ACTA ARITHMETICA
LXXXVII.2 (1998)

An example in Beurling's theory of primes

by

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1. Introduction. Let $P = \{p_1, p_2, \ldots\}$ be a set of numbers satisfying the following three conditions:

$$1 < p_1, \qquad p_k \le p_{k+1}, \qquad \lim_{k\to\infty} p_k = \infty.$$

Let N be the set of numbers which are finite products of elements of P. Following A. Beurling [1], we call P a set of *generalized primes* (g-primes) and N a set of *generalized integers* (g-integers). We denote by $P(x)$ and $N(x)$ the counting functions of the sets P and N. Two interesting problems arise. First, we have the *Inverse Problem*: given the knowledge of the distribution of the set N of g-integers, obtain information about the distribution of the set P of g-primes. The classical Prime Number Theorem is an example of an inverse problem. See also the above cited reference to Beurling [1], as well as H. G. Diamond [2], where the Prime Number Theorem is proved in the setting of the Theory of Generalized Numbers. We also have the *Direct Problem*: given the knowledge of the distribution of the elements in P, obtain information about the distribution of those of N. The Direct Problem has been studied by (among others) P. Malliavin [4] and H. G. Diamond [3]. We now summarize their results.

Assume that the distribution of the primes is given by

$$P(x) = \mathrm{Li}(x) + O(xe^{-\sqrt{\log x}}). \tag{1}$$

We are interested in estimating $N(x)$. P. Malliavin [4] proved that

$$N(x) = cx + O(x\exp\{-\theta(\log x)^{0.2}\})$$

for some positive constants c and θ. If we let

$$A = \text{l.u.b.}\{a : N(x) = cx + O(x\exp\{-\theta(\log x)^a\})\},$$

1991 *Mathematics Subject Classification*: Primary 11N80.

This article, with minor changes, is part of the author's Ph.D. dissertation, written at the University of Illinois, Urbana-Champaign, under the direction of Prof. H. G. Diamond.

then the following estimates are true:

$$\begin{aligned} &A \geq 0.2 && \text{(Malliavin, 1961)}, \\ &A \geq 0.333\ldots && \text{(Diamond, 1970)}. \end{aligned}$$

In this article we construct a generalized number system to prove that $A \leq 0.5$. That is, we prove the following

THEOREM. *There exists a continuous distribution $P(x)$ of prime numbers for which* (1) *holds and furthermore*

$$N(x) = cx + \Omega(xe^{-\theta\sqrt{\log x}}) \tag{2}$$

for some positive constants c and θ.

The words "continuous distribution" in the theorem deserve an explanation. They are to mean that there exists a measure dP (see (4) below) whose integral $\int_1^x dP$ is a continuous function in the interval $(1, \infty)$. We use this measure as input in the exponential formula

$$dN = e^{dP} = \delta + dP + \frac{dP * dP}{2!} + \frac{(dP)^{*3}}{3!} + \ldots, \tag{3}$$

where δ is the Dirac measure placed at the point 1 and $*$ is the multiplicative (Dirichlet) convolution for measures; the measure $dP * dP$ assigns to each Borel set E the numerical value

$$\iint_{st \in E} dP(s) \cdot dP(t).$$

Moreover, the "set of integers" N is given by $N(x) = \int_1^x e^{dP}$. We will also make use of the zeta function of this "set of integers" as given by

$$\zeta(s) = \int_1^\infty x^{-s}\, dN(x) = \int_1^\infty x^{-s} e^{dP} = \exp\Big\{\int_1^\infty x^{-s}\, dP(x)\Big\}.$$

Finally, we point out that the exponential formula (3) gives the counting measure for the set of ordinary integers when we take dP to be the measure $d\Pi$ where

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \ldots,$$

and $\pi(x)$ is the number of ordinary primes not greater than x. For a more detailed treatment of these notions see Diamond [3].

2. The example. We start by letting

$$P(x) = \int_1^x \frac{1 - t^{-k}}{\log t} \gamma(t)\, dt, \tag{4}$$

where

$$\gamma(t) = 1 - \sum_{n>n_0} \alpha_n \frac{\cos(b_n \log t)}{t^{a_n}}, \qquad t \ge 1. \tag{5}$$

We think of k, n_0, $\{\alpha_n\}$, $\{a_n\}$ and $\{b_n\}$ as parameters to be chosen. The following constants serve our purposes (however, we will carry out some of our calculations for an arbitrary choice of these parameters):

$$\begin{aligned} k &= 4, & n_0 &= 3, \\ b_n &= \exp\{(\log x_n)^{1/2}\}, & a_n &= 1/\log b_n = 1/\sqrt{\log x_n}, \\ x_1 &= e^{10}, & x_{n+1} &= \exp\{(\log x_n)^2\}, \\ T_n &= \exp\{(\log x_n)^{3/4}\}, & \alpha_n &= 2/n^2. \end{aligned}$$

We also define $\alpha = \sum_{n>n_0} \alpha_n$, and for $n > n_0$ we let $\alpha_{-n} = \alpha_n$, $a_{-n} = a_n$ and $b_{-n} = -b_n$. We make constant use of this notation.

In order that we may consider $P(x)$ to be a counting function, it is necessary that it be non-decreasing. This is a consequence of the following easy lemma.

LEMMA 1. *With γ as in* (5), *we have $\gamma(t) \ge 0$.*

3. Estimation of $P(x)$. Now we show that (1) holds. In fact we have a slightly better estimate for $P(x)$.

PROPOSITION 2. *If $P(x)$ is given by* (4) *and $k > 1$, then*

$$P(x) = \mathrm{Li}(x) + O(xe^{-2\sqrt{\log x}}).$$

Proof. We have

$$\begin{aligned} P(x) &= \int_1^x \frac{1-t^{-k}}{\log t}\Big(1 - \sum_{n>n_0} \alpha_n \frac{\cos(b_n \log t)}{t^{a_n}}\Big)\,dt \\ &= \int_e^x \frac{1-t^{-k}}{\log t}\,dt - \sum_{n>n_0} \alpha_n \int_e^x \frac{1-t^{-k}}{\log t} \cdot \frac{\cos(b_n \log t)}{t^{a_n}}\,dt + O(1) \\ &= \int_e^x \frac{dt}{\log t} - \sum_{n>n_0} \alpha_n \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}\,dt + O(1), \end{aligned}$$

because $k > 1$. Now we show that

$$\sum_{n>n_0} \alpha_n \int_e^x \frac{\cos(b_n \log t)}{t^{a_n} \log t}\,dt = O(xe^{-2\sqrt{\log x}}).$$

