Asymptotic density of \( A \subset \mathbb{N} \) and density of the ratio set \( R(A) \)

by

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Dedicated to the memory of Professor Paul Erdős

1. Introduction. Denote by \( \mathbb{N} \) the set of all positive integers and if a subset \( A \subset \mathbb{N} \) is given, define the ratio set by

\[
R(A) = \{a/b : a, b \in A\}.
\]

The lower and upper asymptotic density of \( A \), denoted by \( \underline{d}(A) \) and \( \overline{d}(A) \) respectively, are defined as

\[
\underline{d}(A) = \liminf_{x \to \infty} \frac{A(x)}{x}, \quad \overline{d}(A) = \limsup_{x \to \infty} \frac{A(x)}{x},
\]

where \( A(x) = \#\{a \leq x : a \in A\} \).

In the present paper we are concerned with certain relations between the asymptotic densities of a set \( A \) as well as with density of \( R(A) \) in \([0, \infty)\). T. Šalát [6] showed that \( \underline{d}(A) = \overline{d}(A) > 0 \) or \( \overline{d}(A) = 1 \) implies that \( R(A) \) is everywhere dense in \([0, \infty)\) and for every sufficiently small \( \varepsilon > 0 \) there exists a subset \( A \subset \mathbb{N} \) such that \( \overline{d}(A) = 1 - \varepsilon \) and \( R(A) \) is not everywhere dense in \([0, \infty)\). He gave an example of \( A \subset \mathbb{N} \) for which \( \underline{d}(A) = 1/4 \) and \( R(A) \cap (5/4, 8/5) = \emptyset \).

We prove that \( 1/2 \) is the lower bound of \( \gamma \)'s for which \( \underline{d}(A) \geq \gamma \) implies that \( R(A) \) is dense in \([0, \infty)\) (Theorem 1). The proof is based on the estimate

\[
\underline{d}(A) \leq \frac{\alpha}{\beta} \min(1 - \overline{d}(A), \overline{d}(A))
\]

where the interval \((\alpha, \beta) \subset [0, \infty)\) is disjoint from \( R(A) \) (Theorem 2). To complete our proof we construct an \( A \subset \mathbb{N} \) for which the complement of the closure of \( R(A) \) is formed by infinitely many pairwise disjoint open intervals.

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On the other hand, we prove that for every given upper and lower asymptotic density there exists an \( A \subset \mathbb{N} \) possessing these densities and having \( R(A) \) everywhere dense (Theorem 3). As an application we give a new class of sets \( A \subset \mathbb{N} \) having dense ratio set \( R(A) \) (Theorem 4). We also prove that the complement of the set \( R(A)^d \) of all limit points of \( R(A) \) is either empty or contains infinitely many open intervals assuming \( \bar{d}(A) > 0 \) (Theorem 5). We generalize our results for any open set \( X \) disjoint from the set \( R(A) \) of all accumulation points of \( R(A) \) (Theorem 6). The paper concludes with some remarks.

Throughout the paper, without loss of generality, we will use only intervals \( (\alpha, \beta) \) contained in \([0, 1]\).

2. Main results

Theorem 1. For every \( A \subset \mathbb{N} \), if the lower asymptotic density \( \underline{d}(A) \geq 1/2 \) then the ratio set \( R(A) \) is everywhere dense in \([0, \infty)\). Conversely, if \( 0 \leq \gamma < 1/2 \) then there exists an \( A \subset \mathbb{N} \) such that \( d(A) = \gamma \) and \( R(A) \) is not everywhere dense in \([0, \infty)\).

The proof immediately follows from the following theorem and example.

Theorem 2. Let \( A \subset \mathbb{N} \) and the interval \((\alpha, \beta), 0 \leq \alpha < \beta \leq 1\), be such that 
\((\alpha, \beta) \cap R(A) = \emptyset\). Then

\[
1. \quad \underline{d}(A) \leq \frac{\alpha}{\beta} \min(1 - \bar{d}(A), \bar{d}(A))
\]

and

\[
2. \quad \overline{d}(A) \leq 1 - (\beta - \alpha).
\]

