+k'')} \in \left]\frac{1}{kT}, \frac{2}{kT}\right[, \tag{7}$$

$$\left|\frac{\frac{h}{k} - \frac{\alpha+\beta}{2}}{\beta-\alpha}\right| = \frac{|k''-k'|}{2(2k+k'+k'')} < \frac{1}{4}, \tag{8}$$

where we define $\alpha = (h' + h)/(k' + k)$ *and* $\beta = (h + h'')/(k + k'')$ *to be the medians.*

Now recall a lemma due to Niederreiter [5] which concerns the number of Farey fractions between 0 and some real α (≤ 1) with given denominator k, that is, the quantity

$$B_k(\alpha) = |\{1 \le h \le k\alpha : \gcd(h, k) = 1\}|.$$

The proof is easy and is essentially a consequence of the Möbius inversion formula.

LEMMA 4. *We have*

$$B_k(\alpha) = \alpha\phi(k) - \sum_{d|k} \mu(k/d)\{d\alpha\}$$

for $k \ge 1$ *and* $\alpha \in [0, 1]$.

We now give estimates concerning usual arithmetical functions (as in [2]).

LEMMA 5. *Let* $r > -2$. *When* T *is large*, *one has*

$$\sum_{1\le k\le T} k^r\phi(k) = \frac{6T^{r+2}}{\pi^2(r+2)} + \begin{cases} O(T^{r+1}\log T + 1) & \text{if } r \ne -1, \\ O(\log^2 T) & \text{if } r = -1, \end{cases}$$

and for $r \ge -1$,

$$\sum_{1\le k\le T} k^r\tau(k) = \begin{cases} \dfrac{1}{r+1}T^{r+1}\log T + O(T^{r+1}) & \text{if } r \ne -1, \\ \dfrac{1}{2}\log^2 T + O(\log T) & \text{if } r = -1. \end{cases}$$

Proof. This is essentially an integration by parts (see [9], I.0):

$$\sum_{1\le k\le T} k^r\phi(k) = \Big(\sum_{1\le k\le T} \phi(k)\Big)T^r - r\int_1^T \Big(\sum_{1\le k\le t} \phi(k)\Big)t^{r-1}\,dt.$$

But the classical result (see once again [9])

$$\sum_{1\le k\le T} \phi(k) = \frac{3T^2}{\pi^2} + \varepsilon(T),$$

with $\varepsilon(T) \ll T\log T$, gives

$$\begin{aligned} \sum_{1\le k\le T} k^r\phi(k) &= \frac{3T^{r+2}}{\pi^2} + O(T^{r+1}\log T) - r\int_1^T \Big(\frac{3t^2}{\pi^2} + \varepsilon(t)\Big)t^{r-1}\,dt \\ &= \frac{6T^{r+2}}{\pi^2(r+2)} + O\Big(T^{r+1}\log T + \int_1^T t^r \log t\,dt\Big), \end{aligned}$$

which implies the result.

The second formula is obtained in the same manner by using the formula (see once again [9])

$$\sum_{1\le k\le T} \tau(k) = T\log T + O(T). \ \blacksquare$$

We now present the results we shall need in the sequel (this is a generalization of the results used in [2]).

LEMMA 6. *Let $0\le\alpha\le\beta\le 1$ and $r\ge -1$. When T is large, one has*

$$K(T,\alpha,\beta,r) = \sum_{h/k\in\mathcal{F}_T\cap]\alpha,\beta[} k^r = \frac{6(\beta-\alpha)T^{r+2}}{\pi^2(r+2)} + \begin{cases} O(T^{r+1}\log T) & \text{if } r>-1,\\ O(\log^2 T) & \text{if } r=-1,\end{cases}$$

and for $r>-1$,

$$H(T,\alpha,\beta,r) = \sum_{h/k\in\mathcal{F}_T\cap]\alpha,\beta[} h^r = \frac{6(\beta^{r+1}-\alpha^{r+1})T^{r+2}}{\pi^2(r+1)(r+2)} + \begin{cases} O(T^{r+1}\log T + T) & \text{if } r\ne 0,\\ O(T\log^2 T) & \text{if } r=0.\end{cases}$$

Proof. An elementary argument shows that it is sufficient to prove the formula for $\alpha=0$. For simplicity, we write K and H for $K(T,0,\beta,r)$ and $H(T,0,\beta,r)$.

We have

$$K = \sum_{1\le k\le T} k^r B_k(\beta) = \beta \sum_{1\le k\le T} k^r\phi(k) + O\Big(\sum_{1\le k\le T} k^r\tau(k)\Big),$$

because of Lemma 4 and $|\sum_{d|k}\mu(k/d)\{d\alpha\}|\le\tau(k)$. Using Lemma 5, we get the first estimate.

