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Cohomology groups of units in $\mathbb{Z}_p^d$-extensions

by

Mingzhi Xu (Columbus, Ohio)

In this paper, K is an arbitrary number field and p is a prime number. Let $\mathbb{Z}_p$ be the p-adic integers and let K_∞ be a Galois extension of K such that $\mathcal{G} = \mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p^d$, $d \in \mathbb{Z}$, $d \geq 1$. For an arbitrary field F between K and K_∞, let $\mathcal{E}(F)$ be the group of global units of F and let $\mathcal{E}(F)^{\mathrm{univ}}$ be the intersection $\bigcap_{L \subset K_\infty, L/F \text{ finite}} N_{L/F}(\mathcal{E}(L))$. The Iwasawa algebra $\mathbb{Z}_p[[\mathcal{G}]]$ will be denoted by Λ. An ideal in Λ that contains two elements that are relatively prime will be called an ideal *of height at least two*. For a set S of primes in K above p, $M_S(F)$ denotes the maximal abelian p-extension of F which is unramified outside of S, and let $X_S(F) = \mathrm{Gal}(M_S(F)/F)$.

If F is finite over K, then $A(F)$ will be the p-part of the ideal class group of F, and for a prime $\wp \subset K$, $U_\wp(F)$ will be the group of local units of $F \otimes_K K_\wp$ which are congruent to 1 modulo the primes above $\wp$. The product $\prod_{\wp \in S} U_\wp(F)$ is denoted by $U(F)$. The closure of $\mathcal{E}(F) \cap U(F)$ in $U(F)$ is written as $\overline{\mathcal{E}}(F)$. If F is infinite over K, we define $A(F)$, $\overline{\mathcal{E}}(F)$ and $U(F)$ to be the inverse limits $\varprojlim A(L)$, $\varprojlim \overline{\mathcal{E}}(L)$ and $\varprojlim U(L)$ respectively, where the inverse limits are over finite extensions L of K such that $L \subset F$, and are with respect to norm maps. Define $T(F)$ to be the set of primes of K which ramify in K_∞/F, and let r_1 and r_2 be the numbers of real and complex primes of K.

Suppose F is finite over K, and let $r_1(F)$ and $r_2(F)$ be the numbers of real and complex primes of F. Then $\mathrm{rank}_{\mathbb{Z}} \mathcal{E}(F) = r_1(F) + r_2(F) - 1$. Hence we must have $\overline{\mathcal{E}}(F) \cong \mathbb{Z}_p^c \times B$, where $c \leq r_1(F) + r_2(F) - 1$ and B is finite. Let $\delta_F = r_1(F) + r_2(F) - 1 - c$. For a general F, if the set $\{\delta_L : L \subset F, L/K \text{ finite}\}$ is bounded, then we say that the *weak Leopoldt hypothesis* holds for F and S.

Fix a set S of primes in K above p. If $\wp$ is any prime in S and F is finite over K, let v be a prime of F lying above $\wp$ and let F_v^* be the multiplicative group of F_v, the completion of F at v. Following Wintenberger ([12]), we

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define $Z(F_v)$ to be $\varprojlim_n F_v^*/(F_v^*)^{p^n}$. If F/K is an infinite extension, $Z(F_v)$ is defined to be $\varprojlim Z(L_q)$, where the inverse limit is over finite extensions L of K such that $L \subset F$, and q is the prime of L lying under v. We also define $Z_\wp(F) = \varprojlim \prod_{q|\wp, q\subset L} Z(L_q)$, where the inverse limit is over finite extensions L of K such that $L \subset F$. Observe that for any F with $K \subset F \subset K_\infty$, we have $U_\wp(F) \subset Z_\wp(F)$.

If H is a closed subgroup of $\mathcal{G}$, define $I(H)$ to be the ideal of Λ generated by $\{\gamma - 1 : \gamma \in H\}$. If $H = \mathrm{Gal}(K_\infty/F)$, we also write $I(H)$ as $I(F)$, and we define Λ_H to be $\Lambda/I(H) = \mathbb{Z}_p[[\mathcal{G}/H]]$. For convenience, we let $X(F) = X_S(F)$. The maps

$$\pi_X : X(K_\infty)/(I(F)X(K_\infty)) \to X(F),$$
$$\pi_A : A(K_\infty)/(I(F)A(K_\infty)) \to A(F),$$
$$\pi_U : U(K_\infty)/(I(F)U(K_\infty)) \to U(F),$$
$$\pi_{\mathcal{E}} : \overline{\mathcal{E}}(K_\infty)/(I(F)\overline{\mathcal{E}}(K_\infty)) \to \overline{\mathcal{E}}(F)$$

will be the natural projection maps.

Before we state the main results, let us state the exact assumptions. We assume that the Iwasawa-μ-conjecture is true for K. We also assume that for every $\mathbb{Z}_p$-extension F of K such that $F \subset K_\infty$, the weak Leopoldt hypothesis holds for F and S. In addition, we assume that for any finite extension F of K such that $F \subset K_\infty$, Leopoldt's conjecture holds for F.