Proof of (1). Let \( A \subset \mathbb{N} \) be listed in strictly increasing order as \( a_1 < a_2 < \ldots < a_n < \ldots \). If \((\alpha, \beta) \cap R(A) = \emptyset\), then the intervals

\[(\alpha a_n, \beta a_n), \quad n = 1, 2, \ldots,
\]
cannot intersect \( A \) but they may have mutually nonempty intersections. We can select pairwise disjoint subintervals

\[(3) \quad (\alpha a_{[\theta n]}, \alpha a_{[\theta n]} + \alpha), (\alpha a_{[\theta n] + 1}, \alpha a_{[\theta n] + 1} + \alpha), \ldots, \]
\[(\alpha a_{n-1}, \alpha a_{n-1} + \alpha), (\alpha a_n, \beta a_n)
\]
for some \( 0 \leq \theta \leq 1 \) (here we put \( a_{[\theta n]} = 0 \) if \( [\theta n] = 0 \)). Define \( B = \mathbb{N} - A \) and \( B(x) = \#\{b \leq x : b \in B\} \). Counting the number of integer points belonging to (3) we obtain

\[B(\beta a_n) \geq (n - [\theta n])(\alpha - 1) + ((\beta - \alpha)a_n - 1) + B(\alpha a_{[\theta n]})
\]
for all sufficiently large \( n \). To eliminate \( 1 \) in \( \alpha - 1 \) we replace \( n \) with \( nk \) and \( \alpha \) with \( k\alpha \). Then (3) transforms into pairwise disjoint subintervals of the
Density of the ratio set $R(A)$

(4) \((\alpha a[\theta n]k, \alpha a[\theta n]k + k\alpha), (\alpha a[\theta n]+1)k, \alpha a[\theta n]+1)k + k\alpha), \ldots, (\alpha a(n-1)k, \alpha a(n-1)k + k\alpha), (\alpha a nk, \beta a nk)\).

Thus, we have

\[
\frac{B(\beta a nk)}{\beta a nk} \geq \frac{(n - [\theta n])(k\alpha - 1)}{[\theta n]k} + \frac{((\beta - \alpha) a nk - 1)}{\alpha a nk} + \frac{B(\alpha a [\theta n]k)}{\alpha a [\theta n]k} \cdot \frac{\alpha a[\theta n]k}{\beta a nk}.
\]

To compute the limit superior of the left and right hand sides, respectively, use the fact that

(i) \(\limsup_{n \to \infty} \frac{B(\beta a nk)}{\beta a nk} \leq \bar{d}(B) = 1 - \underline{d}(A)\),

(ii) \(\limsup_{n \to \infty} nk/a nk = \overline{d}(A)\),

(iii) \(\liminf_{n \to \infty} \frac{B(\alpha a[\theta n]k)}{\alpha a[\theta n]k} \geq \overline{d}(B) = 1 - \underline{d}(A)\), and

(iv) by selecting indices \(n\) for which \(\lim_{n \to \infty} nk/a nk = \overline{d}(A)\) we have

(assuming \(\underline{d}(A) > 0\))

\[
\liminf_{n \to \infty} \frac{a[\theta n]k}{a nk} = \liminf_{n \to \infty} \frac{a[\theta n]k}{[\theta n]k} \cdot \lim_{n \to \infty} \frac{[\theta n]k}{a nk} \geq \frac{1}{\overline{d}(A)} \bar{d}(A)\theta.
\]

Thus, letting \(k \to \infty\) we get

\[
1 - \underline{d}(A) \geq (1 - \theta)\frac{\alpha}{\beta} \bar{d}(A) + \frac{\beta - \alpha}{\beta} + (1 - \bar{d}(A))\frac{\alpha}{\beta} \theta.
\]

Computing the maximum of the right hand side for \(0 \leq \theta \leq 1\) yields

\[
1 - \underline{d}(A) \geq \frac{\beta - \alpha}{\beta} + \frac{\alpha}{\beta} \max(\bar{d}(A), 1 - \bar{d}(A)),
\]

which justifies (1).

Proof of (2). Every infinite set \(A \subset \mathbb{N}\) with infinite complement \(\mathbb{N} - A\) can be expressed as the set of the integer points lying in the intervals

(5) \([b_1, c_1], [b_2, c_2], \ldots, [b_n, c_n], \ldots,\)

whose endpoints form two integer sequences ordered as

\[b_1 \leq c_1 < b_2 \leq c_2 < \ldots < b_n \leq c_n < \ldots\]

Clearly

(6) \(\underline{d}(A) = \liminf_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n-1} (c_i - b_i + 1),\)

