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On inhomogeneous Diophantine approximation and the Nishioka–Shiokawa–Tamura algorithm

by

Takao Komatsu (Nagaoka)

In memory of Professor Takayoshi Mitsui

We obtain the values concerning $\mathcal{M}(\theta,\phi)=\liminf_{|q|\to\infty}|q|\|q\theta-\phi\|$ using the algorithm by Nishioka, Shiokawa and Tamura. In application, we give the values $\mathcal{M}(\theta,1/2)$, $\mathcal{M}(\theta,1/a)$, $\mathcal{M}(\theta,1/\sqrt{ab(ab+4)})$ and so on when $\theta=(\sqrt{ab(ab+4)}-ab)/(2a)=[0;a,b,a,b,\ldots]$.

1. Introduction. Let θ be irrational and ϕ real. We suppose throughout that $q\theta-\phi$ is never integral for any integer q. Define

$$\mathcal{M}(\theta,\phi)=\liminf_{|q|\to\infty}|q|\|q\theta-\phi\|,$$

which is called the *inhomogeneous approximation constant* for the pair θ, ϕ. It is convenient to introduce the functions

$$\mathcal{M}_+(\theta,\phi)=\liminf_{q\to+\infty}q\|q\theta-\phi\|$$

and

$$\mathcal{M}_-(\theta,\phi)=\liminf_{q\to+\infty}q\|q\theta+\phi\|=\liminf_{q\to-\infty}|q|\|q\theta-\phi\|.$$

Then $\mathcal{M}(\theta,\phi)=\min(\mathcal{M}_+(\theta,\phi),\mathcal{M}_-(\theta,\phi))$. These notations are introduced by Cusick, Rockett and Szüsz [2]. $\mathcal{M}(\theta,\phi)$ or $\mathcal{M}_+(\theta,\phi)$ has been treated by Cassels [1], Descombes [3], Sós [9], Cusick *et al.* [2] and the author [5] by using several algorithms for inhomogeneous Diophantine approximation in which ϕ is expressed by the continued fraction expansion of θ. However, it is not easy to evaluate $\mathcal{M}(\theta,\phi)$ if it exists for any given pair of θ and ϕ.

In this paper we establish the relationship between $\mathcal{M}(\theta,\phi)$ and the algorithm of Nishioka, Shiokawa and Tamura. Indeed, this was hinted at in [5] but has not been proved yet. If we use this result, we can find the

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exact value of $\mathcal{M}(\theta,\phi)$ for any pair of θ and ϕ at least when θ is a positive real root of a quadratic equation and $\phi\in\mathbb{Q}(\theta)$. In fact, we give several applications when $\theta=(\sqrt{ab(ab+4)}-ab)/(2a)=[0;a,b,a,b,\ldots]=[0;\overline{a,b}]$ for fixed positive integers a and b.

2. Algorithm by Nishioka, Shiokawa and Tamura. We first introduce several notations and show how Nishioka, Shiokawa and Tamura [6] represent ϕ through the continued fraction expansion of θ. As usual $\theta=[a_0;a_1,a_2,\ldots]$ denotes the continued fraction expansion of θ, where

$$\theta=a_0+\theta_0,\qquad a_0=\lfloor\theta\rfloor,$$
$$1/\theta_{n-1}=a_n+\theta_n,\qquad a_n=\lfloor 1/\theta_{n-1}\rfloor\qquad(n=1,2,\ldots).$$

The kth convergent $p_k/q_k=[a_0;a_1,\ldots,a_k]$ of θ is then given by the recurrence relations

$$p_k=a_kp_{k-1}+p_{k-2}\qquad(k=0,1,\ldots),\qquad p_{-2}=0,\ \ p_{-1}=1,$$
$$q_k=a_kq_{k-1}+q_{k-2}\qquad(k=0,1,\ldots),\qquad q_{-2}=1,\ \ q_{-1}=0.$$

Denote by $\phi={}_\theta[b_0;b_1,b_2,\ldots]$ the expansion of ϕ in terms of the sequence $\{\theta_0,\theta_1,\ldots\}$, where

$$\phi=b_0-\phi_0,\qquad b_0=\lceil\phi\rceil,$$
$$\phi_{n-1}/\theta_{n-1}=b_n-\phi_n,\qquad b_n=\lceil\phi_{n-1}/\theta_{n-1}\rceil\qquad(n=1,2,\ldots).$$

Then ϕ is represented by

$$\begin{aligned}\phi&=b_0-b_1\theta_0+b_2\theta_0\theta_1-\ldots+(-1)^kb_k\theta_0\theta_1\ldots\theta_{k-1}-(-1)^k\theta_0\theta_1\ldots\theta_{k-1}\phi_k\\&=b_0-\sum_{k=1}^{\infty}(-1)^{k-1}b_k\theta_0\theta_1\ldots\theta_{k-1}=b_0-\sum_{k=1}^{\infty}b_kD_{k-1},\end{aligned}$$

where $D_k=q_k\theta-p_k=(-1)^k\theta_0\theta_1\ldots\theta_k$. As usual $\overline{a_k,a_{k+1},\ldots,a_n}$ is the periodic sequence with period $a_k,a_{k+1},\ldots,a_n$. We interpret $\overline{b_k,b_{k+1},\ldots,b_n}$ similarly.

Now, we state our main theorem.

Theorem 1.

$$\mathcal{M}_-(\theta,\phi)=\liminf_{n\to+\infty}\min(B_n\|B_n\theta+\phi\|,B_n^*\|B_n^*\theta+\phi\|),$$

where $B_n=\sum_{k=1}^n b_kq_{k-1}$ *and* $B_n^*=B_n-q_{n-1}$.

Proof. Let n be odd. Then $\|B_n\theta+\phi\|=\{B_n\theta+\phi\}=\phi_nD_{n-1}$. Take any integer k with $0<k<q_n$. Since $\{(q_n-jq_{n-1})\theta\}=1-|D_n|-jD_{n-1}$ is the jth largest value ($j=1,\ldots,q_n-1$) among $\{k\theta\}$ (see e.g. [4], [8]

for details),

$$\begin{aligned}\{(B_n + q_n - jq_{n-1})\theta + \phi\} &= \{B_n\theta + \phi\} + \{(q_n - jq_{n-1})\theta\}\\ &= 1 - (|D_n| + (j - \phi_n)D_{n-1})\\ &< 1 - \phi_n D_{n-1} = 1 - \|B_n\theta + \phi\|\end{aligned}$$

if $j \neq 1$. However, $\{(B_n + q_n - q_{n-1})\theta + \phi\}$ is less than 1, but less than $1 - \|B_n\theta + \phi\|$ if and only if $\theta_n + 1 > 2\phi_n$. Thus, for any integer k with $0 < k < q_n$ and $k \neq q_n - q_{n-1}$ we have $\|B_n\theta + \phi\| < \|(B_n + k)\theta + \phi\|$, yielding $B_n\|B_n\theta + \phi\| < (B_n + k)\|(B_n + k)\theta + \phi\|$.

Next, consider a positive integer $B_n - k$ with $0 < k < q_n$. When $b_n \le a_n$, then $B_n - (q_n - q_{n-1}) < B_{n-1} + q_{n-1}$. Hence, if $B_n - q_n + q_{n-1} < B_{n-1}$, then

$$\begin{aligned}&(B_n - q_n + q_{n-1})\|(B_n - q_n + q_{n-1})\theta + \phi\|\\ &\le \min(B_{n-1}\|B_{n-1}\theta + \phi\|, (B_{n-1} - q_{n-1} + q_{n-2})\|(B_{n-1} - q_{n-1} + q_{n-2})\theta + \phi\|).\end{aligned}$$

Otherwise, it is sufficient to consider smaller n. When $b_n = a_n + 1$, we obtain

$$\begin{aligned}\|(B_n - q_n + q_{n-1})\theta + \phi\| &= 1 + \|B_{n-1}\theta + \phi\| - \{(q_n - q_{n-1})\theta\}\\ &= 1 + \phi_n|D_{n-1}| - (1 - |D_n| - |D_{n-1}|)\\ &= (1 + \phi_n + \theta_n)|D_{n-1}|\\ &> |D_{n-1}| + (1 - \phi_{n-1})|D_{n-2}|\\ &= \|(B_{n-1} + q_{n-1} - q_{n-2})\theta + \phi\|,\end{aligned}$$

yielding

$$\begin{aligned}&(B_n - q_n + q_{n-1})\|(B_n - q_n + q_{n-1})\theta + \phi\|\\ &\qquad > (B_{n-1} + q_{n-1} - q_{n-2})\|(B_{n-1} + q_{n-1} - q_{n-2})\theta + \phi\|.\end{aligned}$$