Our main result is: Let F be any field between K and K_∞. For any integer $i \ge 0$, there exist a positive integer n and an ideal $\mathcal{A}$ of height at least two in Λ, both independent of F, such that

$$I^n_{T(K)}\mathcal{A}H^i(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)) = 0.$$

When $d = 1$, this was proved by Iwasawa ([5]). Greenberg ([3]) proved many fundamental results when $d \ge 2$ and S is the set of all primes above p. In [9], Rubin proved a key result (Theorem 7.6(i)) for the case when $d = 2$ and K is an imaginary quadratic field, which will be generalized to prove our result.

In addition, the rank of $X_S(K_\infty)$ will be given by a formula which generalizes a result of Greenberg. The more general module $X_S(F)$ is also considered and the result can be found in Theorem 2.2.

1. The Λ-modules $U(K_\infty)$, $X(K_\infty)$ and $A(K_\infty)$

LEMMA 1.1. *For $\wp \in S$, let $D_\wp$ be the decomposition group of $\wp$ in K_∞/K. Let $\pi_{Z,\wp}$ be the natural projection*: $Z_\wp(K_\infty)/I(F)Z_\wp(K_\infty) \to Z_\wp(F)$. *Then* $I(D_\wp)^{d-1}\ker(\pi_{Z,\wp}) = 0$.

Proof. This follows from Lemma 5.2 in [12] and induction.

If Q is any set of primes of K above p, then the product $\prod_{\wp\in Q} I(D_\wp)$ will be written as I_Q.

THEOREM 1.2. *We have*

$$I_{T(F)\cap S}\operatorname{coker}(\pi_U)=0 \quad \textit{and} \quad I^d_{T(F)\cap S}\ker(\pi_U)=0.$$

Proof. When $d=2$ and K is imaginary quadratic, this was proved by Rubin in Theorem 5.1(i) of [9]. The proof here is similar. More precisely, it follows from Lemma 1.1 and class field theory by looking at $\operatorname{coker}(\pi_U)_\wp$ and $\ker(\pi_U)_\wp$ for each $\wp\in S$.

LEMMA 1.3. *Let L_∞ be an abelian extension of K_∞ that is Galois over K and let $Z=\operatorname{Gal}(L_\infty/K_\infty)$. Suppose L_1 is the fixed field of $I(F)Z$ and L_2 is the maximal abelian extension of F in L_∞. Then $L_2\subset L_1$, and $\operatorname{Gal}(L_1/L_2)$ is finitely generated over $\mathbb{Z}_p$. Also $\mathcal{G}$ acts trivially on $\operatorname{Gal}(L_1/L_2)$. If $\operatorname{Gal}(K_\infty/F)$ is cyclic, then $L_1=L_2$.*

This is exactly Lemma 5.2 of [9]. From the proof given there, we see that if $\alpha_1,\ldots,\alpha_n\in\operatorname{Gal}(L_1/F)$ generate $\operatorname{Gal}(K_\infty/F)$, then $\operatorname{Gal}(L_1/L_2)$ is generated by the commutators $[\alpha_i,\alpha_j]$, $1\le i\le n$, $1\le j\le n$.

A Λ_H-module M is called a *torsion* Λ_H-module if M can be annihilated by an element α in Λ_H which is not a zero divisor. For any Λ-module Y, let $Y^H=\{y\in Y: hy=y \text{ for all } h\in H\}$ and $Y_H=Y/I(H)Y$.

LEMMA 1.4. *Suppose $H\subset\mathcal{G}$ and $0\to Y\to Z\to W\to 0$ is an exact sequence of Λ-modules. Then there is an exact sequence*

$$H_1(H,Z)\to H_1(H,W)\to Y_H\to Z_H\to W_H\to 0.$$

If $H=\operatorname{Gal}(K_\infty/F)$ is cyclic, then the sequence

$$0\to Y^H\to Z^H\to W^H\to Y_H\to Z_H\to W_H\to 0$$

is exact.

Proof. The first sequence is just part of the long exact homology sequence. The second is a straightforward consequence of the Snake Lemma.

LEMMA 1.5. *If M is a finitely generated torsion-free Λ-module of rank ϱ, then for any $f\in\Lambda$, $f\ne 0$, there is an exact sequence*

$$0\to M\to\Lambda^\varrho\to N\to 0,$$

such that N is a torsion Λ-module with an annihilator g such that $(g,f)=1$, where (g,f) is the greatest common divisor of g and f.

Proof. Let $\Lambda_f=\{a/b: a \text{ and } b\in\Lambda, (b,f)=1\}$. Since Λ_f is a principal ideal domain, $M\otimes\Lambda_f$ is a free Λ_f-module. The lemma follows.

LEMMA 1.6. *Let $s=\sum_{\wp\in S}[K_\wp:\mathbb{Q}_p]-r_1-r_2$. If L/K is a finite extension such that $L\subset K_\infty$, let $S_1=\{q: q$ is a prime in L, and there* *exists* $\wp \in S$ *such that* $q \mid \wp\}$ *and let* $s(L) = \sum_{q\in S_1}[F_q : \mathbb{Q}_p] - r_1(L) - r_2(L)$. *Then* $s(L) = s[L : K]$.