(7) \(\overline{d}(A) = \limsup_{n \to \infty} \frac{1}{c_n} \sum_{i=1}^{n} (c_i - b_i + 1).\)
The points of $A \cap [1, c_n]$ divided by $i$, $i \in [b_n, c_n]$, form a subset $R_n \subset R(A)$; we obtain the intervals
\[
\left[\frac{b_1}{i}, \frac{c_1}{i}\right], \left[\frac{b_2}{i}, \frac{c_2}{i}\right], \ldots, \left[\frac{b_{n-1}}{i}, \frac{c_{n-1}}{i}\right], \left[\frac{b_n}{i}, \frac{c_n}{i}\right]
\]
which have the following property: the distance of any two neighbouring points of $R_n$ lying in $[b_{n-k}/i, c_{n-k}/i]$ is less than $1/b_n$ and the same holds for the union
\[
\bigcup_{i=b_n}^{c_n} \left[\frac{b_{n-k}}{i}, \frac{c_{n-k}}{i}\right] = \left[\frac{b_{n-k}}{c_n}, \frac{c_{n-k}}{b_n}\right].
\]
Thus, for sufficiently large $n$, every interval $(\alpha, \beta) \subset [0,1]$ satisfying $(\alpha, \beta) \cap R(A) = \emptyset$ must lie in the complement of $[b_{n-k}/c_n, c_{n-k}/b_n]$, $k = 0,1,\ldots, n-1$, which is formed by the pairwise disjoint intervals
\[
(\frac{c_{n-k}}{b_n}, \frac{b_{n-k+1}}{c_n}), \quad k = 1,\ldots, n-1,
\]
some of which may be empty. Hence, a necessary condition for $(\alpha, \beta) \cap R(A) = \emptyset$ is the existence of an integer sequence $k_n$, $k_n < n$, such that
\[
(\alpha, \beta) \subset \left(\frac{c_{n-k_n}}{b_n}, \frac{b_{n-k_n+1}}{c_n}\right)
\]
for all sufficiently large $n$. This also gives
\[
\frac{b_{n-k_n+1}}{c_n} - \frac{c_{n-k_n}}{c_n} \geq \beta - \alpha.
\]

Now we can express the upper asymptotic density as
\[
\overline{d}(A) = \limsup_{n \to \infty} \left(\frac{c_n - b_n}{c_n} + \frac{n}{c_n} - \left(\frac{b_2 - c_1}{c_n} + \frac{b_3 - c_2}{c_n} + \ldots + \frac{b_n - c_{n-1}}{c_n}\right)\right)
\]
whence
\[
\overline{d}(A) - \overline{d}(C) \leq 1 - (\beta - \alpha),
\]
where $C$ is the range of $c_n$. 

For sufficiency of (9) we need the set $R(A)^l$ of all limit points of $R(A)$ (cf. Section 4). By the above reasoning we see that $(\alpha, \beta) \cap R(A)^l = \emptyset$ if and only if there exists $k_n < n$ satisfying (9) for all sufficiently large $n$. Thus, inequality (11) holds for $(\alpha, \beta)$ satisfying $(\alpha, \beta) \cap R(A)^l = \emptyset$ as well.

Now, for a positive integer $k$, transform
\[
[b_n, c_n] \to [kb_n, kc_n + k - 1]
\]
and denote by $A_k$ the set of all integer points lying in $[kb_n, kc_n + k - 1]$, $n = 1,2,\ldots$. Similarly, $C_k$ is the set of all $kc_n + k - 1$. Evidently
\[
\overline{d}(A_k) = \overline{d}(A), \quad \overline{d}(C_k) = \overline{d}(C)/k, \quad R(A_k)^l = R(A)^l,
\]
which gives $\overline{d}(A) - \overline{d}(C)/k \leq 1 - (\beta - \alpha)$ and (2) follows. $\blacksquare$
Using (2) and the part \(d(A) \leq (\alpha/\beta)(1 - \overline{d}(A))\) of (1) we have

**Corollary.** For every subset \(A \subseteq \mathbb{N}\), if \(d(A) + \overline{d}(A) \geq 1\) then \(R(A)\) is everywhere dense in \([0, \infty)\).

To complete our proof of Theorem 1 consider

**Example 1.** Let \(\gamma, \delta\) and \(a\) be given positive real numbers satisfying \(\gamma < \delta\) and \(a > 1\). Let \(A\) be the set of all integer points lying in the intervals

\[(\gamma, \delta), (\gamma a, \delta a), (\gamma a^2, \delta a^2), \ldots, (\gamma a^n, \delta a^n), \ldots\]