Therefore, it is sufficient to pay attention only to the small fractional parts $\{k\theta\}$. Since $\{jq_{n-1}\theta\} = jD_{n-1}$ is the jth least ($j = 1, \dots, q_n - 1$) (see [4], [8]), we have

$$\begin{aligned}\|(B_n - jq_{n-1})\theta + \phi\| &= \{jq_{n-1}\theta\} - \{B_n\theta + \phi\} = (j - \phi_n)D_{n-1}\\ &> \phi_n D_{n-1} = \|B_n\theta + \phi\|\end{aligned}$$

if $j \neq 1$. However, $\|(B_n - q_{n-1})\theta + \phi\| > \|B_n\theta + \phi\|$ if and only if $\phi_n < 1/2$. Thus, we have $\|B_n\theta + \phi_n\| < \|(B_n - k)\theta + \phi\|$ for any integer k with $0 < k < q_n$ and $k \neq q_{n-1}$. When $b_n \le 2$, the assertion holds because $B_n - q_{n-1} \le B_{n-1} + q_{n-1}$. Thus, we can assume that $b_n \ge 3$. Since $\|(B_n - jq_{n-1})\theta + \phi\|$ is the jth least value among $\|(B_n - k)\theta + \phi\|$ for all integers k with $0 < k < q_n$,

$$(B_n - jq_{n-1})\|(B_n - jq_{n-1})\theta + \phi\| < (B_n - k)\|(B_n - k)\theta + \phi\|$$

for $(j - 1)q_{n-1} < k < jq_{n-1}$ ($j = 1, \dots, b_n - 1$). Therefore, from $B_n =$

$B_{n-1} + b_n q_{n-1} > ((j+1) - \phi_n) q_{n-1}$ we have

$$\begin{aligned}(B_n - jq_{n-1})\|(B_n - jq_{n-1})\theta + \phi\| &= (B_n - jq_{n-1})(j - \phi_n)D_{n-1} \\ &> (B_n - q_{n-1})(1 - \phi_n)D_{n-1} \\ &= (B_n - q_{n-1})\|(B_n - q_{n-1})\theta + \phi\|.\end{aligned}$$

Finally, since

$$\begin{aligned}\|(B_n - q_{n-1})\theta + \phi\| &= (1 - \phi_n)D_{n-1} \\ &< |D_n| + (1 - \phi_n)D_{n-1} = \|(B_n + q_n - q_{n-1})\theta + \phi\|\end{aligned}$$

and $B_n - q_{n-1} < B_n + q_{n-1} - q_{n-2}$, we conclude that

$$(B_n - q_{n-1})\|(B_n - q_{n-1})\theta + \phi\| < (B_n + q_n - q_{n-1})\|(B_n + q_n - q_{n-1})\theta + \phi\|.$$

Let n be even. Then $\|B_n\theta + \phi\| = \phi_n|D_{n-1}| = 1 - \{B_n\theta + \phi\}$. The rest of the part is similar to the odd case. ∎

REMARK. (1) From the proof above, $B_n^*\|B_n^*\theta + \phi\| < B_n\|B_n\theta + \phi\|$ if $\phi_n > 1/2$. But, $B_n^*\|B_n^*\theta + \phi\| > B_n\|B_n\theta + \phi\|$ does not always hold even though $\phi_n < 1/2$.

(2) Together with $\mathcal{M}_+(\theta, \phi) = \mathcal{M}_-(\theta, 1 - \phi)$, one can obtain the value $\mathcal{M}(\theta, \phi)$.

3. A special case. Without loss of generality, we assume hereafter that $0 < \theta < 1$ and $0 < \phi \le 1/2$. Set

$$\theta = \frac{\sqrt{D} - ab}{2a} = [0; a, b, a, b, \ldots] = [0; \overline{a, b}]$$

where $D = ab(ab + 4)$ is the discriminant of the quadratic equation $a\theta^2 + ab\theta - b = 0$. First of all, we consider an artificially made ϕ.

THEOREM 2. *Let* $\phi = (1 - s) + s\theta = {}_\theta[1; \overline{sa, sb}]$ *with* $0 < s < 1$ *satisfying* $sa, sb \in \mathbb{N}$. *Then*

$$\mathcal{M}(\theta, \phi) = \frac{\|s\|^2(ab - |a - b|)}{\sqrt{D}}.$$

Proof. From the theory of continued fractions,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \right\}^n = \begin{pmatrix} p_{2n} & p_{2n-1} \\ q_{2n} & q_{2n-1} \end{pmatrix}.$$

It follows that

$$\begin{aligned}\begin{pmatrix} q_{2n} & q_{2n-1} \\ p_{2n} & p_{2n-1} \end{pmatrix} &= \begin{pmatrix} ab+1 & a \\ b & 1 \end{pmatrix}^n \\ &= \frac{1}{\alpha-\beta}\begin{pmatrix} (\alpha^{n+1}-\beta^{n+1})-(\alpha^n-\beta^n) & a(\alpha^n-\beta^n) \\ b(\alpha^n-\beta^n) & (\alpha^n-\beta^n)-(\alpha^{n-1}-\beta^{n-1}) \end{pmatrix}\end{aligned}$$

where

$$\alpha = \frac{(ab+2)+\sqrt{D}}{2}, \qquad \beta = \frac{(ab+2)-\sqrt{D}}{2}$$

satisfying $\alpha+\beta = ab+2$, $\alpha\beta = 1$ and $\alpha-\beta = \sqrt{D}$. Notice that

$$\phi_{2n-2} = s(1-\theta), \qquad \phi_{2n-1} = s(1-\theta_1) \qquad (n = 1, 2, \ldots)$$

where $\theta_1 = (\sqrt{D}-ab)/(2b) = [0; \overline{b, a}]$.

Now, the relations

$$\begin{aligned} B_{2n} &= \sum_{i=1}^{n} (saq_{2i-2} + sbq_{2i-1}) \\ &= \frac{s}{\sqrt{D}} \sum_{i=1}^{n} (a(\alpha^i-\beta^i) - a(\alpha^{i-1}-\beta^{i-1}) + ba(\alpha^i-\beta^i)) \\ &\sim \frac{sa}{\sqrt{D}} \cdot \frac{(1+b)\alpha-1}{\alpha-1}\alpha^n \end{aligned}$$

and $\|B_{2n}\theta+\phi\| = \phi_{2n}|D_{2n-1}| = s(1-\theta)\beta^n$ entail that

$$B_{2n}\|B_{2n}\theta+\phi\| \to \frac{s^2a}{\sqrt{D}} \cdot \frac{(1+b)\alpha-1}{\alpha-1}(1-\theta) = \frac{s^2(ab+a-b)}{\sqrt{D}} \qquad (n\to\infty).$$

In a similar manner,

$$B^*_{2n} = B_{2n} - q_{2n-1} \sim \frac{a}{\sqrt{D}} \cdot \frac{s(1+b)\alpha - s - (\alpha-1)}{\alpha-1}\alpha^n$$

and $\|B^*_{2n}\theta+\phi\| = (1-\phi_{2n})|D_{2n-1}| = (s\theta-s+1)\beta^n$ entail that

$$\lim_{n\to\infty} B^*_{2n}\|B^*_{2n}\theta+\phi\| = \frac{sab+(2s-1)a-s^2(ab+a-b)}{\sqrt{D}}.$$

Moreover,

$$B_{2n-1} = \sum_{i=1}^{n}(saq_{2i-2}+sbq_{2i-1}) - sbq_{2n-1} \sim \frac{sa}{\sqrt{D}} \cdot \frac{\alpha-1+b}{\alpha-1}\alpha^n$$

and $\|B_{2n-1}\theta+\phi\| = \phi_{2n-1}|D_{2n-2}| = s(1-\theta_1)\beta^n$ entail that

$$\lim_{n\to\infty} B_{2n-1}\|B_{2n-1}\theta+\phi\| = \frac{s^2(ab-a+b)}{\sqrt{D}}.$$