The second sum H requires a more careful treatment. For an integer h we define the function ψ_h for any positive integer n by

$$\psi_h(n) = \sum_{1\le k\le n,\ \gcd(h,k)=1} 1.$$

If $n=qh+r$ is the Euclidean division of n by h, we have

$$\begin{aligned}\psi_h(n) &= \sum_{i=1}^{q}\ \sum_{(i-1)h<k\le ih,\ \gcd(h,k)=1} 1 + \sum_{qh+1\le k\le qh+r,\ \gcd(h,k)=1} 1\\ &= q\phi(h) + B_h(n/h-q)\\ &= q\phi(h) + (n/h-q)\phi(h) - \sum_{d|h}\mu(h/d)\{d(n/h-q)\},\end{aligned}$$

by Lemma 4. We thus have

$$\psi_h(n) = \frac{n}{h}\phi(h) + O(\tau(h)). \tag{9}$$

Returning to H, we can write

$$\begin{aligned} H &= \sum_{h/k\in\mathcal{F}_T\cap]0,\beta[} h^r = \sum_{k=1}^{T} \sum_{1\le h<\beta k,\, \gcd(h,k)=1} h^r \\ &= \sum_{1\le h\le\beta T} h^r \sum_{h/\beta<k\le T,\, \gcd(h,k)=1} 1 = \sum_{1\le h\le\beta T} h^r(\psi_h(T)-\psi_h(h/\beta)) \\ &= \sum_{1\le h\le\beta T} h^r\Big(\frac{T}{h}\phi(h)+O(\tau(h))-\frac{1}{\beta}\phi(h)-O(\tau(h))\Big) \\ &= T\sum_{1\le h\le\beta T} h^{r-1}\phi(h) - \frac{1}{\beta}\sum_{1\le h\le\beta T} h^r\phi(h) + O\Big(\sum_{1\le h\le\beta T} h^r\tau(h)\Big) \end{aligned}$$

in view of (9). This gives the second formula by Lemma 5. ∎

By writing the difference $K(T,\alpha,\beta,-1)-K(T/2,\alpha,\beta,-1)$, we deduce immediately the following estimate which we will need in the sequel:

Corollary 1. *We have*

$$\sum_{h/k\in\mathcal{F}_T\cap]\alpha,\beta[,\, k>T/2} \frac{1}{k} = \frac{3(\beta-\alpha)T}{\pi^2} + O(\log^2 T).$$

4. Construction of approximations for f. We define $q_0 = 2, \mathcal{E}_0 = \{0\}$ and $\mathcal{C}_0 = \{f_0, (S_{0,0}, \alpha_{0,0}, \beta_{0,0})\}$ with $f_0(u) = u/2$, $S_{0,0} = [0,1]$, $\alpha_{0,0} = 0, \beta_{0,0} = 1$. Clearly, $\mathcal{C}_0$ is a q_0-well weighted curve and $q_0 \ge r_0$ if $r_0 = 1$, a situation to which the proof easily reduces.

Now we build by induction a sequence $(\mathcal{E}_n)_{n\ge1}$ of finite families of indices, a sequence $(q_n)_{n\ge1}$ of integers tending to infinity and a sequence $\mathcal{C}_n = \{f_n, (S_{n,i}, \alpha_{n,i}, \beta_{n,i})_{i\in\mathcal{E}_n}\}$ of q_n-well weighted curves such that for each n the following conditions hold:

$$q_n \text{ divides strictly } q_{n+1}, \tag{10}$$

$$\mathcal{C}_{n+1} \text{ refines } \mathcal{C}_n, \tag{11}$$

$$\nu(\mathcal{C}_n) \ge c_1^n, \tag{12}$$

$$\text{for all } i, \quad \lambda(S_{n,i}) \ge c_2^n(\beta_{n,i}-\alpha_{n,i}), \tag{13}$$

$$\sum_{i\in\mathcal{E}_n} \lambda(S_{n,i}) \le (c_4+\varepsilon)^n \tag{14}$$

and finally

$$\left|\mathcal{C}_n \cap \left(\frac{1}{q_n}\mathbb{Z}\right)^2\right| \geq (c_3 K - \varepsilon)\frac{q_n^{2/3}}{K^n}, \tag{15}$$

with

$$c_1 = \frac{3(1-1/\sqrt{2})}{\pi^2}, \quad c_2 = \frac{\pi^2}{16}, \quad c_3 = \frac{3(1-1/\sqrt{2})}{(2\pi)^{2/3}}, \quad c_4 = 1 - \frac{1}{\sqrt{2}},$$
$$K = \frac{1}{c_1 c_2^{2/3}} = \frac{(16\pi)^{2/3}}{3(1-1/\sqrt{2})} = 15.50\ldots,$$

and any real $\varepsilon > 0$ sufficiently small.