For this \(A\) we see from (5) of \(A\) that \(b_n = \lfloor \gamma a^n \rfloor + 1\), \(c_n = \lfloor \delta a^n \rfloor\) and in order that \(c_n < b_{n+1}\) we need \(\delta/\gamma < a\). In this case, for the intervals in (8) we have

\[
\left(\frac{\delta}{\gamma a^k}, \frac{\delta}{\gamma a^{k-1}}\right) \subseteq \left(\frac{c_{n-k}}{b_n}, \frac{c_{n-k+1}}{c_n}\right), \quad k = 1, \ldots, n-1;
\]

further, \(c_{n-k}/b_n \to \delta/(\gamma a^k), b_{n-k+1}/c_n \to \gamma/(\delta a^{k-1})\) as \(n \to \infty\). Consequently, the closure of \(R(A)\) is \(R(A)^1\). Thus, \([0, 1] - R(A) = \bigcup_{i=1}^{\infty}(\alpha_i, \beta_i)\), where \((\alpha_i, \beta_i) = (a_1/a^{i-1}, b_1/a^{i-1})\) and

\[(\alpha_1, \beta_1) = \left(\frac{\delta}{\gamma a}, \frac{\gamma}{\delta}\right).
\]

This implies that

\([0, 1] - \overline{R(A)} \neq \emptyset \Leftrightarrow \delta/\gamma < \sqrt{a}.

By (6) and (7) we have

\[
d(A) = \frac{\delta - \gamma}{\gamma} \cdot \frac{1}{a-1}, \quad \overline{d}(A) = \frac{\delta - \gamma}{\delta} \cdot \frac{a}{a-1}.
\]

We can also see that for such \(A\) the ratio set \(R(A)\) is everywhere dense in \([0, \infty)\) if and only if \(d(A) + \overline{d}(A) \geq 1\).

Now, if \(\delta/\gamma \to \sqrt{a}\) then \(d(A) \to 1/(\sqrt{a} + 1)\) and if \(\sqrt{a} \to 1 + 0\) then \(d(A) \to 1/2 - 0\). This completes the proof of Theorem 1.

Note that since \(d(A)/(\alpha_1/\beta_1)\overline{d}(A)) \to 1\) as \(\gamma/\delta \to 1\) and \(\overline{d}(A)/(1 - (\beta_1 - \alpha_1)) \to 1\) as \(a \to \infty\), we cannot extend (1) and (2) to

\[
d(A) \leq c(\alpha/\beta) \min(1 - \overline{d}(A), \overline{d}(A)) \quad \text{and} \quad \overline{d}(A) \leq c(1 - (\beta - \alpha))
\]

for some positive constant \(c < 1\). ■

In the sequel we demonstrate that (1) and (2) are necessary but not sufficient conditions for \((\alpha, \beta) \cap R(A) = \emptyset\).

**Theorem 3.** For every pair \((\gamma, \gamma')\) satisfying \(0 \leq \gamma \leq \gamma' \leq 1\) there exists an \(A \subseteq \mathbb{N}\) such that \(d(A) = \gamma, \overline{d}(A) = \gamma'\) and the ratio set \(R(A)\) is everywhere dense in \([0, \infty)\).
Proof. For any infinite set $B \subset \mathbb{N}$ and $\lambda \geq 1$ define $[\lambda B]$ as

$$[\lambda B] = \{[\lambda a] : a \in B\}.$$ 

Clearly,

(i) either both $R(B)$ and $R([\lambda B])$ are everywhere dense in $[0, \infty)$ or neither is;

(ii) $\underline{d}([\lambda B]) = \underline{d}(B)/\lambda$, $\overline{d}([\lambda B]) = \overline{d}(B)/\lambda$.

If $\gamma' > 0$, put $\lambda = 1/\gamma'$ and then use the well-known fact that for every pair $(\delta, \delta')$ satisfying $0 \leq \delta \leq \delta' \leq 1$ there exists $B \subset \mathbb{N}$ such that $\underline{d}(B) = \delta$ and $\overline{d}(B) = \delta'$. Applying this for $(\delta, \delta') = (\lambda \gamma, \lambda \gamma')$, bearing in mind that $\lambda \gamma' = 1$ and using [6, Th. 1] we find that $R(B)$ is dense in $[0, \infty)$. Accordingly, $A = [\lambda B]$ is the desired set.