Proof. Because L/K is unramified outside of p, we have $r_1(L) = [L : K]r_1$ and $r_2(L) = [L : K]r_2$. Also for each $\wp \in S$, $\sum_{q|\wp, q\subset L}[F_q : \mathbb{Q}_p] = [L : K]$. It follows that $s(L) = s[L : K]$.

From now on, we assume that for every $\mathbb{Z}_p$-extension F of K such that $F \subset K_\infty$, the weak Leopoldt hypothesis holds for F and S. Fix such an F. Then for any field L between K and F, by class field theory and Lemma 1.6, $\operatorname{rank}_{\mathbb{Z}_p} X(L) = [L : K]s + \delta_L$. Since δ_L is bounded, if s were negative, then we could choose an L such that $[L : K]$ is large enough that $\operatorname{rank}_{\mathbb{Z}_p} X(L) = [L : K]s + \delta_L$ is negative, which is a contradiction. Therefore, $s \geq 0$.

Theorem 1.7. *Let* S *be as above. Then*

(i) $I(\mathcal{G})\operatorname{coker}(\pi_X) = 0$ *and* $I(\mathcal{G})I_{T(F)-S}\ker(\pi_X) = 0$. *Furthermore*, $\operatorname{coker}(\pi_X) = \operatorname{Gal}(F_\infty/F)$ *where* F_∞ *is the maximal extension of* F *in* K_∞ *which is unramified outside of* S, *and* $\ker(\pi_X)$ *is finitely generated over* $\mathbb{Z}_p$ *when* F/K *is finite.*

(ii) $I(\mathcal{G})\operatorname{coker}(\pi_A) = 0$ *and* $I(\mathcal{G})I_{T(F)}\ker(\pi_A) = 0$. *Further*, $\operatorname{coker}(\pi_A) = \operatorname{Gal}(F_{\mathrm{unr}}/F)$ *where* F_{unr} *is the maximal extension of* F *in* K_∞ *which is everywhere unramified*, *and* $\ker(\pi_A)$ *is finitely generated over* $\mathbb{Z}_p$ *when* F/K *is finite.*

Proof. For K imaginary quadratic, this was proved by Rubin [9]. We follow his procedures.

Since $\operatorname{coker}(\pi_X) = \operatorname{Gal}(M_S(F)\cap K_\infty/F)$, assertion (i) for $\operatorname{coker}(\pi_X)$ is clear. Let M_1 be $M_S(K_\infty)^{I(F)X(K_\infty)}$ and let M_2 be the maximal abelian extension of F in $M_S(K_\infty)$. Then $\operatorname{Gal}(M_1/K_\infty) = X(K_\infty)/I(F)X(K_\infty)$ and $\ker(\pi_X) = \operatorname{Gal}(M_1/K_\infty M_S(F))$. From Lemma 1.3, it follows that $I(\mathcal{G})$ annihilates $\operatorname{Gal}(M_1/M_2)$. Next we consider $\operatorname{Gal}(M_2/K_\infty M_S(F))$.

Since $\operatorname{Gal}(M_2/F)$ is abelian, we have

$$\operatorname{Gal}(M_2/M_S(F)) = \prod_{v\in S'} I_v,$$

where S' is the set of primes of F lying above $T(F) - S$, and for each $v \in S'$, I_v is the inertia group of v in $\operatorname{Gal}(M_2/F)$. If $T(F) - S$ is empty, then $M_2 = M_S(F)$. For $v \in S'$, we have $v \mid \wp$, where $\wp \in T(F) - S$. If $\gamma \in D_\wp$ then $\gamma v = v$, so that $\gamma^{-1}I_v\gamma = I_v$. Since M_2/K_∞ is unramified above v, I_v injects into $\operatorname{Gal}(K_\infty/F)$ and it follows that $\gamma-1$ annihilates I_v. Thus $I(D_\wp)$ annihilates I_v. This means $I_{T(F)-S}$ annihilates $\operatorname{Gal}(M_2/M_S(F))$.

Finally, we prove that $\ker(\pi_X)$ is finitely generated over $\mathbb{Z}_p$ when F/K is finite. By Lemma 1.3, $\operatorname{Gal}(M_1/M_2)$ is finitely generated over $\mathbb{Z}_p$. Now by the properties of $\{I_v\}_{v\in S'}$ proved above and since $\mathrm{Gal}(M_2/M_S(F)) = \prod_{v\in S'} I_v$, we find that $\mathrm{Gal}(M_2/M_S(F))$ is finitely generated over $\mathbb{Z}_p$. Because $\ker(\pi_X) = \mathrm{Gal}(M_1/K_\infty M_S(F))$, we have proved (i).

The proof of (ii) is exactly the same as the proof of (i), except that $X(K_\infty)$, $M_S(K_\infty)$ and $M_S(F)$ need to be changed into $A(K_\infty)$, $L(K_\infty)$ and $L(F)$, where $L(K_\infty)$ (resp. $L(F)$) is the maximal abelian unramified p-extension of K_∞ (resp. F).

THEOREM 1.8. *Assume that for every $\mathbb{Z}_p$-extension F of K such that $F \subset K_\infty$, the weak Leopoldt hypothesis holds for F and S. Then $X(K_\infty)$ is a finitely generated Λ-module of rank s.*

Proof. For K imaginary quadratic, this was proved by Rubin in Theorem 5.3(iii) of [9], and for $S = \{\text{all } \wp \text{ above } p\}$ by Greenberg [3]. We basically follow [3].