It is easy to check that $\mathcal{C}_0$ satisfies (12)–(14).

Suppose everything is already built up to n and let us build the corresponding object for $n+1$. First, we define f_{n+1} to be equal to f_n outside the $S_{n,i}$'s. On the $S_{n,i}$'s we proceed as follows: on each of them we refine f_n in f_{n+1} preserving the global convexity (so that f_{n+1} is convex), by using a local version of Jarník's construction. We next show that this refinement can be constructed with local properties that allow us to get the desired global result; more precisely, we show that, on each $S_{n,i}$, we can construct, for every q sufficiently large, a refining qq_n-well weighted curve. Moreover, we establish lower bounds for the free measure ν and the number of qq_n-integer points on the refined curve. As there are only a finite number of $S_{n,i}$, we can find a value of q which is suitable for all $S_{n,i}$. By collecting all local contributions, we get the global result (lower bounds (12) and (15) and upper bound (14)).

4.1. *Refinement: a local Jarník construction.* Now there remains to show how to refine $\mathcal{C}_n$ on a $S_{n,i} = [a,b]$. We simply write α, β for $\alpha_{n,i}, \beta_{n,i}$. Naturally, this refinement defines the function f_{n+1} on the interval $[a,b]$.

We put (for an integer q sufficiently large in a sense which will be clear from the context)

$$T = \left[\left(\frac{\pi^2 q q_n (b-a)}{2(\beta-\alpha)}\right)^{1/3}\right] \tag{16}$$

and $\beta' = \beta - \gamma$ where $\gamma = (\beta - \alpha)/\sqrt{2}$, so that $\alpha < \beta' < \beta$. Define $(P_j)_{0 \leq j \leq m+1}$, the finite sequence of qq_n-integer points in $\mathbb{R}^2$ by: $P_0 = A = (a, f_n(a))$ and, for $1 \leq j \leq m$, $P_j = P_{j-1} + \frac{1}{qq_n}(k_j, h_j)$ where

$$\mathcal{F}_T \cap \,]\alpha, \beta'[\, = \left\{\frac{h_1}{k_1} < \ldots < \frac{h_m}{k_m}\right\}, \tag{17}$$

and finally $P_{m+1} = B = (b, f_n(b))$. Define now

$$\begin{aligned}
(u_1, v_1) &= \left(\alpha, \frac{h_1 + h_2}{k_1 + k_2}\right),\\
(u_2, v_2) &= \left(v_1, \frac{h_2 + h_3}{k_2 + k_3}\right),\\
&\vdots\\
(u_{m-1}, v_{m-1}) &= \left(\frac{h_{m-2} + h_{m-1}}{k_{m-2} + k_{m-1}}, \frac{h_{m-1} + h_m}{k_{m-1} + k_m}\right),\\
(u_m, v_m) &= (v_{m-1}, \beta'),\\
(u_{m+1}, v_{m+1}) &= (\beta', \beta).
\end{aligned}$$

Thus we have

$$\begin{aligned}
u_1 = \alpha < \frac{h_1}{k_1} < v_1 = u_2 = \frac{h_1 + h_2}{k_1 + k_2} < \frac{h_2}{k_2} < v_2 = u_3 = \frac{h_2 + h_3}{k_2 + k_3} < \frac{h_3}{k_3} < v_3\\
= u_4 < \ldots < \frac{h_{m-1}}{k_{m-1}} < v_{m-1} = u_m = \frac{h_{m-1} + h_m}{k_{m-1} + k_m} < \frac{h_m}{k_m} < v_m\\
= u_{m+1} = \beta' < v_{m+1} = \beta.
\end{aligned}$$

Now, we (u_j, v_j)-regularize the segment $P_{j-1}P_j$ for $j = 1, m$ and all other j such that $k_j \le T/2$: this is possible in view of the previous inequalities (and does not spoil the global convexity).