If $\gamma' = 0$ we can put $A = \mathbb{P}$, the set of all primes, since by A. Schinzel (cf. [7, p. 155]) $R(\mathbb{P})$ is everywhere dense in $[0, \infty)$. ■

3. Applications. Applying Theorem 1 we give some new classes of $A \subset \mathbb{N}$ having dense $R(A)$.

Theorem 4. Let $f(t)$, $t \geq 1$, be a strictly increasing continuous function with inverse function $f^{-1}(t)$. Assume that

(i) $\lim_{t \to \infty} f(t) = \infty$,

(ii) $\lim_{n \to \infty} (f^{-1}(n+1) - f^{-1}(n)) = \infty$,

(iii) $\lim_{n \to \infty} \frac{f^{-1}(n+x) - f^{-1}(n)}{f^{-1}(n+1) - f^{-1}(n)} = \psi(x)$ exists for every $x \in [0, 1]$,

and for $x \in [0, 1]$ put

(iv) $\lim \inf_{n \to \infty} f^{-1}(n)/f^{-1}(n+x) = \chi(x),$

(v) $A_x = \{n \in \mathbb{N} : \{f(n)\} \in (0, x]\}$, where $\{f(n)\}$ is the fractional part of $f(n)$.

If $\psi(x) + 1 - \chi(x)(1 - \psi(x)) \geq 1$, then $R(A_x)$ is everywhere dense in $[0, \infty)$.

Proof. Observe that $\underline{d}(A_x)$ and $\overline{d}(A_x)$ have the same meaning as the lower and upper distribution functions of $f(n)$ mod 1 (cf. [5, Def. 7.1, p. 53]), hence the theorem follows from [5, Th. 7.7, p. 58] and our Corollary. ■

Applying Theorem 4 to $f(t) = \log t$ we deduce that $x \geq 1/2$ implies the density of $R(A_x)$. Since in this case the set $A_x$ has the form described in Example 1 with $\gamma = 1$, $\delta = e^x$ and $a = e$, it follows that $x \geq 1/2$ is also necessary for the density of $R(A_x)$ to hold.

For another application of Theorem 1 we make use of [4]. Let $a > 1$ be an integer and $A$ consist of all $A \subset \mathbb{N}$ containing no 3-term progressions of the form $k, kq, kq^2$, where $k \in \mathbb{N}$ and $q \in \{a, a^2, a^3, a^4\}$. It is proved in [4, Ex. 2] that $\sup_{A \in \mathcal{A}} \underline{d}(A) \geq (1 - a^{-1})(1 + a^{-1} + a^{-3} + a^{-4})(a^3/(a^3 + 1))$. 

Density of the ratio set \( R(A) \)

which, together with our Theorem 1, implies that \( R(A) \) is everywhere dense in \([0, \infty)\) for some \( A \in \mathcal{A} \).

4. Complement of the limit points of the ratio set. As before, assume that \( A \subset \mathbb{N} \) is ordered into the sequence \( a_1 < a_2 < \ldots \) and consider the ratio set \( R(A) \) as a double sequence \( a_m/a_n \), \( m, n = 1, 2, \ldots \) We introduce two further sets:

(i) \( R(A)^l \) is the set of all limit points \( x = \lim_{i \to \infty} a_{m_i}/a_{n_i} \) of \( R(A) \).

(ii) \( R(A)^d \) is the set of all accumulation points of \( R(A) \), i.e. the points \( x \) which can be expressed as a limit \( x = \lim_{i \to \infty} a_{m_i}/a_{n_i} \) of a one-to-one sequence \( a_{m_i}/a_{n_i} \).

Clearly, \( R(A)^l \) and \( R(A)^d \) are closed. It is shown in [1] that for every system of pairwise disjoint open intervals \( (\alpha_i, \beta_i), i \in \mathcal{I} \), there exists \( A \subset \mathbb{N} \) such that \( [0,1] - R(A)^d = \bigcup_{i \in \mathcal{I}} (\alpha_i, \beta_i) \) and the same proof applies to \( R(A)^l \).

To extend the above result of [1] we prove

Theorem 5. If \( d(A) > 0 \) and \( [0,1] - R(A)^l \neq \emptyset \), then

\[ [0,1] - R(A)^l = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i), \]

where \( \alpha_i < \beta_i \) and \( (\alpha_i, \beta_i) \cap (\alpha_j, \beta_j) = \emptyset \) for \( i \neq j \).

Proof. We divide the proof into three steps.

1. Let \( \gamma > 0 \) be a limit point of the form

\[ \gamma = \lim_{n \to \infty} a_{g(n)}/a_n, \]

where \( g(n) \) is a suitable integer sequence. Then

\[ (\alpha, \beta) \cap R(A)^l = \emptyset \Rightarrow (\gamma \alpha, \gamma \beta) \cap R(A)^l = \emptyset = (\alpha/\gamma, \beta/\gamma) \cap R(A)^l. \]

Indeed, assuming \( \gamma \alpha < \delta < \gamma \beta \) and

\[ \delta = \lim_{i \to \infty} a_{m_i}/a_{n_i}, \]

we have

\[ \frac{\delta}{\gamma} = \frac{\lim_{i \to \infty} a_{m_i}/a_{n_i}}{\lim_{i \to \infty} a_{g(n_i)}/a_{n_i}} = \lim_{i \to \infty} \frac{a_{m_i}}{a_{g(n_i)}}, \]

which is a contradiction. Repeating (13) yields \( (\gamma^k \alpha, \gamma^k \beta) \cap R(A)^l = \emptyset \) for all \( k \in \mathbb{Z} \).

