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On the Iwasawa λ-invariants of quaternion extensions

by

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Dedicated to the memory of
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For a prime number l and a number field k, denote by $\lambda_l(k)$ the Iwasawa λ-invariant associated with the ideal class group of the cyclotomic $\mathbb{Z}_l$-extension $k_\infty(l)$ over k. It is conjectured that this invariant is zero for any prime l and any totally real number field k (cf. [7]). Several authors have given some sufficient conditions for the conjecture when k is a real abelian field (cf. [1]–[9]). Using them, many examples of the vanishing of λ-invariants for real abelian number fields are given. However, it seems that an example of a totally real non-abelian field has not yet been given. In this paper we give quaternion extensions K over the rational number field $\mathbb{Q}$ with $\lambda_2(K) = 0$. A Galois extension K over $\mathbb{Q}$ is called a *quaternion extension* if the Galois group $G(K/\mathbb{Q})$ of K over $\mathbb{Q}$ is isomorphic to the quaternion group H_8 of order 8. The quaternion group H_8 is a group $H_8 = \langle\sigma,\tau\rangle$ of order 8 with $\sigma^4 = 1$, $\sigma^2 = \tau^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$.

The main purpose of this paper is to prove the following:

THEOREM. *Let p be a prime number with $p \equiv 3 \pmod 8$, $k = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ and $k_\infty(2)$ the cyclotomic $\mathbb{Z}_2$-extension of k. Then there exist natural numbers x, y with $x^2 - y^2p = 2p$. Let $K_\infty(2)$ be the cyclotomic $\mathbb{Z}_2$-extension of $K = k(\sqrt{(x + y\sqrt{p})(2 + \sqrt{2})})$. Then the Galois group $G(K/\mathbb{Q})$ of K over $\mathbb{Q}$ is isomorphic to the quaternion group H_8 and the λ-invariant $\lambda(K_\infty(2)/K)$ of $K_\infty(2)$ over K vanishes.*

First we recall the following lemma which plays an important role in our proof of this theorem:

LEMMA (cf. [2]). *Let l be a prime number, k a totally real number field of finite degree and K a real cyclic extension of degree l over k. Assume that*

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$k_\infty(l)$ has only one prime ideal lying over l and that the class number h_k of k is not divisible by l. Then the following are equivalent:

(1) $\lambda(K_\infty(l)/K) = 0$.

(2) *For any prime ideal w of $K_\infty(l)$ which is prime to l and ramified in $K_\infty(l)/k_\infty(l)$, the order of the ideal class of w is prime to l.*

Proof (of Theorem). Since $p \equiv 3 \pmod 8$, we have $N_{\mathbb{Q}(\sqrt{p})/\mathbb{Q}}(\mathbb{Q}(\sqrt{p})^\times) \not\ni -1$. Hence the cardinality of the ambiguous classes of $\mathbb{Q}(\sqrt{p})$ is equal to one, which shows that a prime ideal of $\mathbb{Q}(\sqrt{p})$ lying above 2 is principal. Therefore there exist integers x, y with $x^2 - py^2 = 2p$ by $\left(\frac{-2}{p}\right) = 1$. We put $\alpha = \sqrt{(2+\sqrt{2})(x+y\sqrt{p})}$. Now, let σ, τ be elements of the Galois group $G(k/\mathbb{Q})$ with $\sqrt{2}^\sigma = -\sqrt{2}$, $\sqrt{p}^\sigma = \sqrt{p}$, $\sqrt{2}^\tau = \sqrt{2}$ and $\sqrt{p}^\tau = -\sqrt{p}$. Then we have $(\alpha^2)(\alpha^2)^\sigma = 2(x+y\sqrt{p})^2$ and $(\alpha^2)(\alpha^2)^\tau = 2p(2+\sqrt{2})^2$, which shows that K is a Galois extension over $\mathbb{Q}$. For simplicity, we denote by σ, τ extensions of σ, τ to K with $\alpha^\sigma = \sqrt{2}\alpha^{-1}(x+y\sqrt{p})$ and $\alpha^\tau = \sqrt{2p}\alpha^{-1}(2+\sqrt{2})$. Then we can easily see $G(K/\mathbb{Q}) = \langle\sigma,\tau\rangle$, $\sigma^4 = 1$, $\sigma^2 = \tau^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. Hence $G(K/\mathbb{Q})$ is isomorphic to H_8.

Now, we prove $\lambda(K_\infty(2)/K) = 0$. First we notice that the class number $h_{\mathbb{Q}(\sqrt{p})}$ is not divisible by 2. Therefore h_k is not divisible by 2, since 2 is fully ramified in k over $\mathbb{Q}$ and since p is unramified in k over $\mathbb{Q}(\sqrt{p})$. One should also remark that the infinite primes are unramified. Let $\mathfrak{P}_p$ be a prime ideal of K lying above p, $\mathfrak{p}_2$ a prime ideal of k lying above 2 and $\mathfrak{p}_p$ a prime ideal of k lying above p. Then we can see $((x+y\sqrt{p})(2+\sqrt{2})) = \mathfrak{p}_p^2\mathfrak{p}_2^2$. Hence we have $(\alpha) = \mathfrak{P}_p(2+\sqrt{2})$ in K. This shows that $\mathfrak{P}_p$ is a principal ideal of K. Therefore $\lambda(K_\infty(2)/K) = 0$ follows from the Lemma or [7, Lemma 3]. This completes our proof.

REMARK. Since there exist infinitely many prime numbers p with $p \equiv 3 \pmod 8$ which are unramified in $k/\mathbb{Q}(\sqrt{p})$, there exist infinitely many quaternion extensions K with $\lambda(K_\infty(2)/K) = 0$.

References

[1] T. Fukuda and K. Komatsu, *On $\mathbb{Z}_p$-extensions of real quadratic fields*, J. Math. Soc. Japan 38 (1986), 95–102.

[2] T. Fukuda, K. Komatsu, M. Ozaki and H. Taya, *Iwasawa λ_p-invariants of relative cyclic extensions of degree p*, Tokyo J. Math. 20 (1997), 475–480.

[3] T. Fukuda and H. Taya, *The Iwasawa λ-invariants of $\mathbb{Z}_p$-extensions of real quadratic fields*, Acta Arith. 69 (1995), 277–292.

[4] R. Greenberg, *On the Iwasawa invariants of totally real number fields*, Amer. J. Math. 98 (1976), 263–284.

[5] H. Ichimura and H. Sumida, *On the Iwasawa λ-invariants of certain real abelian fields*, to appear.
[6] —, —, *On the Iwasawa λ-invariants of certain real abelian fields II*, to appear.
[7] K. Iwasawa, *A note on capitulation problem for number fields II*, Proc. Japan Acad. 65 (1989), 183–186.
[8] J. S. Kraft and R. Schoof, *Computing Iwasawa modules of real quadratic number fields*, Compositio Math. 97 (1995), 135–155.
[9] M. Ozaki and H. Taya, *On the Iwasawa λ_2-invariants of certain families of real quadratic fields*, submitted for publication (1996).

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