Finally,

$$B^*_{2n-1} = B_{2n-1} - q_{2n-2} \sim \frac{a}{\sqrt{D}}\,(s-(1-s)\theta)\,\alpha^n$$

and $\|B^*_{2n-1}\theta+\phi\| = (1-\phi_{2n-1})|D_{2n-2}| = (1-s+s\theta_1)\beta^n$ entail that

$$\lim_{n\to\infty} B^*_{2n-1}\|B^*_{2n-1}\theta+\phi\| = \frac{sab+(2s-1)b-s^2(ab-a+b)}{\sqrt{D}}\,.$$

Since $sb(1-2s)+(2s-1) \ge 1\cdot(1-2s)+(2s-1)=0$ and $sa(1-2s)+(2s-1) \ge 1\cdot(1-2s)+(2s-1)=0$, one has

$$\mathcal{M}_-(\theta,\phi) = \frac{s^2(ab-|a-b|)}{\sqrt{D}}.$$

Next, one expands

$$1-\phi = {}_\theta[1;(1-s)a+1,\overline{(1-s)b,(1-s)a}]$$

and

$$\phi_0 = (1-s)+s\theta,$$
$$\phi_{2n-1} = (1-s)(1-\theta_1),\qquad \phi_{2n} = (1-s)(1-\theta)\qquad (n=1,2,\ldots).$$

It follows that

$$\lim_{n\to\infty} B_{2n-1}\|B_{2n-1}\theta-\phi\| = \frac{(1-s)^2(ab-a+b)}{\sqrt{D}},$$
$$\lim_{n\to\infty} B^*_{2n-1}\|B^*_{2n-1}\theta-\phi\| = \frac{(1-s)ab+(1-2s)b-(1-s)^2(ab-a+b)}{\sqrt{D}},$$
$$\lim_{n\to\infty} B_{2n}\|B_{2n}\theta-\phi\| = \frac{(1-s)^2(ab+a-b)}{\sqrt{D}},$$
$$\lim_{n\to\infty} B^*_{2n}\|B^*_{2n}\theta-\phi\| = \frac{(1-s)ab+(1-2s)a-(1-s)^2(ab+a-b)}{\sqrt{D}},$$

yielding

$$\mathcal{M}_+(\theta,\phi) = \mathcal{M}_-(\theta,1-\phi) = \frac{(1-s)^2(ab-|a-b|)}{\sqrt{D}}.\ \blacksquare$$

Example 1 ([7]). Let $a=b$ be even, and $s=1/2$. Then the pair of $\theta=[0;\overline{a}]$ and $\phi=(1+\theta)/2={}_\theta[1;\overline{a/2}]$ gives

$$\lim_{n\to\infty} B_n\|B_n\theta+\phi\| = \lim_{n\to\infty} B^*_n\|B^*_n\theta+\phi\| = \frac{a}{4\sqrt{a^2+4}}.$$

Since $1-\phi=(1-\theta)/2={}_\theta[1;a/2+1,\overline{a/2}]$,

$$\lim_{n\to\infty} B_n\|B_n\theta-\phi\| = \lim_{n\to\infty} B^*_n\|B^*_n\theta-\phi\| = \frac{a}{4\sqrt{a^2+4}}.$$

Therefore,

$$\mathcal{M}(\theta,\phi)=\mathcal{M}_{\pm}(\theta,\phi)=\frac{a}{4\sqrt{a^2+4}},$$

which gives an answer to an open problem related to Khinchin's results ([5]).

EXAMPLE 2. Let $a=b$ be a composite odd number, say $a=a_1a_2$, where a_1 and a_2 are also odd numbers with $a_1\le a_2$ and put $s=1/a_1$. Then

$$\mathcal{M}(\theta,\phi)=\mathcal{M}_{\pm}(\theta,\phi)=\frac{a_2}{a_1\sqrt{a^2+4}},$$

which answers another open problem related to Khinchin's results ([5]). However, the case of odd prime a is not settled yet.

4. Some basic applications. In this section we compute $\mathcal{M}(\theta,\phi)$ for some basic ϕ's as seen in [2] and [5]. Put $\Xi_n=B_n\|B_n\theta-\phi\|\sqrt{D}/\phi^2$, $\Xi_n^*=B_n^*\|B_n^*\theta-\phi\|\sqrt{D}/\phi^2$, $\Psi_n=B_n\|B_n\theta+\phi\|\sqrt{D}/\phi^2$ and $\Psi_n^*=B_n^*\|B_n^*\theta+\phi\|\sqrt{D}/\phi^2$ for simplicity.

THEOREM 3.

$$\mathcal{M}\left(\theta,\frac{1}{2}\right)=\begin{cases}\dfrac{\min(a,b)}{4\sqrt{D}} & \textit{if both } a \textit{ and } b \textit{ are odd},\\[2mm] \dfrac{a}{4\sqrt{D}} & \textit{otherwise}.\end{cases}$$

Proof. It is clear that $\Xi_n=\Psi_n$ and $\Xi_n^*=\Psi_n^*$ when $\phi=1/2$.

If $a=1$ and b is even with $b\ge 4$, then $\phi=1/2$ is expanded as

$$\frac{1}{2}={}_{\theta}\left[1;1,\frac{b}{2},\overline{1,b+1,1,\frac{b}{2}-1}\right]$$

and for $n=1,2,\ldots,$

$$\phi_{4n-2}=\frac{b+2-\sqrt{D}}{4},\qquad \phi_{4n-1}=\frac{5b-\sqrt{D}}{4b},$$
$$\phi_{4n}=\frac{b+3-\sqrt{D}}{2},\qquad \phi_{4n+1}=\frac{5b-3\sqrt{D}}{4b}.$$

By the inhomogeneous continued fraction expansion above, for $n=1,2,\ldots,$

$$B_{4n-2}=1+\frac{b}{2}q_1+\sum_{i=1}^{n-1}\left(q_{4i-2}+(b+1)q_{4i-1}+q_{4i}+\left(\frac{b}{2}-1\right)q_{4i+1}\right),$$
$$B_{4n-1}=B_{4n-2}+q_{4n-2},\quad B_{4n}=B_{4n-2}+q_{4n-2}+(b+1)q_{4n-1},$$
$$B_{4n+1}=B_{4n-2}+q_{4n-2}+(b+1)q_{4n-1}+q_{4n}.$$

Now,

$$\begin{aligned}\|B_{4n-2}\theta+\phi\|&=\phi_{4n-2}|D_{4n-3}|\\&=\phi_{4n-2}\theta_0\theta_1\ldots\theta_{4n-3}=\tfrac{1}{2}\beta(\theta_0\theta_1)^{2n-1}=\tfrac{1}{2}\beta^{2n},\end{aligned}$$

and

$$\sqrt{D}B_{4n-2} = \sqrt{D}\left(1+\frac{b}{2}\right) + \sum_{i=1}^{n-1}\biggl((\alpha^{2i}-\beta^{2i}) - (\alpha^{2i-1}-\beta^{2i-1})$$
$$+ (b+1)(\alpha^{2i}-\beta^{2i}) + (\alpha^{2i+1}-\beta^{2i+1}) - (\alpha^{2i}-\beta^{2i})$$
$$+ \left(\frac{b}{2}-1\right)(\alpha^{2i+1}-\beta^{2i+1})\biggr)$$
$$= \sqrt{D}\left(1+\frac{b}{2}\right) + \left(\frac{b}{2}\alpha^2+(b+1)\alpha-1\right)\alpha\cdot\frac{\alpha^{2n-2}-1}{\alpha^2-1}$$
$$- \left(\frac{b}{2}\beta^2+(b+1)\beta-1\right)\beta\cdot\frac{\beta^{2n-2}-1}{\beta^2-1}$$
$$= \frac{1}{2}(b\alpha^2+2(b+1)\alpha-2)\frac{\alpha^{2n-1}}{\alpha^2-1} + (\text{others}).$$

Here "others" tend to 0 in Ψ_{4n-2} as n tends to infinity. Therefore, we obtain

$$\lim_{n\to\infty}\Psi_{4n-2} = \frac{b\alpha+2(b+1)-2\beta}{\alpha^2-1} = 1.$$

Similarly, one finds $\lim_{n\to\infty}\Psi^*_{4n-2} = 2b-1$. Since $B^*_{4n-1}\|B^*_{4n-1}\theta+\phi\| < B_{4n-1}\|B_{4n-1}\theta+\phi\|$ or $\Psi^*_{4n-1} < \Psi_{4n-1}$ as $\phi_{4n-1} > 1/2$, there is no necessity to evaluate the right-hand side (in fact, one can find $\lim_{n\to\infty}\Psi_{4n-1} = 6b-1$). And $B^*_{4n-1} = B_{4n-2}$ entails $\Psi^*_{4n-1} = \Psi_{4n-2} \to 1$ $(n\to\infty)$.