2. Using all points \( \gamma, \delta, \eta, \ldots \) of the form (12) we can define a group

\[ G(A) = \{ \gamma^i \delta^j \eta^k : i, j, k, \ldots \in \mathbb{N} \}. \]

Let \( [0,1] - R(A)^l = \bigcup_{i \in \mathcal{I}} (\alpha_i, \beta_i) \). Applying (13) for \( t \in G(A) \cap [0,1] \) and \( i \in \mathcal{I} \), we get some \( j, k \in \mathcal{I} \) such that \( (t \alpha_i, t \beta_i) \subset (\alpha_j, \beta_j) \) and \( (t^{-1} \alpha_j, t^{-1} \beta_j) \subset \).
\((\alpha_k, \beta_k)\). This implies \(i = k\) and
\[(t\alpha_i, t\beta_i) = (\alpha_j, \beta_j).\]

For a fixed \((\alpha_{i_0}, \beta_{i_0})\), the intervals \((t\alpha_{i_0}, t\beta_{i_0}), t \in G(A) \cap (0, 1)\), are nonoverlapping, which implies that \(\mathcal{I}\) is infinite. Moreover, \(G(A)\) must be discrete and thus cyclic.

3. Assuming \(d(A) > 0\), we prove that \(G(A) \cap (0, 1)\) is nonempty. Let \(n/a_n > \theta > 0\) for all sufficiently large \(n\). For any \(u, v\) satisfying \(0 < u < v < \theta\) we have
\[
\frac{a_{[vn]}}{a_n} \geq \frac{[vn]}{a_n} > u\theta, \quad \frac{a_{[vn]}}{a_n} \leq \frac{a_{[vn]}}{vn} \leq \frac{v}{\theta}
\]
for all sufficiently large \(n\). Thus, we obtain
\[
\frac{a_i}{a_n} \in \left(u\theta, \frac{v}{\theta}\right)\quad \text{for } i \in [vn], [vn],
\]
which implies the existence of \(t \in G(A)\) satisfying \(t \in [u\theta, v/\theta) \subset (0, 1)\).

Note that as the proof of (2) shows, \(t \in G(A) \cap [0, 1]\) if and only if there exists \(k_n < n\) such that \(t \in \left[\frac{b_{n-k_n}}{b_n}, \frac{c_{n-k_n}}{c_n}\right]\) for all sufficiently large \(n\).

In Example 1 the group \(G(A)\) is generated by \(1/a\) and the complement of \(R(A)^l\) has a simple structure. For general \(A\) the complement of \(R(A)^l\) may be more complicated.

**Example 2.** In this example we abbreviate \((\gamma a, \delta a)\) as \((\gamma, \delta)a\). Assume that
\[
0 < \gamma_1 < \delta_1 < \gamma_2 < \delta_2 < a\gamma_1 < a\delta_1 \quad \text{and} \quad a > 1
\]
and let \(A\) be the set of all integer points lying in the pairwise disjoint open intervals
\[
(\gamma_1, \delta_1), (\gamma_2, \delta_2), (\gamma_1, \delta_1)a, (\gamma_2, \delta_2)a, \ldots, (\gamma_1, \delta_1)a^n, (\gamma_2, \delta_2)a^n, (\gamma_1, \delta_1)a^{n+1}, \ldots
\]
Then, in (5) for this \(A\) we get two types of intervals \([b_n, c_n]\), which give (asymptotically) two types of intervals in (8) and which form two sequences of pairwise disjoint open intervals
\[
I_2a^{i-1}, I_1a^{i-1}, \quad i = 1, 2, \ldots, \quad \text{and} \quad J_2a^{i-1}, J_1a^{i-1}, \quad i = 1, 2, \ldots,
\]
where
\[
I_2 = \left(\frac{\delta_2}{\gamma_2a}, \frac{\gamma_1}{\delta_2}\right), \quad I_1 = \left(\frac{\delta_1}{\gamma_1a}, \frac{\gamma_2}{\delta_1}\right), \quad J_2 = \left(\frac{\delta_1}{\gamma_1a}, \frac{\gamma_2}{\delta_1}\right), \quad J_1 = \left(\frac{\delta_2}{\gamma_2a}, \frac{\gamma_1}{\delta_1}\right).
\]
Moreover, there are inclusions between the intervals in (8) and the above intervals, respectively. This guarantees \(R(A) = R(A)^l\) as well as
The intersection \((I_2 \cup I_1) \cap (J_2 \cup J_1)\) consists of at most three pairwise disjoint open intervals \((\alpha_1, \beta_1), (\alpha_1', \beta_1')\) and \((\alpha_2', \beta_2')\).