To this end, notice that

$$\begin{aligned}
\Big|\int_e^x \frac{\cos(b_n \log t)}{t^{a_n}\log t}\,dt\Big| &= \Big|\int_1^{\log x} \frac{\cos(b_n t)}{t}\, e^{t(1-a_n)}\,dt\Big| \\
&= \Big|\Big[\frac{\sin(b_n t)}{b_n t}\, e^{t(1-a_n)}\Big]_1^{\log x} \\
&\quad - \frac{1}{b_n}\int_1^{\log x} \sin(b_n t)\,\frac{e^{t(1-a_n)}}{t}\Big(1 - a_n - \frac{1}{t}\Big)\,dt\Big| \\
&\le 2\frac{x^{1-a_n}}{b_n \log x} + \frac{1}{b_n}\int_1^{\log x} \frac{e^{t(1-a_n)}}{t}\,dt.
\end{aligned}$$

To estimate the last integral we notice that $e^{t(1-a_n)}/t$ reaches a minimum value at $t = 1/(1-a_n) \to 1$ as $n \to \infty$. After attaining its minimum the integrand increases to infinity, thus, the largest value of the integrand is achieved when $t = \log x$. Hence

$$\Big|\int_e^x \frac{\cos(b_n \log t)}{t^{a_n}\log t}\,dt\Big| \le 2\,\frac{x^{1-a_n}}{b_n \log x} + \frac{1}{b_n}\cdot\frac{x^{1-a_n}}{\log x}\log x \le 3\,\frac{x^{1-a_n}}{b_n}.$$

By the definition of a_n and b_n we have

$$\frac{x^{1-a_n}}{b_n} = x\,\exp\Big\{-\frac{\log x}{\sqrt{\log x_n}} - \sqrt{\log x_n}\Big\}.$$

Let $A^2 = \log x$ and $B^2 = \log x_n$. From $(A-B)^2 \ge 0$, we deduce that $-A^2/B - B \le -2A$. Hence

$$\frac{x^{1-a_n}}{b_n} = x\,\exp\Big\{-\frac{A^2}{B} - B\Big\} \le xe^{-2A} = xe^{-2\sqrt{\log x}}.$$

Therefore

$$\sum_{n>n_0} \alpha_n \Big|\int_e^x \frac{\cos(b_n \log t)}{t^{a_n}\log t}\,dt\Big| \le \sum_{n>n_0} 3\alpha_n x e^{-2\sqrt{\log x}} = 3\alpha x e^{-2\sqrt{\log x}}.$$

This proves Proposition 2. ∎

4. Estimation of $N(x)$. We now define our zeta function for $s = \sigma + it$ with $\sigma > 1$ as

$$\zeta(s) = \int_1^\infty x^{-s}\,dN(x) = \exp\Big\{\int_1^\infty x^{-s}\,dP(x)\Big\}. \tag{6}$$

By inverting the first equation in (6) we get

$$N(x)=\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\zeta(s)\frac{x^s}{s}\,ds,\qquad b>1.$$

Furthermore, if

$$M(x)=\int\limits_1^x N(t)\,dt,$$

then

$$M(x)=\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\zeta(s)\frac{x^{s+1}}{s(s+1)}\,ds,\qquad b>1. \tag{7}$$

To prove that (2) is true it suffices to show that for some $\theta^*<\theta$,

$$M(x)=\frac{c}{2}x^2+\Omega(x^2e^{-\theta^*\sqrt{\log x}}). \tag{8}$$

Indeed, if (2) is not true then

$$N(x)=cx+o(xe^{-\theta\sqrt{\log x}}),$$

so that

$$M(x)=\int\limits_1^x\{ct+o(t\,e^{-\theta\sqrt{\log t}})\}\,dt=\frac{c}{2}x^2+o(x^2e^{-\theta^*\sqrt{\log x}}).$$

This is a contradiction.

We will prove that (8) is true for $\theta^*=3+\varepsilon$ and hence that (2) holds for $\theta>3$.

The next proposition is important for our purposes because it expresses our zeta function as defined in (6) as an infinite product, from which we can read off its singularities. As a matter of fact, we have chosen the parameters $\{a_n\}$ and $\{b_n\}$ in such a way as to give us a distribution of the singularities of $\zeta(s)$ in the complex plane, from which we can deduce the oscillation statement (8). In order to state it recall that for $n>n_0$, we have $a_{-n}=a_n$, $b_{-n}=-b_n$ and $\alpha_{-n}=\alpha_n$.

Proposition 3. *Consider the zeta function*

$$\zeta(s)=\exp\Big\{\int\limits_1^\infty x^{-s}\,dP(x)\Big\},$$

where dP is given by (4) *above. Then, for $\Re(s)>1$,*

$$\zeta(s)=\frac{s+k-1}{s-1}\prod_{|n|>n_0}\left(1-\frac{k}{s-1+a_n-ib_n+k}\right)^{\alpha_n/2}. \tag{9}$$

Lemma 4. *If we define $\gamma(t)$ by* (5) *and*

$$\gamma_N(t)=1-\sum_{n_0<n\le N}\alpha_n\frac{\cos(b_n\log t)}{t^{a_n}},\qquad t\ge 1,$$

then $\gamma_N(t)$ converges uniformly to $\gamma(t)$ for $t \ge 1$ and in fact $|\gamma(t) - \gamma_N(t)| \le 2/N$.