Since $B^*_{4n}\|B^*_{4n}\theta+\phi\| < B_{4n}\|B_{4n}\theta+\phi\|$ or $\Psi^*_{4n} < \Psi_{4n}$ as $\phi_{4n} > 1/2$, it is sufficient to evaluate the left-hand side (the right-hand side tends to $2b+9$). The relations

$$\|B^*_{4n}\theta+\phi\| = (1-\phi_{4n})|D_{4n-1}| = \left(\tfrac{1}{2}-\beta\right)\theta_0^{2n}\theta_1^{2n} = \left(\tfrac{1}{2}-\beta\right)\beta^{2n}$$

and

$$\sqrt{D}\,B^*_{4n} = \sqrt{D}(B_{4n-2}+q_{4n-2}+bq_{4n-1})$$
$$= \frac{1}{2}(b\alpha^2+2(b+1)\alpha-2)\frac{\alpha^{2n-1}}{\alpha^2-1}$$
$$+ (\alpha^{2n}-\beta^{2n}) - (\alpha^{2n-1}-\beta^{2n-1}) + b(\alpha^{2n}-\beta^{2n}) + (\text{others})$$

entail that

$$\lim_{n\to\infty}\Psi^*_{4n} = 2\left(\frac{1}{2}-\beta\right)\left(\frac{b\alpha+2(b+1)-2\beta}{\alpha^2-1} + (2-2\beta)+2b\right)$$
$$= (1-2\beta)(3-2\beta+2b) = 2b-1.$$

Since $B^*_{4n+1} = B_{4n}$, one obtains $\Psi^*_{4n+1} = \Psi_{4n} \to 2b+9$ $(n\to\infty)$. On the other hand, the relations

$$\|B_{4n+1}\theta+\phi\| = \phi_{4n+1}|D_{4n}| = \frac{5b-3\sqrt{D}}{4b}\theta_0^{2n+1}\theta_1^{2n} = \left(\frac{1}{2}-2\beta\right)\beta^{2n}$$

and

$$\sqrt{D}B_{4n+1} = \frac{1}{2}(b\alpha^2 + 2(b+1)\alpha - 2)\frac{\alpha^{2n-1}}{\alpha^2-1} + (\alpha^{2n} - \alpha^{2n-1}) + (b+1)\alpha^{2n} + (\alpha^{2n+1} - \alpha^{2n}) + (\text{others})$$

entail that

$$\lim_{n\to\infty} \Psi_{4n+1} = 4\left(\frac{1}{2} - 2\beta\right) \times \left(\frac{b\alpha + 2(b+1) - 2\beta}{2(\alpha^2-1)} + (1-\beta) + (b+1) + (\alpha - 1)\right) = (-(2b+3) + 2\sqrt{D})(2b+3+2\sqrt{D}) = 4b - 9.$$

If $a = 1$ and b is odd with $b \geq 3$, then $\phi = 1/2$ is expanded as

$$\frac{1}{2} = {}_{\theta}\left[1; 1, \frac{b+1}{2}, \overline{1, \frac{b-1}{2}, 1, b+1, 1, \frac{b-1}{2}}\right]$$

and for $n = 1, 2, \ldots,$

$$\phi_{6n-4} = \frac{b+4-\sqrt{D}}{4}, \quad \phi_{6n-3} = \frac{2b-\sqrt{D}}{2b}, \quad \phi_{6n-2} = \frac{b+2-\sqrt{D}}{4},$$
$$\phi_{6n-1} = \frac{5b-\sqrt{D}}{4b}, \quad \phi_{6n} = \frac{b+3-\sqrt{D}}{2}, \quad \phi_{6n+1} = \frac{5b-3\sqrt{D}}{4b}.$$

Hence, for $n = 1, 2, \ldots$ one can similarly find

$$\Psi^*_{6n-4} \to b, \quad \Psi_{6n-3} \to 3b-4, \quad \Psi^*_{6n-3} = \Psi_{6n-4} \to b+4,$$
$$\Psi_{6n-2} = \Psi^*_{6n-1} \to 1, \quad \Psi^*_{6n-2} \to 2b-1, \quad \Psi_{6n-1} \to 6b-1,$$
$$\Psi_{6n} = \Psi^*_{6n+1} \to 2b+9, \quad \Psi^*_{6n} \to 2b-1, \quad \Psi_{6n+1} \to 4b-9 \quad (n \to \infty).$$

If $a = 1$ and $b = 2$, then $\phi = 1/2$ is expanded as

$$\frac{1}{2} = {}_{\theta}[1; 1, 1, 1, 3, \overline{2, 3, 1, 2}]$$

and for $n = 1, 2, \ldots,$

$$\phi_{4n} = \frac{5-2\sqrt{3}}{2}, \quad \phi_{4n+1} = \frac{9-3\sqrt{3}}{4}, \quad \phi_{4n+2} = \frac{6-3\sqrt{3}}{2}, \quad \phi_{4n+3} = \frac{7-3\sqrt{3}}{4}.$$

In a similar manner, one finds

$$\Psi_{4n} \to 13, \quad \Psi^*_{4n} \to 3, \quad \Psi_{4n+1} \to 27, \quad \Psi^*_{4n+1} \to 1,$$
$$\Psi_{4n+2} = \Psi^*_{4n+3} \to 9, \quad \Psi^*_{4n+2} \to 11, \quad \Psi_{4n+3} \to 11 \quad (n \to \infty).$$

If $a = b = 1$, then $\phi = 1/2$ is expanded as

$$\frac{1}{2} = {}_{\theta}[1; 1, 1, 2, \overline{2, 1, 1}]$$

and for $n = 1, 2, \dots,$

$$\phi_{3n+1} = \frac{9-3\sqrt{5}}{4}, \quad \phi_{3n+2} = \frac{7-3\sqrt{5}}{4}, \quad \phi_{3n+3} = \frac{4-\sqrt{5}}{2}.$$

Similarly one finds that for $n = 1, 2, \dots,$

$$\Psi_{3n+1} = \Psi^*_{3n+2} \to 9, \quad \Psi^*_{3n+1} \to 5, \quad \Psi_{3n+3} \to 11, \quad \Psi_{3n+2} = \Psi^*_{3n+3} \to 1$$

as n tends to infinity. Therefore, we obtain

$$\mathcal{M}(\theta,\phi) = \mathcal{M}_{\pm}(\theta,\phi) = \frac{1}{4\sqrt{5}}.$$

If a is odd with $a \ge 3$ and b is even, then $\phi = 1/2$ is represented as

$$\frac{1}{2} = {}_{\theta}\left[1; \frac{a+1}{2}, \overline{\frac{b}{2}, \frac{a+1}{2}, b, \frac{a-1}{2}}\right].$$

Then $\phi_0 = 1/2$, and for $n = 1, 2, \dots,$

$$\phi_{4n-3} = \frac{a+2}{4} - \frac{\sqrt{D}}{4b}, \quad \phi_{4n-2} = \frac{b+2}{4} - \frac{\sqrt{D}}{4a},$$
$$\phi_{4n-1} = \frac{a+4}{4} - \frac{\sqrt{D}}{4b}, \quad \phi_{4n} = \frac{b+1}{2} - \frac{\sqrt{D}}{2a}.$$

It follows that

$$\begin{aligned}
\Psi_{4n-1} &\to (2b-1)a + 4b, & \Psi^*_{4n-1} &\to a,\\
\Psi_{4n} &\to (2b+1)a - 4b, & \Psi^*_{4n} &\to (2b-1)a + 4b,\\
\Psi_{4n+1} &\to (b-1)a + b, & \Psi^*_{4n+1} &\to (b+1)a - b,\\
\Psi_{4n+2} &\to (b+1)a - b, & \Psi^*_{4n+2} &\to (b-1)a + b \quad (n \to \infty).
\end{aligned}$$