In all cases the group \(G(A)\) is cyclic with generator \(1/a\) or \(1/\sqrt{a}\) depending on \((\gamma_2, \delta_2) = (\gamma_1, \delta_1)\sqrt{a}\).

Applying (6) and (7) we have

\[
\begin{align*}
d(A) &= \min \left( \frac{(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2)}{\gamma_1} \cdot \frac{1}{a - 1}, \right. \\
&\quad \left. \frac{1}{\gamma_2} \left( \frac{(\delta_1 - \gamma_1)}{a - 1} + \frac{(\delta_2 - \gamma_2)}{a - 1} \right) \right), \\
\overline{d}(A) &= \max \left( \frac{(\delta_1 - \gamma_1) + (\delta_2 - \gamma_2)}{\delta_2} \cdot \frac{a}{a - 1}, \right. \\
&\quad \left. \frac{1}{\delta_1} \left( \frac{(\delta_1 - \gamma_1)}{a - 1} + \frac{(\delta_2 - \gamma_2)}{a - 1} \right) \right). 
\end{align*}
\]

For example, putting \(\gamma_1 = 1, \delta_1 = 2, \gamma_2 = 5\) and \(a = 40\), we have
\((\alpha_1, \beta_1) = (2/5, 1/2), (\alpha_1', \beta_1') = (6/40, 1/6), (\alpha_2', \beta_2') = (2/40, 5/80)\).

Further, \(\overline{d}(A) = 2/39, \overline{d}(A) = 41/78, \| [0, 1] - \overline{R}(A) \| = 31/234\) and \(G(A)\) is generated by \(1/40\).

5. Extension of Theorem 2. In this part we extend (1) and (2) to intervals \((\alpha, \beta) \subset [0, 1]\) satisfying

\[(\alpha, \beta) \cap R(A)^d = \emptyset,\]

which does not follow from Theorem 2 directly. Clearly, if \((\alpha, \beta) \cap R(A)^d = \emptyset\) then for every \(\varepsilon > 0\) there exist finitely many pairwise disjoint open intervals \((\alpha_i, \beta_i), i = 1, \ldots, s,\) such that

(i) \(\bigcup_{i=1}^{s} (\alpha_i, \beta_i) \subset (\alpha, \beta),\)

(ii) \(\beta - \alpha - \sum_{i=1}^{s} (\beta_i - \alpha_i) < \varepsilon,\)

(iii) \(\forall (1 \leq i \leq s) (\alpha_i, \beta_i) \cap R(A) = \emptyset.\)

So, Theorem 2 only implies

\[
\begin{align*}
d(A) &\leq \min \frac{\alpha_i}{\beta_i} \min(1 - \overline{d}(A), \overline{d}(A)) \\
\overline{d}(A) &\leq \min \left(1 - (\beta_i - \alpha_i)\right).
\end{align*}
\]
Finally, in what follows we will replace the open interval \((\alpha, \beta)\) with an open set \(X \subset [0, 1]\) and prove estimates better than \((14)\) and \((15)\). Here \(|X|\) denotes the Lebesgue measure of \(X\).

**Theorem 6.** Let \(X\) be an open set in \([0, 1]\) and write \(g(x) = |X \cap [0, x]|\). If \(X \cap R(A)^d = \emptyset\), then

\[
\overline{d}(A) \leq \frac{x}{y} \min(1 - \overline{d}(A), \overline{d}(A)) + \frac{(y - g(y)) - (x - g(x))}{y}
\]

for every \(x, y\) satisfying

(i) \(0 \leq x < y \leq 1\),

(ii) there exist two sequences \(x_k, \delta_k > 0\) such that \((x_k, x_k + \delta_k) \cap R(A)^d = \emptyset\) for every \(k\) and \(x_k \to x\) as \(k \to \infty\).