Proof (of Proposition 3). Notice first that

$$\frac{\cos(b\log t)}{t^a} = \frac{1}{2t^a}(e^{ib\log t} + e^{-ib\log t}) = \frac{1}{2}(t^{-a+ib} + t^{-a-ib}).$$

Hence

$$\frac{\cos(b\log t)}{t^a} \cdot \frac{1-t^{-k}}{\log t}\,dt = \frac{1}{2}(t^{-a-ib} + t^{-a+ib})\frac{1-t^{-k}}{\log t}\,dt.$$

Thus, for $\Re(s) > 1$, we have

$$-\frac{d}{ds}\int_1^\infty t^{-s}\frac{\cos(b\log t)}{t^a} \cdot \frac{1-t^{-k}}{\log t}\,dt$$
$$= \frac{1}{2}\int_1^\infty (t^{-s-a-ib} + t^{-s-a+ib} - t^{-s-a-ib-k} - t^{-s-a+ib-k})\,dt$$
$$= \frac{d}{ds}\log\left[\left(1 - \frac{k}{s-1+a+ib+k}\right)^{1/2}\left(1 - \frac{k}{s-1+a-ib+k}\right)^{1/2}\right].$$

Therefore, we have

$$-\int_1^\infty t^{-s}\frac{\cos(b\log t)}{t^a} \cdot \frac{1-t^{-k}}{\log t}\,dt$$
$$= \log\left\{\left(1 - \frac{k}{s-1+a+ib+k}\right)^{1/2}\left(1 - \frac{k}{s-1+a-ib+k}\right)^{1/2}\right\} + \text{Const.}$$

By taking the limit as $\Re(s)$ tends to infinity we see that the constant of integration is zero.

Similarly, we have

$$\int_1^\infty t^{-s}\frac{1-t^{-k}}{\log t}\,dt = \log\frac{s+k-1}{s-1}.$$

Thus, from the definition of γ_N, we get

$$\int_1^\infty t^{-s}\frac{1-t^{-k}}{\log t}\,\gamma_N(t)\,dt$$
$$= \log\left(\frac{s+k-1}{s-1}\right) + \sum_{n_0<|n|\le N}\alpha_n\log\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{1/2}$$
$$= \log\left\{\frac{s+k-1}{s-1}\prod_{n_0<|n|\le N}\left(1 - \frac{k}{s-1+a_n-ib_n+k}\right)^{\alpha_n/2}\right\}.$$

Taking the limit as $N \to \infty$ yields the assertion. ∎

The infinite product representation of $\zeta(s)$ given in (9) holds not only in the half plane $\Re(s) > 1$, but in a larger region. To show this, we prove the following proposition, which will also give us a useful upper bound for $|\zeta(s)|$ in the extended domain of definition.

PROPOSITION 5. *If* $s = \sigma + it$ *is such that* $\sigma > -k + 2$, $\alpha = \sum_{n > n_0} \alpha_n$, *and if*

$$\varphi(s) = \prod_{|n| > n_0} \left(1 - \frac{k}{s - 1 + a_n - ib_n + k}\right)^{\alpha_n/2}$$

then

$$|\varphi(s)| \le (k+1)e^{\alpha}.$$

Proof. Let $s = \sigma + it$ be given. Since $b_n = \exp\{\sqrt{\log x_n}\}$ we have

$$\begin{aligned} b_{n+1} - b_n &= \exp\{\sqrt{\log x_{n+1}}\} - \exp\{\sqrt{\log x_n}\} \\ &= \exp\{\log x_n\} - \exp\{\sqrt{\log x_n}\} \ge x_n - \sqrt{x_n} \ge 4k \end{aligned}$$

for all n, since $x_n \ge x_1 = e^{10}$ and $k < 5000$. Therefore the interval $(t - 2k, t + 2k)$ contains at most one element of $\{b_n\}$. Denote this element (if it exists!) by $b_{n(t)}$. We now write

$$|\varphi(s)| = \left|1 - \frac{k}{s - 1 + a_{n(t)} - ib_{n(t)} + k}\right|^{\alpha_{n(t)}/2} \prod_{\substack{|n| > n_0 \\ n \ne n(t)}} \left|1 - \frac{k}{s - 1 + a_n - ib_n + k}\right|^{\alpha_n/2}.$$

Since $a_n > 0$, we have $\sigma - 1 + a_n + k > \sigma - 1 + k > 1$ and hence

$$\left|1 - \frac{k}{s - 1 + a_{n(t)} - ib_{n(t)} + k}\right|^{\alpha_{n(t)}/2} \le \left|1 + \frac{k}{\sigma - 1 + k}\right|^{\alpha_{n(t)}/2} \le 1 + k.$$

On the other hand, when $n \ne n(t)$,

$$\begin{aligned} \left|1 - \frac{k}{s - 1 + a_n - ib_n + k}\right|^{\alpha_n/2} &= \exp\left\{\frac{\alpha_n}{2} \log\left|1 - \frac{k}{s - 1 + a_n - ib_n + k}\right|\right\} \\ &= \exp\left\{\frac{\alpha_n}{2} \Re \log\left(1 - \frac{k}{s - 1 + a_n - ib_n + k}\right)\right\} \\ &= \exp\left\{\frac{\alpha_n}{2} \Re\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \dots\right)\right\}, \end{aligned}$$

where

$$|z| = \left|\frac{k}{s-1+a_n-ib_n+k}\right| \le \frac{k}{|\Im(s)-b_n|} = \frac{k}{|t-b_n|} \le \frac{k}{2k} = \frac{1}{2}.$$

Therefore

$$\begin{aligned}|\varphi(s)| &\le (k+1)\prod_{\substack{|n|>n_0\\ n\ne n(t)}} \exp\left\{\frac{\alpha_n}{2}\left(|z| + \frac{|z|^2}{2} + \frac{|z|^3}{3} + \ldots\right)\right\}\\ &\le (k+1)\exp\left\{\frac{1}{4}\sum_{|n|>n_0}\alpha_n\left(1+\frac{1}{2}+\frac{1}{4}+\ldots\right)\right\}\\ &= (k+1)\exp\left\{\frac{1}{2}\sum_{|n|>n_0}\alpha_n\right\} = (k+1)e^{\alpha}\end{aligned}$$

as claimed. ■

For the particular case $k=4$, $n_0=3$, and $\alpha_n = 2n^{-2}$ we have

$$|\varphi(s)| \le 5\exp\left\{2\sum_{n>3}\frac{1}{n^2}\right\} < 9 \quad \text{if } \sigma > -2.$$

Let D_ζ be the region defined by

$$\begin{aligned}D_\zeta = \{s = \sigma+it \in \mathbb{C} : \sigma > -k+2,&\\ s \ne \xi(1-a_n+ib_n) + (1-\xi)(1-a_n+ib_n-k)&\\ \text{for any } 0\le\xi\le 1,\ |n|\ge n_0\}.&\end{aligned}$$

By a theorem of Weierstrass on the uniform convergence of analytic functions, we know that $\varphi(s)$ is analytic on D_ζ. The equation

$$\zeta(s) = \frac{s+k-1}{s-1}\varphi(s), \quad \sigma > 1,$$

gives us an analytic continuation of $\zeta(s)$ to D_ζ with $s=1$ removed, where $\zeta(s)$ has a simple pole. Notice that, since the zeros of $\varphi(s)$ are of fractional order, we avoid problems of multiple-valuedness by restricting the domain of definition of $\zeta(s)$ to D_ζ.