If a is odd with $a \ge 3$ and b is odd with $b \ge 3$, then $\phi = 1/2$ is represented as

$$\frac{1}{2} = {}_{\theta}\left[1; \frac{a+1}{2}, \overline{\frac{b+1}{2}, a, \frac{b-1}{2}, \frac{a+1}{2}, b, \frac{a-1}{2}}\right].$$

Then $\phi_0 = 1/2$, and for $n = 1, 2, \dots,$

$$\phi_{6n-5} = \frac{a+2}{4} - \frac{\sqrt{D}}{4b}, \quad \phi_{6n-4} = \frac{b+4}{4} - \frac{\sqrt{D}}{4a}, \quad \phi_{6n-3} = \frac{a+1}{2} - \frac{\sqrt{D}}{2b},$$
$$\phi_{6n-2} = \frac{b+2}{4} - \frac{\sqrt{D}}{4a}, \quad \phi_{6n-1} = \frac{a+4}{4} - \frac{\sqrt{D}}{4b}, \quad \phi_{6n} = \frac{b+1}{2} - \frac{\sqrt{D}}{2a}.$$

It follows that

$$\begin{aligned}
\Psi_{6n-4} &\to (2b+4)a - b, & \Psi^*_{6n-4} &\to b,\\
\Psi_{6n-3} &\to (2b-4)a + b, & \Psi^*_{6n-3} &\to (2b+4)a - b,\\
\Psi_{6n-2} &\to (b+1)a - b, & \Psi^*_{6n-2} &\to (b-1)a + b,
\end{aligned}$$

$$\Psi_{6n-1} \to (2b-1)a+4b, \quad \Psi^*_{6n-1} \to a,$$
$$\Psi_{6n} \to (2b+1)a-4b, \quad \Psi^*_{6n} \to (2b-1)a+4b,$$
$$\Psi_{6n+1} \to (b-1)a+b, \quad \Psi^*_{6n+1} \to (b+1)a-b \quad (n \to \infty).$$

If a is even, then $\phi = 1/2$ is expanded as

$$\frac{1}{2} = {}_\theta\Big[1; \frac{a}{2}+1, \overline{b, \frac{a}{2}}\Big].$$

Then $\phi_0 = 1/2$, and for $n = 1, 2, \ldots,$

$$\phi_{2n-1} = \frac{a+4}{4} - \frac{\sqrt{D}}{4b} \quad \text{and} \quad \phi_{2n} = \frac{b+1}{2} - \frac{\sqrt{D}}{2a}.$$

It follows that

$$\Psi_{2n} \to (2b+1)a-4b, \quad \Psi^*_{2n} \to (2b-1)a+4b,$$
$$\Psi_{2n+1} \to (2b-1)a+4b, \quad \Psi^*_{2n+1} \to a \quad (n \to \infty). \ \blacksquare$$

THEOREM 4.

$$\mathcal{M}\Big(\theta, \frac{1}{\sqrt{D}}\Big) = \frac{a}{D\sqrt{D}}.$$

Proof. If $a \ge 2$, then $\phi = 1/\sqrt{D}$ is represented as

$$\frac{1}{\sqrt{D}} = {}_\theta[1; a, \overline{1, a+1, b, a-1}].$$

Then $\phi_0 = 1 - 1/\sqrt{D}$, and for $n = 1, 2, \ldots,$

$$\phi_{4n-3} = \frac{ab+1}{2b} - \frac{a(ab+3)}{2\sqrt{D}}, \quad \phi_{4n-2} = \frac{3}{2} - \frac{ab+2}{2\sqrt{D}},$$
$$\phi_{4n-1} = \frac{ab+2b+1}{2b} - \frac{a(ab+5)}{2\sqrt{D}}, \quad \phi_{4n} = \frac{b+2}{2} - \frac{ab^2+4b+2}{2\sqrt{D}}.$$

It follows that

$$\Psi_{4n} \to a(b^2(b+1)a^2 - b(b+1)(b-4)a - (2b+1)^2),$$
$$\Psi^*_{4n} \to a(b^2(b+1)a + (2b+1)^2),$$
$$\Psi_{4n+1} \to a,$$
$$\Psi^*_{4n+1} \to a(b^3a^2 - b^2(b-5)a - (2b-1)^2),$$
$$\Psi_{4n+2} \to a(2b^2a^2 + 8ba - 1),$$
$$\Psi^*_{4n+2} \to a,$$
$$\Psi_{4n+3} \to a(b^2(b-1)a^2 + b(b^2+5b-4)a + (2b+1)^2),$$
$$\Psi^*_{4n+3} \to a(b^2a^2 + 4ba - 1) \quad (n \to \infty).$$

If $a=1$, then $\phi=1/\sqrt{D}$ is expanded as

$$\frac{1}{\sqrt{D}}={}_\theta[1;1,1,2,\overline{b+1,2,b,1}]$$

and for $n=1,2,\dots,$

$$\phi_{4n}=\frac{(b+4)\sqrt{D}-(b^2+4b+2)}{2\sqrt{D}},\qquad \phi_{4n+1}=\frac{(4b+1)\sqrt{D}-(2b^2+7b)}{2b\sqrt{D}},$$
$$\phi_{4n+2}=\frac{(b+3)\sqrt{D}-(b^2+5b+2)}{2\sqrt{D}},\qquad \phi_{4n+3}=\frac{(3b+1)\sqrt{D}-(b^2+5b)}{2b\sqrt{D}}.$$

Thus, one can find

$$\Psi_{4n}\to b^3+7b^2+12b-1,\quad \Psi_{4n}^*\to 1,\quad \Psi_{4n+1}\to 3b^3+11b^2-4b+1,$$
$$\Psi_{4n+1}^*\to 2b^3+8b-1,\quad \Psi_{4n+2}=\Psi_{4n+3}^*\to b^2+4b-1,$$
$$\Psi_{4n+2}^*\to b^3+5b^2+4b+1,\quad \Psi_{4n+3}\to 2b^3+8b-1\quad (n\to\infty).$$

If $b\ge 2$, then $1-\phi=1-1/\sqrt{D}$ is represented as

$$1-\frac{1}{\sqrt{D}}={}_\theta[1;1,b,\overline{a,b,a,b-1}].$$

Then $\phi_0=1/\sqrt{D}$, $\phi_1=(2b-1)/(2b)-a/(2\sqrt{D})$, and for $n=1,2,\dots,$

$$\phi_{4n-2}=\frac{b+1}{2b}-\frac{ab^2-ab+4b-2}{2\sqrt{D}},\qquad \phi_{4n-1}=\frac{ab+2b-1}{2b}-\frac{a(ab+3)}{2\sqrt{D}},$$
$$\phi_{4n}=\frac{b+2}{2}-\frac{ab^2+4b-2}{2\sqrt{D}},\qquad \phi_{4n+1}=\frac{ab+2b-1}{2b}-\frac{a(ab+5)}{2\sqrt{D}}.$$

In a similar manner, one finds that

$$\begin{aligned}
\Xi_{4n}&\to a(b^2(b+1)a^2-b(b^2-5b-4)a-(2b-1)^2),\\
\Xi_{4n}^*&\to a(b^2(b-1)a+(2b-1)^2),\\
\Xi_{4n+1}&\to a(b^2(b-2)a^2+b(b^2+3b-8)a+(2b-1)^2),\\
\Xi_{4n+1}^*&\to a(2b^2a^2+8ba-1),\\
\Xi_{4n+2}&\to a(b^3a^2-b^2(b-5)a-(2b-1)^2),\\
\Xi_{4n+2}^*&\to a(b^2(b-1)a+(2b-1)^2),\\
\Xi_{4n+3}&\to a(b^2(b-1)a^2+b(b+4)(b-1)a+(2b-1)^2),\\
\Xi_{4n+3}^*&\to a(b^2a^2+4ba-1)\qquad (n\to\infty).
\end{aligned}$$