If F is a finite extension of K, then the exact sequence

$$0 \to \ker(\pi_X) \to X(K_\infty)_F \to X(F)$$

shows that, because $\ker(\pi_X)$ and $X(F)$ are finitely generated over $\mathbb{Z}_p$, so is $X(K_\infty)_F$. This implies that $X(K_\infty)$ is a finitely generated Λ-module. The statement about $\mathrm{rank}_\Lambda X(K_\infty)$ can be proved by induction. We shall use τ to denote $\mathrm{rank}_\Lambda X(K_\infty)$. Let Y be the torsion Λ-submodule of $X(K_\infty)$ and let $Z = X(K_\infty)/Y$. We use induction on d to prove $\tau = s$.

If $d = 1$, then K_∞ is a $\mathbb{Z}_p$-extension of K. Let F be a field between K and K_∞. Let $M(F)$ be the maximal abelian extension of F contained in $M_S(K_\infty)$ so it corresponds to the commutator subgroup of $\mathrm{Gal}(M_S(K_\infty)/F)$. Thus

$$\mathrm{rank}_{\mathbb{Z}_p}(X(K_\infty)/I(F)X(K_\infty)) = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(M(F)/K_\infty).$$

By the same argument as in the proof of Theorem 1.7(i), we find that $\xi_F = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(M(F)/M_S(F))$ is bounded by a number independent of F, and

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}_p}(X(K_\infty)/I(F)X(K_\infty)) &= \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(M_S(F)/K_\infty) + \xi_F \\ &= \mathrm{rank}_{\mathbb{Z}_p} X(F) - 1 + \xi_F.\end{aligned}$$

However, $\mathrm{rank}_{\mathbb{Z}_p} X(F) = [F:K]s + \delta_F$. Thus

$$\mathrm{rank}_{\mathbb{Z}_p}(X(K_\infty)/I(F)X(K_\infty)) = [F:K]s - 1 + \xi_F + \delta_F.$$

On the other hand, it follows from the structure theory of Λ-modules that

$$\mathrm{rank}_{\mathbb{Z}_p}(X(K_\infty)/I(F)X(K_\infty)) = \tau[F:K] + \varepsilon_F,$$

where $\varepsilon_F = \mathrm{rank}_{\mathbb{Z}_p}(Y/I(F)Y)$, so it is bounded. We now have $[F:K]s-1+\xi_F + \delta_F = \tau[F:K] + \varepsilon_F$, which means $\tau = s$, since δ_F is bounded because of the weak Leopoldt hypothesis. This proves that $\tau = s$ when $d = 1$.

If $d \geq 2$, we assume that the conclusion is true for $d-1$. Let H be a direct summand of $\mathcal{G}$ isomorphic to $\mathbb{Z}_p$ and let h be a topological generator of H. From the exact sequence $0 \to Y \to X(K_\infty) \to Z \to 0$ and Lemma 1.4, we get

$$0 \to Y_H \to X(K_\infty)_H \to Z_H \to 0,$$

since $Z^H = 0$. This implies

$$\operatorname{rank}_{\Lambda_H} X(K_\infty)_H = \operatorname{rank}_{\Lambda_H}(Z_H) + \operatorname{rank}_{\Lambda_H}(Y_H).$$

But from Lemma 1.5, we have an exact sequence

$$0 \to Z \to \Lambda^\tau \to N \to 0,$$

in which N has an annihilator g such that $(g, h-1) = 1$. This gives us the exact sequence

$$N^H \to Z_H \to \Lambda_H^\tau \to N_H \to 0.$$

Since the image of g in Λ_H, which is not zero, annihilates N_H and N^H, we know that $\operatorname{rank}_{\Lambda_H}(Z_H) = \tau$. Combining the above, we get

$$\operatorname{rank}_{\Lambda_H} X(K_\infty)_H = \tau + \operatorname{rank}_{\Lambda_H}(Y_H).$$

Let $\Phi \in \Lambda$ be a nonzero annihilator of Y and for all $\wp \in T(K') - S$ such that $D_\wp$ is cyclic, let $h_\wp$ be a topological generator of $D_\wp$. The fixed field of H will be denoted by K'. We choose H so that $h-1$ does not divide Φ or $h_\wp - 1$ for all $\wp \in T(K') - S$ such that $D_\wp$ is cyclic. For such H, Y_H is a torsion Λ_H-module, since the projection $\overline{\Phi}$ of Φ in Λ_H is a nonzero annihilator of Y_H. Hence $\operatorname{rank}_{\Lambda_H} X(K_\infty)_H = \tau$. Now we consider the following exact sequence of Λ_H-modules:

$$0 \to \ker(\pi_X) \to X(K_\infty)_H \to X(K') \to \operatorname{coker}(\pi_X) \to 0.$$

Because of the way H was chosen, there exists $\alpha \in I_{T(K')-S}$ such that α is not a zero divisor in Λ_H. Since $I(\mathcal{G})I_{T(K')-S}\ker(\pi_X) = 0$ and $I(\mathcal{G})\operatorname{coker}(\pi_X) = 0$, we conclude that both $\ker(\pi_X)$ and $\operatorname{coker}(\pi_X)$ are torsion Λ_H-modules. This means

$$\tau = \operatorname{rank}_{\Lambda_H} X(K_\infty)_H = \operatorname{rank}_{\Lambda_H} X(K').$$

By the induction hypothesis, $\operatorname{rank}_{\Lambda_H} X(K') = s$. This completes the proof of Theorem 1.8.