COROLLARY 6. *For $s \in D_\zeta$ such that $|s-1|>1$ we have $|\zeta(s)| \le 45$.*

Proof. We have

$$|\zeta(s)| = \left|\frac{s+k-1}{s-1}\varphi(s)\right| \le 9\left|1+\frac{k}{s-1}\right| \le 9\left(1+\frac{4}{|s-1|}\right). \ ■$$

Our next step is to estimate $M(x)$ (cf. (7)) where

$$x = x_n\left(1+\frac{\theta_1}{\log x_n}\right), \quad |\theta_1| < 1. \tag{10}$$

REMARK. It would be more accurate to write w_n, for example, in place of x. We prefer to write x in order to keep our formulas simple. We will choose θ_1 (and hence x) in such a way that $M(x)$ equals the main term cx, plus a large "error term".

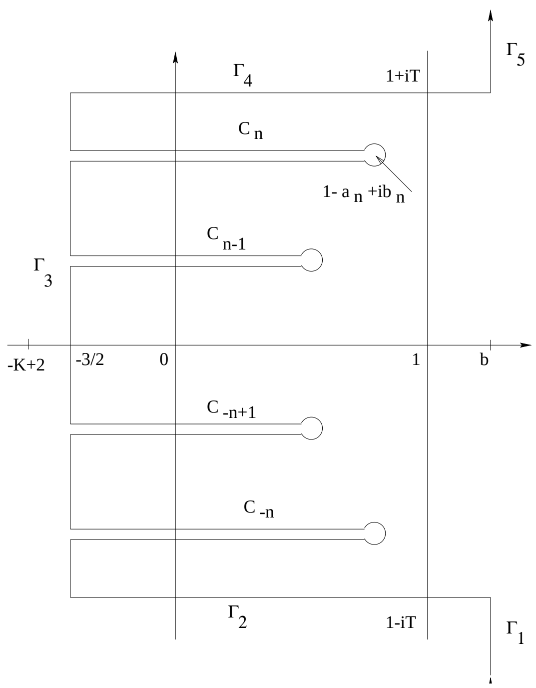

Fig. 1

We deform the vertical path of integration in the inversion formula (7) from the path $\Re(s) = b > 1$ to a path $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3^* \cup \Gamma_4 \cup \Gamma_5 \subset D_\zeta$ (see Figure 1). Here Γ_1 joins $b - i\infty$ to $b - iT_n$, where $T_n = \exp\{(\log x_n)^{3/4}\}$. The points $b - iT_n$ to $-3/2 + iT_n$ are joined by Γ_2. The segments Γ_5 and Γ_4 are symmetric to Γ_1 and Γ_2 with respect to the horizontal axis. We denote by Γ_3^* a comb formed by horizontal loops C_m, $n_0 < |m| \le n$, each going around the singular point $1 - a_m + ib_m$. The collection of vertical line segments joining one loop to the next one is denoted by Γ_3. The points on Γ_3 have real part equal to $-3/2$. Furthermore, each C_m is made up of two horizontal line segments joined at the right hand side by a small circle with center at $1 - a_m + ib_m$. The two horizontal line segments of C_m are extended to the left until they meet Γ_3.

Now we write

$$M(x) = I_1 + \ldots + I_5 + J_{-n} + \ldots + J_{+n} + \{\text{residues at } s = 0, 1\}$$

where

$$I_m = \frac{1}{2\pi i} \int_{\Gamma_m} \zeta(s) \frac{x^{s+1}}{s(s+1)}\, ds, \qquad m = 1, \ldots, 5,$$

$$J_m = \frac{1}{2\pi i} \int_{C_m} \zeta(s) \frac{x^{s+1}}{s(s+1)}\, ds, \qquad n_0 < |m| \le n.$$

Here, as above, C_m is the mth horizontal loop with imaginary part equal to b_m.

Consider first the integral I_3. As a matter of fact, in this case we do not have just one integral but many of them. This is because the vertical segment Γ_3 is broken at each horizontal loop C_m. However, on each vertical component of Γ_3 the integrand is bounded by the same constant: 45. Thus, since $\Re(s) = -3/2$ on Γ_3, we have

$$|I_3| \le \frac{1}{2\pi} \int_{-\infty}^{\infty} 45 \frac{x^{-3/2+1}}{|(-3/2+it)(1/2+it)|}\, dt \le \frac{8}{\sqrt{x}} \int_0^{\infty} \frac{dt}{1/4+t^2}$$
$$\le \frac{8}{\sqrt{x}} \Big\{ \int_0^{1/2} 4\, dt + \int_{1/2}^{\infty} \frac{dt}{t^2} \Big\} = \frac{8}{\sqrt{x}} \Big(2 + \frac{1}{1/2} \Big) = \frac{32}{\sqrt{x}}.$$

Let $b = 1 + 1/\log x_n$. Then $|I_2|$ and $|I_4|$ are both less than $58(x/T_n)^2$. Indeed, each is at most

$$\frac{1}{2\pi} \int_{-3/2}^{1+(\log x_n)^{-1}} 45 \frac{x^{2+(\log x_n)^{-1}}}{T_n^2}\, d\sigma$$
$$\le \frac{8}{T_n^2} x^{2+(\log x_n)^{-1}} \int_{-3/2}^{1+1/10} d\sigma$$
$$\le 21 \Big(\frac{x}{T_n} \Big)^2 \exp\Big\{ \frac{1}{\log x_n} \log\Big(x_n \Big(1 + \frac{\theta_1}{\log x_n} \Big) \Big) \Big\}$$
$$\le 21 \Big(\frac{x}{T_n} \Big)^2 \exp\Big\{ 1 + \Big(\frac{1}{\log x_n} \Big)^2 \Big\} \le 58 \Big(\frac{x}{T_n} \Big)^2.$$