Moreover, we (β', β)-regularize the segment P_mP_{m+1} (that is, P_mB). In order to show it is possible, we only have to prove, by Lemma 2, that

$$x_{P_m} < x_B, \tag{18}$$

$$\beta' < s_{P_mB} < \beta. \tag{19}$$

But the vector joining P_0 to P_m is

$$(b - a)\left(\frac{\beta' - \alpha}{\beta - \alpha}, \frac{\beta'^2 - \alpha^2}{2(\beta - \alpha)}\right) + o(1)$$

because

$$\sum_{j=1}^{m} k_j = \frac{2(\beta' - \alpha)T^3}{\pi^2} + O(T^2 \log T) = \frac{\beta' - \alpha}{\beta - \alpha}(b - a)qq_n + o(q),$$

and

$$\sum_{j=1}^{m} h_j = \frac{(\beta'^2 - \alpha^2)T^3}{\pi^2} + O(T^2 \log T) = \frac{\beta'^2 - \alpha^2}{2(\beta - \alpha)}(b - a)qq_n + o(q)$$

which are consequences of Lemma 6 and the definition of T. Thus (18) results from

$$\frac{\beta' - \alpha}{\beta - \alpha} = 1 - 1/\sqrt{2} = c_4 < 1.$$

More precisely, this shows that, for any $\varepsilon > 0$ and q large enough,

$$\lambda([x_A, x_{P_m}]) \le (c_4 + \varepsilon)\lambda([x_A, x_B]). \tag{20}$$

For (19), we write

$$\lim_{q\to\infty} s_{P_m B} = \lim_{q\to\infty} \frac{y_B - \big(y_A + (x_B - x_A)\frac{\beta'^2-\alpha^2}{2(\beta-\alpha)} + o(1)\big)}{x_B - \big(x_A + (x_B - x_A)\frac{\beta'-\alpha}{\beta-\alpha} + o(1)\big)} = \frac{s_{A,B}(\beta-\alpha) - (\beta'^2-\alpha^2)/2}{\beta-\beta'}.$$

This shows that condition (19) becomes, on taking q sufficiently large,

$$\left|\left(s_{A,B} - \frac{\alpha+\beta}{2}\right)(\beta-\alpha)\right| < \frac{\gamma^2}{2},$$

which holds in view of condition (5). Thus our choice for γ is suitable and we can regularize as required.

As there are only a finite number of such constructions to be done, we can build in that way, and for any q large enough, a curve C_{n+1} by collecting all local constructions: the $S_{n+1,i}$'s are the collection of all intervals constructed above of the form $[x_{P_{j-1}}, x_{P_j}]$ with $k_j > T/2$ (and $j \ne 1, m$), the $\alpha_{n+1,i}$'s and $\beta_{n+1,i}$'s being the corresponding u_j and v_j. Condition (4) is an easy consequence of our construction: refining the curve, we have not affected the global convexity because every slope of the refinement stays strictly between α and β (see (17) and (19) and look at our way to regularize). Condition (5) is a consequence of Lemma 3 (formula (8)) on Farey fractions.

We have to prove that condition (13) is satisfied for our new curve. But, for every i,

$$\frac{\lambda(S_{n+1,i})}{\beta_{n+1,i} - \alpha_{n+1,i}} = \left(\frac{k_j}{qq_n}\right)\frac{k_j(k_{j-1}+k_j)(k_{j+1}+k_j)}{k_{j-1}+2k_j+k_{j+1}}$$

for some j such that $k_j > T/2$. By (7) this is

$$= \frac{k_j^2}{qq_n}\left(\frac{1}{k_{j-1}+k_j} + \frac{1}{k_{j+1}+k_j}\right)^{-1} > \frac{Tk_j^2}{2qq_n} > \frac{T^3}{8qq_n}.$$

Now, in view of (16), for large T (that is, large q),

$$\frac{T^3}{8qq_n} \sim \frac{\pi^2}{16}\left(\frac{b-a}{\beta-\alpha}\right) \ge \frac{\pi^2}{16}c_2^n = c_2^{n+1},$$

because of (13) for n, and so

$$\frac{\lambda(S_{n+1,i})}{\beta_{n+1,i} - \alpha_{n+1,i}} \ge c_2^{n+1},$$

as required.

Thus C_{n+1} is a qq_n-well weighted curve and condition (11) holds.

4.2. *Global bounds.* We keep the notations of the preceding section.