If $a\ge 2$ and $b=1$, then $1-\phi=1-1/\sqrt{D}$ is expanded as

$$1-\frac{1}{\sqrt{D}}={}_\theta[1;1,1,a,\overline{1,a+1,1,a-1}]$$

and

$$\phi_{4n-1} = \frac{(a+1)\sqrt{D} - (a^2+3a)}{2\sqrt{D}}, \qquad \phi_{4n} = \frac{3\sqrt{D} - (a+2)}{2\sqrt{D}},$$
$$\phi_{4n+1} = \frac{(a+3)\sqrt{D} - (a^2+5a)}{2\sqrt{D}}, \qquad \phi_{4n+2} = \frac{3\sqrt{D} - (a+6)}{2\sqrt{D}}.$$

Hence, one can obtain

$$\begin{aligned}&\Xi_{4n-1} = \Xi^*_{4n} \to a, \quad \Xi^*_{4n-1} \to a(a^2+4a-1),\\ &\Xi_{4n} \to a(2a^2+8a-1), \quad \Xi_{4n+1} = \Xi^*_{4n+2} \to a(2a+9),\\ &\Xi^*_{4n+1} \to a(a^2+4a-1), \quad \Xi_{4n+2} \to a(2a^2+6a-9) \quad (n\to\infty).\end{aligned}$$

If $a = b = 1$, then $1-\phi = 1-1/\sqrt{5}$ is expanded as

$$1 - \frac{1}{\sqrt{5}} = {}_\theta[1;1,1,1,1,2,\overline{2,2,1,1}]$$

and

$$\phi_{4n+2} = \frac{5\sqrt{5}-7}{2\sqrt{5}}, \qquad \phi_{4n+3} = \frac{5\sqrt{5}-9}{2\sqrt{5}},$$
$$\phi_{4n+4} = \frac{2\sqrt{5}-4}{\sqrt{5}}, \qquad \phi_{4n+5} = \frac{2\sqrt{5}-3}{\sqrt{5}}.$$

Hence, one can find

$$\begin{aligned}&\Xi_{4n+2} \to 19, \quad \Xi^*_{4n+2} \to 1, \quad \Xi_{4n+3} = \Xi^*_{4n+4} \to 11,\\ &\Xi^*_{4n+3} \to 9, \quad \Xi_{4n+5} \to 11, \quad \Xi_{4n+4} = \Xi^*_{4n+5} \to 4 \quad (n\to\infty). \ \blacksquare\end{aligned}$$

Next, we find the value $\mathcal{M}(\theta, 1/a)$. In fact, we can do more. Concerning an arbitrary divisor of a, say d (≥ 2), we have the following theorem:

THEOREM 5.
$$\mathcal{M}\left(\theta, \frac{1}{d}\right) = \frac{a}{d^2\sqrt{D}}.$$

Proof. $\phi = 1/d$ is represented as

$$\frac{1}{d} = {}_\theta\left[1; \left(1-\frac{1}{d}\right)a+1, \overline{b, \left(1-\frac{1}{d}\right)a}\right]$$

and

$$\begin{aligned}&\phi_0 = 1 - 1/d,\\ &\phi_{2n-1} = 1 - \left(1-\frac{1}{d}\right)\theta_1, \quad \phi_{2n} = \left(1-\frac{1}{d}\right) - \theta \quad (n = 1,2,\ldots).\end{aligned}$$

It follows that

$$\begin{aligned}&\Psi_{2n} \to a(bd+d-1)(d-1) - bd^2, \qquad \Psi^*_{2n} \to (bd-1)a + bd^2,\\ &\Psi_{2n+1} \to a(bd-d+1)(d-1) + bd^2, \quad \Psi^*_{2n+1} \to a(d-1)^2 \quad (n\to\infty).\end{aligned}$$

$1-\phi = 1-1/d$ is represented as

$$1-\frac{1}{d} = {}_\theta\Big[1; \frac{a}{d}+1, \overline{b, \frac{a}{d}}\Big]$$

and

$$\phi_0 = 1/d, \quad \phi_{2n-1} = 1-\frac{1}{d}\theta_1, \quad \phi_{2n} = \frac{1}{d}-\theta \quad (n=1,2,\ldots).$$

In a similar manner, one finds that

$$\Xi_{2n} \to (bd+1)a - bd^2, \qquad \Xi_{2n}^* \to a(bd-d+1)(d-1)+bd^2,$$
$$\Xi_{2n+1} \to (bd-1)a + bd^2, \quad \Xi_{2n+1}^* \to a \quad (n\to\infty). \ \blacksquare$$

Putting $a=d$ yields the desired result.

Corollary 1. *For* $a \ge 2$,

$$\mathcal{M}\Big(\theta, \frac{1}{a}\Big) = \frac{1}{a\sqrt{D}}.$$

Compared with this result, it is not easy to find $\mathcal{M}(\theta, 1/b)$, $\mathcal{M}(\theta, 1/(2a))$ or $\mathcal{M}(\theta, 1/k)$ with $k \in \mathbb{N}$ because these inhomogeneous continued fraction expansions are not so simple.

Theorem 6. *If* $a=1$ *and* $b>1$, *then*

$$\mathcal{M}\Big(\theta, \frac{1}{b}\Big) = \mathcal{M}_+\Big(\theta, \frac{1}{b}\Big) = \frac{1}{b^2\sqrt{D}}.$$

Proof. Let $b \ge 4$. Then $\phi = 1/b$ is expanded as

$$\frac{1}{b} = {}_\theta[1; 1,1,1,2, \underbrace{\overline{1,3,1,4,\ldots,1,b-1,1,b+1,2,b+1,1,1}}_{2b}]$$

and

$$\phi_{2i} = \begin{cases} \dfrac{(i+2)b-2-i\sqrt{D}}{2b} & (i=0,1,2,\ldots,b-1), \\ \dfrac{(i+4)b-2-i\sqrt{D}}{2b} & (i=b,b+1), \end{cases}$$

$$\phi_{2i+1} = \frac{b^2+(2i+1)b-(b-1)\sqrt{D}}{2b^2} \qquad (i=0,1,2,\ldots,b-1),$$

$$\phi_{2b+1} = \frac{4b^2+b-(2b-1)\sqrt{D}}{2b^2}, \qquad \phi_{2b+3} = \frac{2b^2+3b-(2b-1)\sqrt{D}}{2b^2},$$

$$\phi_{2bn+k} = \phi_k \quad (k=4,5,\ldots,2b+3;\ n=1,2,\ldots).$$

Hence, for $n=0,1,\ldots$ and $i=2,3,\ldots,b-1$ one gets

$$\Psi_{2bn+2i} = \Psi_{2bn+2i+1}^* \to (i+1)b^2 - (i^2+i+2)b+1,$$
$$\Psi_{2bn+2i}^* \to (i^2+i)b-1, \quad \Psi_{2bn+2i+1} \to ib^2+(i^2+i+2)b-1$$

and for $n=0,1,\dots,$

$$\Psi_{2bn+2b} \to b^3+3b^2-4b+1, \quad \Psi^*_{2bn+2b} \to 2b-1,$$
$$\Psi_{2bn+2b+1} \to 3b^3-b^2+4b-1, \quad \Psi^*_{2bn+2b+1} \to 2b^2-4b+1,$$
$$\Psi_{2bn+2b+2} = \Psi^*_{2bn+2b+3} \to b^3+3b^2-6b+1, \quad \Psi^*_{2bn+2b+2} \to b^2+4b-1,$$
$$\Psi_{2bn+2b+3} = \Psi^*_{2bn+2b+4} \to 6b-1, \quad \Psi_{2bn+2b+4} \to 3b^2-8b+1 \quad (n\to\infty).$$

Therefore, $\mathcal{M}_-(\theta,1/b) = (2b-1)/(b^2\sqrt{b^2+4b})$.