Now we consider the integrals I_1 and I_5: each of $|I_1|$ and $|I_5|$ is at most

$$\frac{1}{2\pi} \int_{T_n}^{\infty} 45 \frac{x^{2+(\log x_n)^{-1}}}{t^2}\, dt \le 8x^2 \exp\Big\{ 1 + \Big(\frac{1}{\log x_n} \Big)^2 \Big\} \frac{1}{T_n}$$
$$\le 8 \frac{x^2}{T_n} e^{1.01} \le 22 \frac{x^2}{T_n}.$$

From the above estimates we get

$$M(x) = k\varphi(1)\tfrac{1}{2}x^2 + (1-k)\varphi(0)x + \sum_{m=-n}^{n} J_m + O(x^2/T_n), \tag{11}$$

where the O-constant is less than $2(58+22)+32 = 192$ (because $1/\sqrt{x} \le x^2/T_n$). Let us examine the expression

$$x^2/T_n = x^2 \exp\{-(\log x_n)^{3/4}\}.$$

We want to write x in place of x_n in the above expression. To do this notice first that, since $x_1 \ge e^{10}$, we have $x_{n+1} = x_n^{\log x_n} \ge x_n^{10}$ so that

$$x_n \ge x_{n-1}^{10} \ge x_{n-2}^{10^2} \ge \ldots \ge x_{n-(n-1)}^{10^{n-1}} \ge e^{10^n}.$$

Thus we get $\log x_n \ge 10^n$ and hence $n \le \log\log x_n$.

LEMMA 7. *If* x_n *is as in* (10) *then* $|\log x - \log x_n| \le 2/10^n$.

Proof. We have

$$\begin{aligned}\log x &= \log\left[x_n\left(1 + \frac{\theta_1}{\log x_n}\right)\right], \quad |\theta_1| < 1,\\ &= \log x_n + \log\left(1 + \frac{\theta_1}{\log x_n}\right).\end{aligned}$$

Thus

$$\begin{aligned}|\log x - \log x_n| &\le \frac{|\theta_1|}{\log x_n}\left(1 + \frac{|\theta_1|}{\log x_n} + \left(\frac{|\theta_1|}{\log x_n}\right)^2 + \ldots\right)\\ &\le \frac{1}{\log x_n}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \ldots\right) \le \frac{2}{\log x_n} \le \frac{2}{10^n}. \ \blacksquare\end{aligned}$$

By the mean value theorem of differential calculus, there is a number ξ_n between x and x_n such that

$$|(\log x)^{3/4} - (\log x_n)^{3/4}| = \frac{3}{4}\left(\frac{1}{\log \xi_n}\right)^{1/4} |\log x - \log x_n| \le \frac{2}{10^n}.$$

Hence

$$\begin{aligned}x^2/T_n &\le x^2 \exp\{-(\log x)^{3/4} + 2/10^n\}\\ &\le x^2 \exp\{-(\log x)^{3/4} + 2/10\} \le 2x^2 e^{-(\log x)^{3/4}}.\end{aligned}$$

From this and from equation (11) we get

$$M(x) = k\varphi(1)\tfrac{1}{2}x^2 + \sum_{m=-n}^{n} J_m + O(x^2 e^{-(\log x)^{3/4}}), \tag{12}$$

where the implied O-constant is less than $2\cdot 192 + |1-k|\varphi(0) \le 384 + 3\cdot 45 = 519$.

Now we show that all the terms in the sum (12) other than J_{-n} and J_n can be put into the error term.

PROPOSITION 8. $|\sum_{n_0<|m|\le n-1} J_m| \le 120x^2 e^{-(\log x)^{3/4}}$.

Proof. It is easy to see that the contribution to the integral J_m due to the loop circle centered at $1-a_m+ib_m$ tends to zero as the radius tends to zero. Hence we can write

$$\begin{aligned}|J_m| &= \left|\frac{1}{2\pi i}\int\limits_{C_m}\zeta(s)\frac{x^{s+1}}{s(s+1)}\,ds\right| \\ &\le \frac{1}{\pi}\int\limits_{-3/2}^{1-a_m} 45\frac{x^{2-a_m}}{b_m^2}\,d\sigma \le \frac{15}{b_m^2}x^2 e^{-a_m\log x}.\end{aligned}$$

But if $|m|\le n-1$ then

$$\begin{aligned}e^{-a_m\log x} &\le e^{-a_{n-1}\log x} \\ &= \exp\left\{-\frac{\log x}{(\log x_n)^{1/4}}\right\} \le \exp\left\{\frac{-\log x_n + 2/10^n}{(\log x_n)^{1/4}}\right\} \\ &\le \exp\{-(\log x_n)^{3/4} + 2/10^n\} \le \exp\{-(\log x)^{3/4} + 4/10^n\} \\ &\le \exp\{-(\log x)^{3/4} + 4/10\} \le 2e^{-(\log x)^{3/4}}.\end{aligned}$$

Hence

$$\Big|\sum_{n_0<|m|\le n-1} J_m\Big| \le 30x^2\, e^{-(\log x)^{3/4}} \sum_{|m|>n_0}\frac{1}{b_m^2}.$$

We finish the proof by noting that the last sum is finite:

$$\sum_{|m|>n_0}\frac{1}{b_m^2} \le \sum_{|m|>n_0} e^{-\sqrt{\log x_m}} \le 2\sum_{m>n_0} e^{-10^{m/2}} \le 2\sum_{m>n_0} e^{-m} \le 4. \ \blacksquare$$

Therefore we now see that

$$M(x) = k\varphi(1)\tfrac{1}{2}x^2 + (J_{-n}+J_n) + O(x^2 e^{-(\log x)^{3/4}}), \tag{13}$$

where the implied O-constant is less than $519 + 120 = 639$.

It remains to study the expression $J_{-n}+J_n$.