Moreover

\[
\overline{d}(A) \leq 1 - |X|.
\]

**Proof.** The proof is similar to the proof of Theorem 2. Instead of \((3)\) we start with the following pairwise disjoint intervals:

\[
(18) \quad (xa_{\theta_n}, xa_{\theta_n} + x), (xa_{\theta_n + 1}, xa_{\theta_n + 1} + x), \ldots, (xa_{n-1}, xa_{n-1} + x), (xa_n, ya_n).
\]

First assume that

(ii)' \((x, x + \delta) \cap R(A) = \emptyset\) for some \(\delta > 0\).

Then for sufficiently large \(i\), the interval \((xa_i, xa_i + x)\) cannot intersect \(A\), since \((xa_i, xa_i + \delta a_i) \cap A = \emptyset\). Moreover, for all sufficiently small \(\varepsilon > 0\), the set \(X \cap (x, y)\) can be approximated by a finite sequence of pairwise disjoint open intervals \((\alpha_i, \beta_i)\), \(i = 1, \ldots, s\), such that \(\bigcup_{i=1}^{s} (\alpha_i, \beta_i) \subset X \cap (x, y)\), \(|X \cap (x, y) - \bigcup_{i=1}^{s} (\alpha_i, \beta_i)| < \varepsilon\) and \(\bigcup_{i=1}^{s} (\alpha_i, \beta_i) \cap R(A) = \emptyset\). Hence, the number of terms of \(B = N - A\) lying in \((xa_n, ya_n)\) is greater than \(a_n(g(y) - g(x) - \varepsilon) - s\) and we have

\[B(ya_n) \geq (n - \theta_n)(x - 1) + (a_n(g(y) - g(x) - \varepsilon) - s) + B(xa_{\theta_n}).\]

Replacing \(n\) by \(nk\) and \(x\) by \(xk\) and letting \(k \to \infty\) we find \((16)\).

In the general case, since \(g(x)\) is continuous, \((ii)'\) can be replaced by \((ii)\).

To prove \((17)\) note only that \((10)\) can be replaced by \(\overline{d}(A)\) by

\[
\frac{b_2 - c_1}{c_n} + \frac{b_3 - c_2}{c_n} + \ldots + \frac{b_n - c_{n-1}}{c_n} \geq \sum_{i=1}^{s} (\beta_i - \alpha_i). \quad \blacksquare
\]

Observe that in Example 1 we have \(\alpha_i/\beta_i = \delta^2/(\gamma^2 a)\) and the minimum of the right hand side of \((16)\) is the same as in \((14)\) and \((1)\). In Example 2,
Density of the ratio set $R(A)$

for $x = \alpha''$ and $y = \beta'$, the right hand side of (16) equals $0.229\ldots$; further, the right hand side of (14) is $0.379\ldots$

6. Concluding remarks

1. The results of T. Šalát mentioned in the introduction can be proved directly by using (1) and (2):

(i) Assume $(\alpha, \beta) \cap R(A) = \emptyset$. If $0 < d(A) = d(A) = \overline{d}(A) < 1/2$ then by (1) we have $d(A) \leq \alpha' \beta' d(A)$ which is a contradiction. If $d(A) \geq 1/2$, then in view of (1) we have $d(A) \leq \frac{d}{2}(1 - d(A)) \leq \frac{1}{2} < \frac{1}{2}$ which also gives a contradiction. Thus (cf. [6, Th. 4]) $d(A) > 0$ implies that $R(A)$ is everywhere dense.

(ii) Assuming $\overline{d}(A) = 1$, (2) implies a contradiction $1 \leq 1 - (\beta - \alpha)$; thus (cf. [6, Th. 1]) $\overline{d}(A) = 1$ implies that $R(A)$ is everywhere dense.

2. It is proved in [3, Th. 2] that if $\mathbb{N} = A \cup B$, then at least one of $R(A)$ or $R(B)$ is everywhere dense in $[0, \infty)$. This can also be proved by using our basic relations (1) and (2).

Assume that $\mathbb{N} = A \cup B$, $A \cap B = \emptyset$, $(\alpha, \beta) \cap R(A) = \emptyset$ and $(\alpha', \beta') \cap R(B) = \emptyset$. Since $d(A) = 1 - \overline{d}(B)$ and $\overline{d}(A) = 1 - d(B)$, applying (1) and (2) we get

(i) $(\beta - \alpha) \leq d(B)$,

(ii) $d(B) \leq \frac{\alpha'}{\beta'}(1 - \overline{d}(B))$,

(iii) $1 - \overline{d}(B) \leq \frac{\alpha}{\beta}d(B)$.

Starting with (i) and then repeatedly applying (ii) and (iii) we get $\beta - \alpha = 0$.

3. A related question is studied in [2].

References


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