2. Results about $X(F)$ and $A(F)$. Let μ_{p^∞} be the discrete group of all p-power roots of unity. We denote by $\mathcal{X}$ the set of continuous characters $\varrho : \mathcal{G} \to \mu_{p^\infty}$. Every $\varrho \in \mathcal{X}$ extends uniquely to a continuous homomorphism on Λ. For $f \in \Lambda$, define $\mathcal{X}(f) = \{\varrho \in \mathcal{X} : \varrho(f) = 0\}$. Let $\gamma_1, \ldots, \gamma_d$ be fixed topological generators of $\mathcal{G}$. We define an injection from $\mathcal{X}(f)$ to $\mu_{p^\infty}^d$ by mapping $\varrho \in \mathcal{X}(f)$ to $(\varrho(\gamma_1), \ldots, \varrho(\gamma_d))$. This identifies $\mathcal{X}(f)$ with the set of zeros of f in $(\mu_{p^\infty})^d$. Also, $I(f)$ will represent the set $\{g \in \Lambda : \varrho(g) = 0 \text{ for all } \varrho \in \mathcal{X}(f)\}$. Following Monsky [8], we let E_d be the free rank d $\mathbb{Z}_p$-module $\mathrm{Hom}((\mu_{p^\infty})^d, \mu_{p^\infty})$. We define closed subsets of $(\mu_{p^\infty})^d$ to be the subsets that are finite unions of subsets of $(\mu_{p^\infty})^d$ each of which is defined by a set of equations $\tau_j(\zeta) = \epsilon_j$, where $\tau_j \in E_d$, $\zeta \in (\mu_{p^\infty})^d$, $\epsilon_j \in \mu_{p^\infty}$. Finally, a $\mathbb{Z}_p$-*flat* in $(\mu_{p^\infty})^d$ is a set T defined by equations $\tau_j(\zeta) = \epsilon_j$, where $\{\tau_j\}$ is a subset of a basis of E_d, $\zeta \in (\mu_{p^\infty})^d$, and $\epsilon_j \in \mu_{p^\infty}$. Suppose $\{\tau_j : 1 \le j \le d\}$ is a basis of E_d and T is defined by τ_j for all j such that $1 \le j \le k$. Then we say that the *dimension* of T is $d-k$. Theorem 2.6 of [8] implies that $\mathcal{X}(f)$, as a subset of $(\mu_{p^\infty})^d$, is closed. This means $\mathcal{X}(f)$ is a finite union of $\mathbb{Z}_p$-flats. We write $\dim \mathcal{X}(f) \le \alpha$ if there is a finite set $\{U_i\}$ of $\mathbb{Z}_p$-flats such that $\bigcup_i U_i$ covers $\mathcal{X}(f)$ and $\dim U_i \le \alpha$ for all i.

LEMMA 2.1. *Suppose $d \ge 2$ and $f \in \Lambda$.*

(i) *If* $\dim \mathcal{X}(f) \le d-2$, *then $I(f)$ is an ideal of height at least two.*

(ii) *If f is relatively prime to $\gamma - 1$ for every $\gamma \ne 1$ in $\mathcal{G}$, then* $\dim \mathcal{X}(f) \le d-2$.

(iii) *Let g be a prime in Λ such that $\mathcal{X}(g)$ has codimension* 1. *There exists a field F such that $K \subset F \subset K_\infty$ and $H = \mathrm{Gal}(K_\infty/F) \cong \mathbb{Z}_p$, with the property $g \mid h-1$, where h is a topological generator of H.*

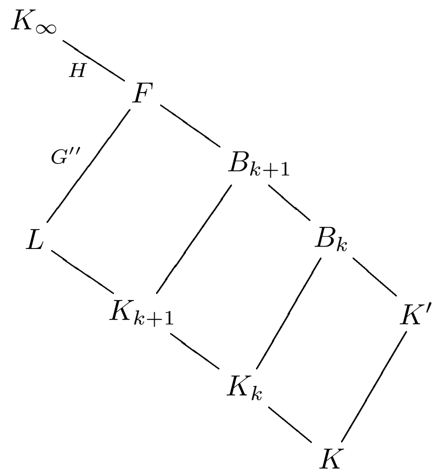
Proof. (i) Since $\dim \mathcal{X}(f) \le d-2$, $\mathcal{X}(f)$ can be written as $\bigcup_{i=1}^m T_i$, where m is a positive integer and for all i, $1 \le i \le m$, T_i is a $\mathbb{Z}_p$-flat such that $\dim T_i \le d-2$. It follows that for each i, $1 \le i \le m$, there exist $f_i, g_i \in \Lambda$ such that $(f_i, g_i) = 1$ and $T_i \subset \mathcal{X}(f_i) \cap \mathcal{X}(g_i)$. Let $\mathcal{A}_i$ be the ideal generated by f_i and g_i, $1 \le i \le m$. Then $\prod_{i=1}^m \mathcal{A}_i \subset I(f)$ and $\prod_{i=1}^m \mathcal{A}_i$ is an ideal of height at least two in Λ. This means $I(f)$ is an ideal of height at least two in Λ.