Denote by J_n' and J_n'' the integrals along the line segments C_n' and C_n'' lying respectively above and below the branch cut C_n so that $J_n = J_n' + J_n''$. Now, if we write

$$s = 1 - a_n + ib_n + te^{i\theta}, \qquad -\pi \le \theta < \pi,$$

then the line segment C_n'' is obtained by letting $\theta = -\pi$ and t run from 0 to

$1 - a_n + 3/2$. In this way we obtain C_n'' with its direction reversed:

$$-C_n'' : \begin{cases} \theta = -\pi, \\ s = 1 - a_n + ib_n - t, \\ ds = -dt, \\ 0 \le t \le 1 - a_n + 3/2. \end{cases}$$

Hence

(14) $\quad J_n''$

$$= \frac{-1}{2\pi i}\Big\{\int\limits_0^{(\log x)^{-1/4}} + \int\limits_{(\log x)^{-1/4}}^{1-a_n+3/2}\Big\} \frac{\zeta(1 - a_n + ib_n - t)x^{2-a_n+ib_n-t}}{(1 - a_n + ib_n - t)(2 - a_n + ib_n - t)}(-dt)$$

$$= \frac{1}{2\pi i}\int\limits_0^{(\log x)^{-1/4}} \frac{\zeta(1 - a_n + ib_n - t)\, x^{2-a_n+ib_n-t}}{(1 - a_n + ib_n - t)(2 - a_n + ib_n - t)}\, dt + O(x^2 e^{-(\log x)^{3/4}}),$$

with the O-constant less than 1. Indeed, since $b_n^2 \ge 10$,

$$\Big|\frac{1}{2\pi i}\int\limits_{(\log x)^{-1/4}}^{1-a_n+3/2} \ldots dt\Big| \le \frac{1}{2\pi} \cdot \frac{45}{b_n^2} x^{2-(\log x)^{-1/4}}(1 - a_n + 3/2 - (\log x)^{-1/4})$$

$$\le \frac{5}{4\pi} \cdot \frac{45}{b_n^2} x^2 \exp\Big\{-\frac{\log x}{(\log x)^{1/4}}\Big\} \le x^2 e^{-(\log x)^{3/4}}.$$

Let us rewrite the integrand in expression (14):

$$\begin{aligned} \frac{\zeta(s)}{s(s+1)} &= \frac{s+k-1}{s(s-1)(s+1)} \prod_{|m|>n_0} \Big(1 - \frac{k}{s - 1 + a_m - ib_m + k}\Big)^{\alpha_m/2} \\ &= \Big(1 - \frac{k}{s - 1 + a_n - ib_n + k}\Big)^{\alpha_n/2} \frac{s+k-1}{s(s-1)(s+1)} \\ &\quad \times \prod_{\substack{|m|>n_0 \\ m\ne n}} \Big(1 - \frac{k}{s - 1 + a_m - ib_m + k}\Big)^{\alpha_m/2} \\ &= (s - 1 + a_n - ib_n)^{\alpha_n/2} \\ &\quad \times \frac{(s+k-1) \prod\limits_{\substack{|m|>n_0 \\ m\ne n}} \Big(1 - \frac{k}{s - 1 + a_m - ib_m + k}\Big)^{\alpha_m/2}}{s(s-1)(s+1)(s - 1 + a_n - ib_n + k)^{\alpha_n/2}} \\ &= (s - 1 + a_n - ib_n)^{\alpha_n/2} f_n(s), \end{aligned}$$

where f_n is an analytic function at $s = 1 - a_n + ib_n$ with a power series expansion having a radius of convergence greater than one:

$$f_n(s) = \sum_{j=0}^{\infty} a_{n,j}(s - 1 + a_n - ib_n)^j.$$

The integrand in (14) can now be written as

$$\begin{aligned}
&x^{2-a_n+ib_n-t}(te^{-i\pi})^{\alpha_n/2} f_n(1 - a_n + ib_n - t)\\
&\qquad = x^{2-a_n+ib_n} \sum_{j=0}^{\infty} a_{n,j} x^{-t} t^{j+\alpha_n/2} e^{-\pi i(j+\alpha_n/2)}.
\end{aligned}$$

Thus, writing E in place of $O(x^2 e^{-(\log x)^{3/4}})$, we have

$$\begin{aligned}
J_n'' &= \frac{1}{2\pi i} \int_0^{(\log x)^{-1/4}} x^{2-a_n+ib_n} \sum_{j=0}^{\infty} a_{n,j} x^{-t} t^{j+\alpha_n/2} e^{-\pi i(j+\alpha_n/2)}\, dt + E\\
&= \frac{1}{2\pi i} x^{2-a_n+ib_n} \sum_{j=0}^{\infty} a_{n,j} e^{-\pi i(j+\alpha_n/2)} \int_0^{(\log x)^{-1/4}} x^{-t} t^{j+\alpha_n/2}\, dt + E\\
&= \frac{1}{2\pi i} x^{2-a_n+ib_n} e^{-\pi i\alpha_n/2} \sum_{j=0}^{\infty} (-1)^j\, a_{n,j} \int_0^{(\log x)^{-1/4}} x^{-t} t^{j+\alpha_n/2}\, dt + E\\
&= \frac{1}{2\pi i} x^{2-a_n+ib_n} e^{-\pi i\alpha_n/2} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1}\\
&\quad \times \sum_{j=0}^{\infty} a_{n,j} \left(\frac{-1}{\log x}\right)^j \int_0^{(\log x)^{3/4}} e^{-t} t^{j+\alpha_n/2}\, dt + E\\
&= \frac{1}{2\pi i} x^{2-a_n+ib_n} e^{-\pi i\alpha_n/2} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1}\\
&\quad \times \int_0^{(\log x)^{3/4}} e^{-t} t^{\alpha_n/2} \sum_{j=0}^{\infty} a_{n,j} \left(\frac{-t}{\log x}\right)^j dt + E\\
&= \frac{1}{2\pi i} x^{2-a_n+ib_n} e^{-\pi i\alpha_n/2} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1} S_n + O(x^2 e^{-(\log x)^{3/4}})
\end{aligned}$$

with

$$S_n = \int_0^{(\log x)^{3/4}} e^{-t} t^{\alpha_n/2} f_n\left(1 - a_n + ib_n - \frac{t}{\log x}\right) dt.$$

In a similar fashion we obtain

$$J_n' = \frac{-1}{2\pi i} x^{2-a_n+ib_n} e^{\pi i\alpha_n/2} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1} S_n + O(x^2 e^{-(\log x)^{3/4}}).$$

We therefore have

$$\begin{aligned}(15)\quad J_n &= J_n' + J_n'' \\ &= -\frac{\sin\frac{\pi}{2}\alpha_n}{\pi} x^{2-a_n+ib_n} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1} S_n + O(x^2 e^{-(\log x)^{3/4}}),\end{aligned}$$

with the O-constant not greater than 10.