$1-\phi = 1-1/b$ is expanded as

$$1-\frac{1}{b} = {}_\theta[1;1,b,\underbrace{\overline{1,b-2,1,b-3,\dots,1,4,1,3,1,2,2,b+1,1,b,1,b-1}}_{2b}]$$

and

$$\phi_{2i} = \begin{cases} \dfrac{b^2-(i-2)b+2-(b-i)\sqrt{D}}{2b} & (i=1,2,\dots,b-2), \\ \dfrac{2b^2-(i-2)b+2-(2b-i)\sqrt{D}}{2b} & (i=b-1,b), \end{cases}$$

$$\phi_{2i+1} = \begin{cases} \dfrac{3b^2-(2i+1)b-(b+1)\sqrt{D}}{2b^2} & (i=1,2,\dots,b-3), \\ \dfrac{5b^2-(2i+1)b-(b+1)\sqrt{D}}{2b^2} & (i=b-2,b-1,b), \end{cases}$$

$$\phi_{2bn+k} = \phi_k \quad (k=2,3,\dots,2b+1;\ n=1,2,\dots).$$

Hence, for $n=0,1,\dots$ and for $i=1,2,\dots,b-3$ one gets

$$\Xi_{2bn+2i} = \Xi^*_{2bn+2i+1} \to (i+2)b^2-(i^2+i-2)b+1,$$
$$\Xi^*_{2bn+2i} \to b^3-(2i+1)b^2+(i^2+i)b-1,$$
$$\Xi_{2bn+2i+1} \to 2b^3-3(i+1)b^2+(i^2+i-2)b-1 \quad (n\to\infty),$$

and for $n=0,1,\dots,$

$$\Xi_{2bn+2b-4} \to 3b^2+1, \quad \Xi^*_{2bn+2b-4} \to 2b-1, \quad \Xi_{2bn+2b-3} \to 2b^3+3b^2-1,$$
$$\Xi^*_{2bn+2b-3} \to 1, \quad \Xi_{2bn+2b-2} = \Xi^*_{2bn+2b-1} \to b^2+2b+1,$$
$$\Xi^*_{2bn+2b-2} \to b^3+b^2-1, \quad \Xi_{2bn+2b-1} \to 2b^3-2b-1,$$
$$\Xi_{2bn+2b} = \Xi^*_{2bn+2b+1} \to 2b^2+2b+1, \quad \Xi^*_{2bn+2b} \to b^3-b^2-1,$$
$$\Xi_{2bn+2b+1} \to 2b^3-3b^2-2b-1 \quad (n\to\infty).$$

If $b=3$, then $\phi=1/3$ is expanded as

$$\frac{1}{3}={}_\theta[1;1,1,1,2,\overline{1,4,2,4,1,1}]$$

and for $n=1,2,\ldots,$

$$\phi_{6n-2}=\frac{5-\sqrt{21}}{3},\qquad \phi_{6n-1}=\frac{12-\sqrt{21}}{9},\qquad \phi_{6n}=\frac{19-3\sqrt{21}}{6},$$
$$\phi_{6n+1}=\frac{39-5\sqrt{21}}{18},\qquad \phi_{6n+2}=\frac{11-2\sqrt{21}}{3},\qquad \phi_{6n+3}=\frac{27-5\sqrt{21}}{18}.$$

It follows that

$$\Psi_{6n-2}=\Psi^*_{6n-1}\to 4,\ \Psi^*_{6n-2}\to 17,\ \Psi_{6n-1}\to 41,\ \Psi_{6n}\to 43,\ \Psi^*_{6n}\to 5,$$
$$\Psi_{6n+1}\to 83,\ \Psi^*_{6n+1}\to 7,\ \Psi_{6n+2}=\Psi^*_{6n+3}\to 37,\ \Psi^*_{6n+2}\to 20,\ \Psi_{6n+3}\to 17$$

as n tends to infinity.

$1-\phi=2/3$ is expanded as

$$\frac{2}{3}={}_\theta[1;1,1,1,2,\overline{1,4,2,4,1,1}]$$

and for $n=1,2,\ldots,$

$$\phi_{6n-4}=\frac{7-\sqrt{21}}{3},\qquad \phi_{6n-3}=\frac{18-2\sqrt{21}}{9},\qquad \phi_{6n-2}=\frac{10-2\sqrt{21}}{3},$$
$$\phi_{6n-1}=\frac{15-2\sqrt{21}}{9},\qquad \phi_{6n}=\frac{17-3\sqrt{21}}{6},\qquad \phi_{6n+1}=\frac{12-2\sqrt{21}}{9}.$$

It follows that

$$\Xi_{6n-4}\to 28,\ \Xi^*_{6n-4}\to 5,\ \Xi_{6n-3}\to 80,\ \Xi^*_{6n-3}\to 1,\ \Xi_{6n-2}=\Xi^*_{6n-1}\to 16,$$
$$\Xi_{6n-2}\to 35,\ \Xi^*_{6n-1}\to 47,\ \Xi_{6n}=\Xi^*_{6n+1}\to 25,\ \Xi^*_{6n}\to 17,\ \Xi_{6n+1}\to 20$$

as n tends to infinity.

The case $b=2$ is included in Theorem 3. ∎

THEOREM 7. *If* $b=1$, *then*

$$\mathcal{M}\left(\theta,\frac{1}{2a}\right)=\mathcal{M}_+\left(\theta,\frac{1}{2a}\right)=\frac{1}{4a\sqrt{a^2+4a}}.$$

Proof. If a is even with $a\ge 4$, then $\phi=1/(2a)$ is expanded as

$$\frac{1}{2a}={}_\theta\left[1;a+1,\overline{1,\frac{a}{2}-1,1,\frac{a}{2}-1,1,a,1,a}\right]$$

and

$$\phi_{8n-7} = \frac{2a^2 + 5a - (2a-1)\sqrt{D}}{4a}, \qquad \phi_{8n-6} = \frac{5a - 2 - 3\sqrt{D}}{4a},$$
$$\phi_{8n-5} = \frac{a^2 + 3a - (a-1)\sqrt{D}}{4a}, \qquad \phi_{8n-4} = \frac{2a - 1 - \sqrt{D}}{2a},$$
$$\phi_{8n-3} = \frac{a^2 + a - (a-1)\sqrt{D}}{4a}, \qquad \phi_{8n-2} = \frac{5a - 2 - \sqrt{D}}{4a},$$
$$\phi_{8n-1} = \frac{2a^2 + 3a - (2a-1)\sqrt{D}}{4a}, \qquad \phi_{8n} = \frac{3a - 1 - \sqrt{D}}{2a}.$$

It follows that

$\Psi_{8n-7} = \Psi^*_{8n-6} \to a(2a^2 + 10a - 1)$, $\quad \Psi^*_{8n-7} \to a(2a^2 - 4a + 1)$,
$\Psi_{8n-6} \to a(4a^2 - 14a + 1)$, $\quad \Psi_{8n-5} = \Psi^*_{8n-4} \to a(a^2 + 4a - 1)$,
$\Psi^*_{8n-5} \to a(a-1)^2$, $\Psi_{8n-4} \to a(3a^2 - 8a + 1)$, $\Psi_{8n-3} = \Psi^*_{8n-2} \to a(2a-1)$,
$\Psi^*_{8n-3} \to a(2a^2 - 4a + 1)$, $\quad \Psi_{8n-2} \to a(6a^2 - 6a + 1)$,
$\Psi_{8n-1} = \Psi^*_{8n} \to a(6a-1)$, $\quad \Psi^*_{8n-1} \to a(2a-1)^2$, $\quad \Psi_{8n} \to a(8a^2 - 10a + 1)$

as n tends to infinity.

If a is odd with $a \geq 5$, then $\phi = 1/(2a)$ is expanded as

$$\frac{1}{2a} = {}_\theta\left[1; a+1, \overline{1, \frac{a-3}{2}, 1, a, 1, a}\right]$$

and for $n = 1, 2, \dots,$

$$\phi_{6n-5} = \frac{2a^2 + 5a - (2a-1)\sqrt{D}}{4a}, \qquad \phi_{6n-4} = \frac{5a - 2 - 3\sqrt{D}}{4a},$$
$$\phi_{6n-3} = \frac{a^2 + a - (a-1)\sqrt{D}}{4a}, \qquad \phi_{6n-2} = \frac{5a - 2 - \sqrt{D}}{4a},$$
$$\phi_{6n-1} = \frac{2a^2 + 3a - (2a-1)\sqrt{D}}{4a}, \qquad \phi_{6n} = \frac{3a - 1 - \sqrt{D}}{2a}.$$

It follows that

$\Psi_{6n-5} = \Psi^*_{6n-4} \to a(2a^2 + 10a - 1)$, $\quad \Psi^*_{6n-5} \to a(2a^2 - 4a + 1)$,
$\Psi_{6n-4} \to a(4a^2 - 14a + 1)$, $\quad \Psi_{6n-3} = \Psi^*_{6n-2} \to a(2a-1)$,
$\Psi^*_{6n-3} \to a(2a^2 - 4a + 1)$, $\quad \Psi_{6n-2} \to a(6a^2 - 6a + 1)$,
$\Psi_{6n-1} = \Psi^*_{6n} \to a(6a-1)$, $\quad \Psi^*_{6n-1} \to a(2a-1)^2$, $\quad \Psi_{6n} \to a(8a^2 - 10a + 1)$

as n tends to infinity.