(ii) can be deduced from Theorem 2.6 of [7].

(iii) By (ii), we could get a $\gamma \in \mathcal{G}$ such that $(g, \gamma - 1) \ne 1$. Since g is prime, $g \mid \gamma - 1$. Let F be the fixed field of γ. Then $H = \mathrm{Gal}(K_\infty/F)$ is generated by γ topologically. This completes the proof of (iii).

THEOREM 2.2. *Let g be a prime element in Λ. Let F be any field such that $K \subset F \subset K_\infty$ and $H = \mathrm{Gal}(K_\infty/F) \cong \mathbb{Z}_p$. If $g \mid h-1$, where h is a topological generator of H, then* $\mathrm{rank}_{\Lambda/g\Lambda}(X(F) \otimes (\Lambda/g\Lambda)) = s$.

Proof. Let G' be a direct summand of $\mathcal{G}$ such that $G' \cong \mathbb{Z}_p$ and $H \subset G'$. We can now write $\mathrm{Gal}(F/K)$ as $V \oplus G''$, where $G'' \cong \mathbb{Z}_p^{d-1}$ and $V \cong G'/H$. Denote by L the fixed field of G'' and by K' the fixed field of V. Let g' be a topological generator of G', and let Λ' be the Iwasawa algebra $\mathbb{Z}_p[[G'']]$.



Since g is a prime and $g \mid h-1$, $g = \omega_{k+1}/\omega_k$, where k is a positive integer and $\omega_j = g'^{p^j} - 1$ for $j = k, k+1$. For the field K_i between K and L corresponding to g'^{p^i}, let B_i be $K'K_i$. Since $\mathrm{Gal}(K'K_i/K_i) \cong G''$, $X(B_i)$ can be considered as a Λ'-module. Consider the exact sequence

$$0 \to \ker(\pi_{X(B_i)}) \to X(F)/I(B_i)X(F) \to X(B_i) \to \mathrm{coker}(\pi_{X(B_i)}) \to 0,$$

where the middle map is the natural projection $\pi_{X(B_i)}$. Let $T'(B_i)$ be the primes of K which ramify in F/B_i. Write $M_2(B_i)$ for the maximal abelian extension of B_i in $M_S(F)$. S'' will denote the set of primes of B_i lying above $T'(B_i) - S$. From the proof of Theorem 1.7(i), we find that $\mathrm{coker}(\pi_{X(B_i)})$ is finite, and that $\ker(\pi_{X(B_i)})$ is a torsion Λ'-module if $\prod_{v\in S''} I_v$ is a torsion Λ'-module, where I_v is the inertia group of v in $\mathrm{Gal}(M_2(B_i)/B_i)$, and I_v can be embedded into $\mathrm{Gal}(F/B_i)$. Since $\mathrm{Gal}(F/B_i)$ is finite, there exists a positive integer j such that $p^j I_v = 0$ for all $v \in S''$, which means $p^j \prod_{v\in S''} I_v = 0$.

This means $\mathrm{rank}_{\Lambda'}(X(F)/I(B_i)X(F)) = \mathrm{rank}_{\Lambda'} X(B_i)$. By Lemma 1.6 and Theorem 1.8, $\mathrm{rank}_{\Lambda'}(X(F)/I(B_i)X(F)) = \mathrm{rank}_{\Lambda'} X(B_i) = sp^i$.

Next consider the exact sequence

$$0 \to I(B_k)X(F)/I(B_{k+1})X(F) \to X(F)/I(B_{k+1})X(F) \to X(F)/I(B_k)X(F) \to 0.$$

Since

$$I(B_k)X(F)/I(B_{k+1})X(F) = \omega_k X(F)/\omega_{k+1}X(F) = \omega_k X(F)/g\omega_k X(F),$$

we have

$$\mathrm{rank}_{\Lambda'}(\omega_k X(F)/g\omega_k X(F)) = s(p^{k+1} - p^k).$$

CLAIM. *$\omega_k X(F)/g\omega_k X(F)$ and $X(F)/gX(F)$ have the same rank as $\Lambda/g\Lambda$-modules.*

If the claim is true, then since Λ' can be embedded into $\Lambda/g\Lambda$ and $\mathrm{rank}_{\Lambda'}(\Lambda/g\Lambda) = p^{k+1} - p^k$, we have

$$\mathrm{rank}_{\Lambda/g\Lambda}(X(F) \otimes (\Lambda/g\Lambda)) = \mathrm{rank}_{\Lambda/g\Lambda}(X(F)/gX(F)) = s.$$

This would complete the proof of the theorem.