When we calculate J_{-n} we obtain the complex conjugate of J_n (this is because $b_{-n} = -b_n$). Therefore

$$J_n + J_{-n} = J_n + \overline{J}_n = 2\Re(J_n).$$

Our next step is to estimate the integral S_n appearing in (15). First we obtain lower and upper bounds for

$$(16)\quad f_n(s) = \frac{s+3}{s(s-1)(s+1)} \cdot \frac{\prod\limits_{\substack{|m|>n_0\\ m\neq n}} \left(1 - \frac{k}{s-1+a_m-ib_m+k}\right)^{\alpha_m/2}}{(s-1+a_n-ib_n+k)^{\alpha_n/2}}$$

when $|s-1+a_n-ib_n| \le 1$.

For the upper bound we notice that

$$|s| > b_n - 1.$$

Thus

$$\left|\frac{s+3}{s(s-1)(s+1)}\right| \le \frac{b_n+6}{(b_n-2)^3} \le \frac{2b_n}{(b_n/2)^3} = \frac{16}{b_n^2}.$$

Also

$$|s-1+a_n-ib_n+k|^{\alpha_n/2} > (k - |s-1+a_n-ib_n|)^{\alpha_n/2} \ge 3^{\alpha_n/2} \ge 1.$$

Now we want to estimate from above the product appearing in the definition of f_n (equation (16)). As in the proof of Proposition 5 we have

$$\left|1 - \frac{k}{s-1+a_m-ib_m+k}\right| \le 1 + \frac{k}{|\Im(s) - b_m|} \le 1 + \frac{k}{2k} = \frac{3}{2}.$$

Thus the product in (16) is less than

$$\prod_{\substack{|m|>n_0\\ m\neq n}} \left(\frac{3}{2}\right)^{\alpha_n/2} \le \prod_{|m|>n_0} \left(\frac{3}{2}\right)^{1/m^2} \le \left(\frac{3}{2}\right)^{2\sum_{j=1}^{\infty} 1/j^2} < 4.$$

Thus we have proved

Proposition 9. *For* $|s-(1-a_n+ib_n)|\le 1$,

$$|f_n(s)|\le 64/b_n^2.$$

This and Cauchy's inequalities give the following

Corollary 10. *For all* $j=1,2,\ldots$,

$$|a_{n,j}|\le 64/b_n^2.$$

Now for the lower bound:

$$|s|\le |s-1+a_n-ib_n|+|1-a_n+ib_n|\le 1+1+|a_n|+|b_n|\le 3+b_n.$$

Thus

$$\left|\frac{s+3}{s(s-1)(s+1)}\right|\ge \frac{|s+3|}{(b_n+4)^3}\ge \frac{|s|-3}{(b_n+4)^3}\ge \frac{b_n-1-3}{(b_n+4)^3}\ge \frac{\frac{1}{2}b_n}{(2b_n)^3}=\frac{1}{16b_n^2}.$$

Each term in the infinite product in (16) is

$$\left|1-\frac{k}{s-1+a_m-ib_m+k}\right|\ge 1-\frac{k}{|s-1+a_m-ib_m+k|}\ge 1-\frac{k}{2k}=\frac{1}{2}.$$

Therefore

$$\prod_{\substack{|m|>n_0\\ m\ne n}}\left(1-\frac{k}{s-1+a_m-ib_m+k}\right)^{\alpha_n/2}$$

$$\ge \prod_{|m|>0}\left(\frac{1}{2}\right)^{1/m^2}=\left(\frac{1}{2}\right)^{2\sum_{m=1}^{\infty}1/m^2}=\left(\frac{1}{2}\right)^{\pi^2/3}>\frac{1}{10}.$$

Thus we have

Proposition 11. *For* $|s-(1-a_n+ib_n)|\le 1$ *we have*

$$|f_n(s)|\ge \frac{1}{16b_n^2}\cdot\frac{1}{5}\cdot\frac{1}{10}=\frac{1}{800b_n^2}.$$

With all these inequalities we can estimate the integral for S_n, the function occurring in (15), as follows:

$$\begin{aligned}S_n&=\int_0^{(\log x)^{3/4}} e^{-t}t^{\alpha_n/2}f_n\left(1-a_n+ib_n-\frac{t}{\log x}\right)dt\\ &=\int_0^{(\log x)^{3/4}} e^{-t}t^{\alpha_n/2}\sum_{j=0}^{\infty}a_{n,j}\left(\frac{-t}{\log x}\right)^j dt\\ &=a_{n,0}\int_0^{(\log x)^{3/4}} e^{-t}t^{\alpha_n/2}\,dt+\sum_{j=1}^{\infty}a_{n,j}\int_0^{(\log x)^{3/4}} e^{-t}t^{\alpha_n/2}\left(\frac{-t}{\log x}\right)^j dt.\end{aligned}$$

For the second term we get, by Corollary 10,

$$\Big|\sum_{j=1}^{\infty} a_{n,j} \int\limits_{0}^{(\log x)^{3/4}} e^{-t} t^{\alpha_n/2} \left(\frac{-t}{\log x}\right)^{j} dt\Big| \le \sum_{j=1}^{\infty} \frac{64}{b_n^2} \left(\frac{1}{\log x}\right)^{j/4} \int\limits_{0}^{\infty} e^{-t} t^{\alpha_n/2}\, dt$$
$$\le \left(\frac{1}{\log x}\right)^{1/4} \frac{64}{b_n^2} \sum_{j=0}^{\infty} \left(\frac{10}{98}\right)^{j/4}$$
$$\le 148 \left(\frac{1}{\log x}\right)^{1/4} \frac{1}{b_n^2}.$$

Now, since

$$\int\limits_{(\log x)^{3/4}}^{\infty} e^{-t} t^{\alpha_n/2}\, dt \le \int\limits_{(\log x)^{3/4}}^{\infty} e^{-t} t\, dt \le 2 \log x\, e^{-(\log x)^{3/4}},$$

the integral S_n in (15) is

$$S_n = a_{n,0}\big(\Gamma\big(\tfrac{1}{2}\alpha_n + 1\big) + \theta_2 \log x\, e^{-(\log x)^{3/4}}\big) + \frac{\theta_3}{b_n^2 (\log x)^{1/4}}, \tag{17}$$

where $|\theta_2| \le 2$ and $|\theta_3| \le 148$. Since $a_{n,0} = f_n(1-a_n+ib_n)$, from Proposition 11 and from $\Gamma\big(\frac{1}{2}\alpha_n + 1\big) > 0.8$ we find that the modulus of S_n is greater than

$$\frac{1}{800 b_n^2}(0.8 - 2 \log x\, e^{-(\log x)^{3/4}}) - \frac{148}{b_n^2 (\log x)^{1/4}} \tag{18}$$
$$= \frac{1}{800 b_n^2}\left(0.8 - 2\, \log x\, e^{-(\log x)^{3/4}} - \frac{118400}{(\log x)^{1/4}}\right) \ge \frac{e^{-2(\log x_n)^{1/2}}}{1600}$$

for $x \ge X_1$, i.e., if x is sufficiently large.