If $a = 3$, then $\phi = 1/6$ is expanded as

$$\frac{1}{6} = {}_\theta[1; 4, \overline{2, 4, 1, 2, 1, 3}]$$

and for $n=1,2,\dots,$

$$\phi_{6n-4}=\frac{25-3\sqrt{21}}{12},\qquad \phi_{6n-3}=\frac{21-4\sqrt{21}}{6},\qquad \phi_{6n-2}=\frac{19-3\sqrt{21}}{12},$$
$$\phi_{6n-1}=\frac{27-5\sqrt{21}}{12},\qquad \phi_{6n}=\frac{8-\sqrt{21}}{6},\qquad \phi_{6n+1}=\frac{33-5\sqrt{21}}{12}.$$

Then one finds that

$\Psi_{6n-4}\to 3\cdot 109$, $\Psi^*_{6n-4}\to 3\cdot 5$, $\Psi_{6n-3}=\Psi^*_{6n-2}\to 3\cdot 35$, $\Psi^*_{6n-3}\to 3\cdot 37$, $\Psi_{6n-2}\to 3\cdot 43$, $\Psi_{6n-1}=\Psi^*_{6n}\to 3\cdot 17$, $\Psi^*_{6n-1}\to 3\cdot 25$, $\Psi_{6n}\to 3\cdot 43$, $\Psi_{6n+1}\to 3\cdot 47$, $\Psi^*_{6n+1}\to 3\cdot 7$ $(n\to\infty)$.

If $a=2$, then $\phi=1/4$ is expanded as

$$\frac{1}{4}={}_\theta[1;3,\overline{2,2,1,3,1,1,1,2}]$$

and for $n=1,2,\dots,$

$$\phi_{8n-6}=\frac{8-3\sqrt{3}}{4},\qquad \phi_{8n-5}=\frac{9-5\sqrt{3}}{4},\qquad \phi_{8n-4}=\frac{7-2\sqrt{3}}{4},$$
$$\phi_{8n-3}=\frac{11-5\sqrt{3}}{4},\qquad \phi_{8n-2}=\frac{6-3\sqrt{3}}{4},\qquad \phi_{8n-1}=\frac{7-3\sqrt{3}}{4},$$
$$\phi_{8n}=\frac{5-2\sqrt{3}}{4},\qquad \phi_{8n+1}=\frac{9-3\sqrt{3}}{4}.$$

Thus, one obtains

$\Psi_{8n-6}\to 2\cdot 37$, $\Psi^*_{8n-6}\to 2\cdot 11$, $\Psi_{8n-5}=\Psi^*_{8n-4}\to 2\cdot 3$, $\Psi^*_{8n-5}\to 2\cdot 25$, $\Psi_{8n-4}\to 2\cdot 37$, $\Psi_{8n-3}=\Psi^*_{8n-2}\to 2\cdot 23$, $\Psi^*_{8n-3}\to 2\cdot 13$, $\Psi_{8n}\to 2\cdot 13$, $\Psi_{8n-2}=\Psi^*_{8n-1}\to 2\cdot 9$, $\Psi_{8n-1}=\Psi^*_{8n}\to 2\cdot 11$, $\Psi_{8n+1}\to 2\cdot 27$, $\Psi^*_{8n+1}\to 2\cdot 1$

as n tends to infinity.

If a is odd, then $1-1/(2a)$ is expanded as

$$1-\frac{1}{2a}={}_\theta\left[1;1,1,\frac{a+3}{2},\overline{2,a,1,a+1,1,\frac{a+1}{2}}\right]$$

and for $n=1,2,\dots,$

$$\phi_{6n-3}=\frac{a^2+7a-(a+1)\sqrt{D}}{4a},\qquad \phi_{6n-2}=\frac{7a+2-3\sqrt{D}}{4a},$$
$$\phi_{6n-1}=\frac{2a^2+5a-(2a+1)\sqrt{D}}{4a},\qquad \phi_{6n}=\frac{3a+1-\sqrt{D}}{2a},$$
$$\phi_{6n+1}=\frac{2a^2+7a-(2a+1)\sqrt{D}}{4a},\qquad \phi_{6n+2}=\frac{5a+2-3\sqrt{D}}{4a}.$$

It follows that

$$\Xi_{6n-3}\to a(2a^2+10a-1),\quad \Xi^*_{6n-3}\to a,\quad \Xi_{6n-2}\to a(10a^2-2a+1),$$
$$\Xi^*_{6n-2}\to a(6a-1),\quad \Xi_{6n-1}=\Xi^*_{6n}\to a(2a-1),\quad \Xi^*_{6n-1}\to a(2a+1)^2,$$
$$\Xi_{6n}\to a(8a^2+2a+1),\quad \Xi_{6n+1}=\Xi^*_{6n+2}\to a(2a^2+8a-1),$$
$$\Xi^*_{6n+1}\to a(2a^2+2a+1),\quad \Xi_{6n+2}\to a(2a-1)^2\quad (n\to\infty).$$

If a is even, then $1-1/(2a)$ is expanded as

$$1-\frac{1}{2a}={}_{\theta}\left[1;1,1,\frac{a}{2}+1,\overline{1,\frac{a}{2}+1,2,a,1,a+1,1,\frac{a}{2}}\right]$$

and for $n=1,2,\ldots,$

$$\phi_{8n-5}=\frac{a^2+5a-(a+1)\sqrt{D}}{4a},\qquad \phi_{8n-4}=\frac{2a+1-\sqrt{D}}{2a},$$
$$\phi_{8n-3}=\frac{a^2+7a-(a+1)\sqrt{D}}{4a},\qquad \phi_{8n-2}=\frac{7a+2-3\sqrt{D}}{4a},$$
$$\phi_{8n-1}=\frac{2a^2+5a-(2a+1)\sqrt{D}}{4a},\qquad \phi_{8n}=\frac{3a+1-\sqrt{D}}{2a},$$
$$\phi_{8n+1}=\frac{2a^2+7a-(2a+1)\sqrt{D}}{4a},\qquad \phi_{8n+2}=\frac{5a+2-3\sqrt{D}}{4a}.$$

It follows that

$$\Xi_{8n-5}=\Xi^*_{8n-4}\to a(a^2+4a-1),\ \Xi^*_{8n-5}\to a(a+1)^2,\ \Xi_{8n-4}\to a(3a^2+1),$$
$$\Xi_{8n-3}\to a(2a^2+10a-1),\quad \Xi^*_{8n-3}\to a,\quad \Xi_{8n-2}\to a(10a^2-2a+1),$$
$$\Xi^*_{8n-2}\to a(6a-1),\quad \Xi_{8n-1}=\Xi^*_{8n}\to a(2a-1),\quad \Xi^*_{8n-1}\to a(2a+1)^2,$$
$$\Xi_{8n}\to a(8a^2+2a+1),\quad \Xi_{8n+1}=\Xi^*_{8n+2}\to a(2a^2+8a-1),$$
$$\Xi^*_{8n+1}\to a(2a^2+2a+1),\quad \Xi_{8n+2}\to a(2a-1)^2\quad (n\to\infty).\ \blacksquare$$

As an analogue to Theorem 7,

$$\mathcal{M}\left(\theta,\frac{1}{2b}\right)=\frac{1}{4b^2\sqrt{D}}\qquad (b\ge 1)$$

may be expected if $a=1$. However, it is difficult to decide whether this holds or not because there is little regularity in the inhomogeneous continued fraction expansion for general b. It is still more difficult for general a and b. Of course, it is not so hard to check this assertion holds for some concrete small a's and b's.

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Nagaoka National College of Technology
888 Nishikatakai
Nagaoka, Niigata 940, Japan
E-mail: komatsu@nagaoka-ct.ac.jp

Present address:
Faculty of Education
Mie University
1515 Kamihama, Tsu
Mie 514-8507, Japan
E-mail: komatsu@edu.mie-u.ac.jp

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