To prove the claim, we consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & W & \to & X(F) & \to & \omega_k X(F) & \to & 0 \\ & & \downarrow g & & \downarrow g & & \downarrow g & & \\ 0 & \to & W & \to & X(F) & \to & \omega_k X(F) & \to & 0 \end{array}$$

where W is the kernel of multiplication by ω_k and the vertical maps are multiplications by g. By the Snake Lemma, we get the exact sequence

$$W/gW \to X(F)/gX(F) \to \omega_k X(F)/g\omega_k X(F) \to 0.$$

Since ω_k is not a zero divisor in $\Lambda/g\Lambda$ and $\omega_k(W/gW) = 0$, we have proved the claim.

From now on, assume that for any field F between K and K_∞ such that F is finite over K, Leopoldt's conjecture holds for F.

According to the classification theorem, for any torsion Λ-module Y, we have exact sequences

$$0 \to \bigoplus \Lambda/f_i\Lambda \to Y \to N \to 0,$$
$$0 \to N_1 \to Y \to \bigoplus \Lambda/f_i\Lambda \to N_2 \to 0,$$

in which $f_i \in \Lambda$ for all i and N, N_1, N_2 can be annihilated by an ideal of height at least two in Λ. We call the ideal generated by $\prod f_i$ the *characteristic ideal* of Y, written $\mathrm{char}(Y)$.

3. Preliminary results

PROPOSITION 3.1. (i) *If $f \in \Lambda$ and $H \subset \mathcal{G}$, then $I(f)H_1(H, \Lambda/f\Lambda) = 0$.*

(ii) *If Y is a finitely generated torsion Λ-module, then there is an ideal $\mathcal{B}$ of height at least two in Λ such that for any $H \subset \mathcal{G}$, $\mathcal{B}I(\mathrm{char}(Y))H_1(H, Y) = 0$.*

Proof. For K imaginary quadratic, this was proved by Rubin in [9], Lemma 7.3. The same argument can be used here.

PROPOSITION 3.2. *Suppose $d \geq 2$. Let $Y = X(K_\infty)_{\mathrm{torsion}}$ be the torsion submodule of the Λ-module $X(K_\infty)$. There is an ideal $\mathcal{C}$ of height at least two in Λ such that $\mathcal{C}I_{T(K)} \subset I(\mathrm{char}(Y))$ and $\mathcal{C}I_{T(K)} \subset I(\mathrm{char}(A(K_\infty)))$.*

Proof. It follows from Theorem 1 of [2] and Lemma 2.1 and Theorem 2.2.

Proposition 3.3. *There is an ideal $\mathcal{B} \subset \Lambda$ of height at least two, such that for every $H \subset \mathrm{Gal}(K_\infty/K)$,*

$$I_{T(K)}\mathcal{B}H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)) = 0 \quad \textit{and} \quad I_{T(K)}\mathcal{B}H_1(H, A(K_\infty)) = 0.$$

Proof. When K is imaginary quadratic and $d = 2$, this is Corollary 7.5 of [9].

By the inclusion $U(K_\infty)/\overline{\mathcal{E}}(K_\infty) \subset X(K_\infty)$ of global class field theory, $(U(K_\infty)/\overline{\mathcal{E}}(K_\infty))_{\mathrm{torsion}} \subset Y$. If $U(K_\infty)/\overline{\mathcal{E}}(K_\infty)$ is torsion, we can use Propositions 3.2 and 3.1 to get $I_{T(K)}\mathcal{B}H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)) = 0$. In general, there is an exact sequence

$$0 \to (U(K_\infty)/\overline{\mathcal{E}}(K_\infty))_{\mathrm{torsion}} \to U(K_\infty)/\overline{\mathcal{E}}(K_\infty) \to Z \to 0,$$

where, by the exact sequence $0 \to U(K_\infty)/\overline{\mathcal{E}}(K_\infty) \to X(K_\infty) \to A(K_\infty) \to 0$ of global class field theory, and by Theorems 1.7(ii) and 1.8, Z is a torsion-free Λ-module of rank s. Now by using Lemma 1.5, one could see that $H_1(H, Z)$ is pseudo-null. Now $I_{T(K)}\mathcal{B}H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)) = 0$ follows from Proposition 3.2 and $I(\mathrm{char}(Y))H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)_{\mathrm{torsion}}) = 0$.

By Propositions 3.1 and 3.2, there is an ideal $\mathcal{B} \subset \Lambda$ of height at least two, such that $H_1(H, A(K_\infty))$ is annihilated by $I_{T(K)}\mathcal{B}$. This proves the second equation.

4. Main theorems. From now on, if M is a Λ-module, we denote $M/I(F)M$ by M_F.

Theorem 4.1. *Suppose F is any extension of K contained in K_∞. There is an ideal $\mathcal{A} \subset \Lambda$ of height at least two, independent of F, such that*

$$I^3_{T(K)}\mathcal{A}\,\mathrm{coker}(\pi_{\mathcal{E}}) = 0 \quad \textit{and} \quad I^{d+1}_{T(K)}\mathcal{A}\ker(\pi_{\mathcal{E}}) = 0.$$

Proof. When $d = 2$ and K is an imaginary quadratic field, this was proved by Rubin in [9], Theorem 7.6(i).