We will use this lower bound for the integral S_n appearing in equation (15). Now consider the other factor in that equation,

$$\frac{\sin \frac{\pi}{2}\alpha_n}{\pi} x^{2-a_n} \left(\frac{1}{\log x}\right)^{\alpha_n/2+1} \ge \frac{\alpha_n}{\pi} x^2 e^{-(\log x_n)^{1/2}} \frac{1}{2(\log x)^2}$$
$$\ge \frac{x^2}{\pi} e^{-(\log x_n)^{1/2}} \frac{1}{2(\log x)^2 (\log\log x_n)^2}.$$

From the above and equation (15),

$$|J_n| \ge \frac{x^2}{\pi 1600} \cdot \frac{e^{-3(\log x_n)^{1/2}}}{4(\log x)^4} \ge \frac{10^{-5}}{(\log x)^4} e^{-3(\log x)^{1/2}} \qquad \text{if } x \ge X_1.$$

We would like $J_n + J_{-n} = 2\Re(J_n)$ to be large. We already know that $|J_n|$ is large, but still it can be that $\Re(J_n) = 0$, say. Let us recall here equation

(15):

$$J_n = -\frac{\sin\frac{\pi}{2}\alpha_n}{\pi} x^{2-a_n+ib_n}\left(\frac{1}{\log x}\right)^{\alpha_n/2+1} S_n + O(x^2 e^{-(\log x)^{3/4}}),$$

where $x = x_n(1+\theta_1/\log x_n)$, $|\theta_1| < 1$. Now express this equation as

$$A := \frac{J_n}{x^{2-a_n}} = Bx^{ib_n} + C,$$

where B and C have obvious definitions. Dividing by $|B|$ we get

$$\begin{aligned}\Re\left(\frac{A-C}{|B|}\right) &= \Re(\exp\{ib_n \log x + i\arg B\}) = \cos(b_n \log x + \arg B)\\ &= \cos\left(b_n\left(\log x_n + \log\left(1+\frac{\theta_1}{\log x_n}\right)\right) + \arg B\right)\\ &= \cos\left(\{b_n \log x_n + \arg B\} + b_n \log\left(1+\frac{\theta_1}{\log x_n}\right)\right).\end{aligned}$$

Notice that $\arg B$ is a function of θ_1. This is because x is a function of θ_1 and S_n is a function of x. Hence we have

$$\arg B = \arg S_n + \pi.$$

The main term on the right hand side of (17) is independent of θ_1. Since the other two terms are much smaller, we see that as θ_1 runs from -1 to $+1$, the argument of S_n (and hence of B) undergoes a change not greater than 2π.

Therefore, as θ_1 runs from -1 to $+1$, the argument of the above cosine runs through an interval centered somewhere in

$$(b_n \log x_n - 2\pi, b_n \log x_n + 2\pi)$$

and having a length greater than

$$\begin{aligned}b_n \log\left(1+\frac{1}{\log x_n}\right) &= b_n \int_1^{1+(\log x_n)^{-1}} \frac{dt}{t} \ge b_n\left(\frac{1}{1+(\log x_n)^{-1}}\right)\left(\frac{1}{\log x_n}\right)\\ &\ge b_n \frac{10}{11}\cdot\frac{1}{\log x_n} = \frac{10}{11}\cdot\frac{e^{\sqrt{\log x_n}}}{\log x_n} \to \infty \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, when n is large we can choose values, $\theta_1(+)$ and $\theta_1(-)$, of θ_1 such that

$$\Re\left(\frac{A-C}{|B|}\right) = +1 \quad\text{and}\quad \Re\left(\frac{A-C}{|B|}\right) = -1.$$

In the first case we get

$$\Re\left(\frac{J_n}{x^{2-a_n}}\right) = \Re(A) = |B| + \Re(C),$$

or, what is the same,

$$\begin{aligned}(19)\qquad \Re(J_n) &= |B|x^{2-a_n} + \Re(C)x^{2-a_n}\\ &\ge |J_n| - |C|x^{2-a_n} + \Re(C)x^{2-a_n}\\ &= |J_n| + O(x^2 e^{-(\log x)^{3/4}})\\ &\ge \frac{10^{-5}}{2(\log x)^4} e^{-3(\log x)^{1/2}} \qquad \text{if } x \ge X_1 \text{ and } \theta_1 = \theta_1(+).\end{aligned}$$

In a similar fashion we obtain

$$(20)\qquad \Re(J_n) \le -\frac{10^{-5}}{2(\log x)^4} e^{-3(\log x)^{1/2}} \qquad \text{if } x \ge X_1 \text{ and } \theta_1 = \theta_1(-).$$

Notice that the inequalities (19) and (20) hold when

$$x \ge X_1, \quad x = x_n\left(1 + \frac{\theta_1}{\log x_n}\right), \quad x_n = \exp\{10^{2^{n-1}}\}, \quad \theta_1 = \theta_1(\pm).$$

These inequalities and the equation

$$(13)\qquad M(x) = 2\varphi(1)x^2 + 2\Re(J_n) + O(x^2 e^{-(\log x)^{3/4}})$$

imply relation (8); in fact we have proved the following stronger statement:

$$M(x) = 2\varphi(1)x^2 + \Omega_\pm(x^2 e^{-(3+\varepsilon)(\log x)^{1/2}}) \qquad \forall \varepsilon > 0.$$

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Received on 3.12.1997
and in revised form on 7.7.1998 (3305)