4.2.1. *Cardinality.* On each $]a,b[$ as above, the number of qq_n-integer points constructed in this way is, by Lemma 6, at least

$$\begin{aligned} m = \sum_{1\le j\le m} 1 &\sim \frac{3}{\pi^2}(\beta'-\alpha)T^2 \\ &\sim c_3(\beta-\alpha)^{1/3}(qq_n(b-a))^{2/3} \ge (c_3-\varepsilon/K)(\beta-\alpha)^{1/3}(qq_n(b-a))^{2/3}, \end{aligned}$$

for q sufficiently large. And condition (13) implies this is

$$\begin{aligned} &\ge (c_3-\varepsilon/K)(\beta-\alpha)^{1/3}(qq_n)^{2/3}(c_2^n(\beta-\alpha))^{2/3} \\ &= (c_3-\varepsilon/K)c_2^{2n/3}(\beta-\alpha)(qq_n)^{2/3}. \end{aligned}$$

Gathering all contributions, we thus get

$$\begin{aligned} \left|\mathcal{C}_{n+1}\cap\left(\frac{1}{qq_n}\mathbb{Z}\right)^2\right| &\ge (c_3-\varepsilon/K)c_2^{2n/3}\sum_{i\in\mathcal{E}_n}(\beta_{n,i}-\alpha_{n,i})(qq_n)^{2/3} \\ &\ge (c_3-\varepsilon/K)c_2^{2n/3}\nu(\mathcal{C}_n)(qq_n)^{2/3} \\ &\ge (c_3K-\varepsilon)(qq_n)^{2/3}/K^{n+1}, \end{aligned}$$

in view of (12) and the definition of K. This shows that (15) holds for $n+1$.

4.2.2. *Free measure.* Now we turn our attention to $\nu(\mathcal{C}_{n+1})$. With Lemma 3, one has

$$\begin{aligned} \nu(\mathcal{C}_{n+1}|_{[a,b]}) &= \sum_{1<j<m,\,k_j>T/2}\frac{k_{j-1}+2k_j+k_{j+1}}{k_j(k_{j-1}+k_j)(k_{j+1}+k_j)} \\ &> \sum_{1<j<m,\,k_j>T/2}\frac{1}{Tk_j} \sim \frac{3(\beta'-\alpha)}{\pi^2} = \frac{3c_4(\beta-\alpha)}{\pi^2}, \end{aligned}$$

in view of Corollary 1. So, for q sufficiently large, we get

$$\nu(\mathcal{C}_{n+1}|_{S_{n,i}}) \ge \frac{3c_4(\beta_{n,i}-\alpha_{n,i})}{\pi^2}.$$

This yields, by adding all contributions,

$$\nu(\mathcal{C}_{n+1}) \ge \sum_{i\in\mathcal{E}_n}\frac{3c_4(\beta_{n,i}-\alpha_{n,i})}{\pi^2} \ge c_1\nu(\mathcal{C}_n) \ge c_1^{n+1}.$$

This shows (12) for $n+1$.

4.2.3. *Upper bound for affine parts.* Inequality (14) follows easily from (20).

4.2.4. *Definition of* q_{n+1}. We take $q_{n+1} = qq_n$ with a q large enough to give a meaning to all previous inequalities and satisfying $qq_n \geq r_{n+1}$. Naturally, condition (10) is trivially checked.

This concludes the proof of the validity of our induction construction.

5. Proof of Theorem 2. For each x in $[0, 1]$, the sequence $(f_n(x))_{n\geq 1}$ is decreasing by construction (see Lemma 1). As it remains positive (because f_n is an increasing function with $f_n(0) = 0$), the sequence $(f_n)_{n\geq 1}$ is convergent. Let f denote its limit and $\mathcal{C}$ the graph of f. Then f is strictly convex: on the one hand, convexity is trivial, on the other hand, no part of $\mathcal{C}$ with a strictly positive Lebesgue measure can be affine by (14) with $\varepsilon < 1/\sqrt{2}$.

Moreover, the number of q_n-integer points on the limit curve cannot be less than that one of q_n-integer points on the graph of f_n, because these points are fixed for $m \geq n$, so we get, for all integer n, the lower bound

$$\left|\mathcal{C} \cap \left(\frac{1}{q_n}\mathbb{Z}\right)^2\right| \geq (c_3K - \varepsilon)\frac{q_n^{2/3}}{K^n} \gg \frac{q_n^{2/3}}{K^n},$$

if ε has been chosen such that $\varepsilon < c_3K$.

6. Conclusion. In this paper, we have proved, for the number of q_n-integer points on a strictly convex curve, a lower bound of the shape

$$\frac{q_n^{2/3}}{D(n)},$$

with a deviation (with respect to Jarník's result) function $D(n) = K^n$.

Naturally, the best deviation one can expect is $D = 1$ because of Jarník's result. Can D be taken equal to 1? More modestly, is it for example possible to have $D(n)$ polynomial (this would mean that the infimum of possible K is 1) or at least, can one give a further improvement on the value of K?

Another question is: what can be said about the q_n's? More generally, what can be said about the relation between the growth order of $(q_n)_{n\geq 1}$ and the deviation D? Arithmetical questions arise too: is it possible to restrict the q_n's to be prime?

Finally, what happens in higher dimensions?

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