Consider the two commutative diagrams with exact rows

$$\begin{array}{ccccccccc}
H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)) & \to & (\overline{\mathcal{E}}(K_\infty))_F & \to & (U(K_\infty))_F & \to & (U(K_\infty)/\overline{\mathcal{E}}(K_\infty))_F & \to & 0 \\
 & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} & & \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_{U/\mathcal{E}}} & & \\
0 & \to & \overline{\mathcal{E}}(F) & \to & U(F) & \to & U(F)/\overline{\mathcal{E}}(F) & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccc} H_1(H, A(K_\infty)) & \to & (U(K_\infty)/\overline{\mathcal{E}}(K_\infty))_F & \to & (X(K_\infty))_F & \to & (A(K_\infty))_F & \to & 0 \\ & & \downarrow{\scriptstyle \pi_{U/\mathcal{E}}} & & \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_A} & & \\ 0 & \to & U(F)/\overline{\mathcal{E}}(F) & \to & X(F) & \to & A(F) & \to & 0 \end{array}$$

in which the top rows come from the exact sequences

$$0 \to \overline{\mathcal{E}}(K_\infty) \to U(K_\infty) \to U(K_\infty)/\overline{\mathcal{E}}(K_\infty) \to 0,$$
$$0 \to U(K_\infty)/\overline{\mathcal{E}}(K_\infty) \to X(K_\infty) \to A(K_\infty) \to 0.$$

By the Snake Lemma, we get the following exact sequences:

$$H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty)) \to \ker(\pi_{\mathcal{E}}) \to \ker(\pi_U) \to \ker(\pi_{U/\mathcal{E}}) \to \operatorname{coker}(\pi_{\mathcal{E}}) \to \operatorname{coker}(\pi_U) \to \operatorname{coker}(\pi_{U/\mathcal{E}}) \to 0$$

and

$$H_1(H, A(K_\infty)) \to \ker(\pi_{U/\mathcal{E}}) \to \ker(\pi_X) \to \ker(\pi_A) \to \operatorname{coker}(\pi_{U/\mathcal{E}}) \to \operatorname{coker}(\pi_X) \to \operatorname{coker}(\pi_A) \to 0.$$

Now the annihilator of $\ker(\pi_{\mathcal{E}})$ comes from the annihilators of $\ker(\pi_U)$ (Theorem 1.2) and $H_1(H, U(K_\infty)/\overline{\mathcal{E}}(K_\infty))$ (Proposition 3.3). Similarly we get the annihilator of $\ker(\pi_{U/\mathcal{E}})$ from the annihilators of $\ker(\pi_X)$ (Theorem 1.7) and $H_1(H, A(K_\infty))$ (Proposition 3.3), and then the annihilator of $\operatorname{coker}(\pi_{\mathcal{E}})$ comes from that of $\ker(\pi_{U/\mathcal{E}})$ and $\operatorname{coker}(\pi_U)$ (Theorem 1.2). This completes the proof of this theorem.

THEOREM 4.2. *Assume that the Iwasawa-μ-conjecture is true for K. Also assume that for any field F between K and K_∞ such that F is finite over K, Leopoldt's conjecture holds for F. Let F, $\mathcal{A}$ be as in Theorem* 4.1 *above. Then*

$$I^3_{T(K)}\mathcal{A}((\mathcal{E}(F)/\mathcal{E}(F)^{\mathrm{univ}}) \otimes \mathbb{Z}_p) = 0.$$

Proof. When $d = 1$, this result is due to Iwasawa ([5]).

If F/K is a finite extension, it follows from Theorem 4.1 that

$$\overline{\mathcal{E}}(F)/\bigcap_{L\subset K_\infty,\, L/F \text{ finite}} N_{L/F}(\overline{\mathcal{E}}(L)) \cong \operatorname{coker}(\pi_{\mathcal{E}})$$

is annihilated by $I^3_{T(K)}\mathcal{A}$. Now from our assumption of Leopoldt's conjecture, we get

$$I^3_{T(K)}\mathcal{A}\Big(\mathcal{E}(F) \otimes \mathbb{Z}_p/\bigcap_{L\subset K_\infty,\, L/F \text{ finite}} N_{L/F}(\mathcal{E}(L) \otimes \mathbb{Z}_p)\Big) = 0.$$

This implies

$$I^3_{T(K)}\mathcal{A} \varprojlim_{L\subset K_\infty,\, L/F \text{ finite}} \mathcal{E}(F) \otimes \mathbb{Z}_p/N_{L/F}(\mathcal{E}(L) \otimes \mathbb{Z}_p) = 0,$$

which implies

$$I_{T(K)}^3\mathcal{A} \varprojlim_{L\subset K_\infty,\, L/F\,\text{finite}} (\mathcal{E}(F)/N_{L/F}\mathcal{E}(L))\otimes\mathbb{Z}_p = 0.$$

Now it is clear that $I_{T(K)}^3\mathcal{A}((\mathcal{E}(F)/\mathcal{E}(F)^{\text{univ}})\otimes\mathbb{Z}_p) = 0$. We proved the conclusion in this case.

If F/K is an infinite extension, then

$$(\mathcal{E}(F)/\mathcal{E}(F)^{\text{univ}})\otimes\mathbb{Z}_p = \varinjlim_{L\subset F,\, L/K\,\text{finite}} ((\mathcal{E}(L)/\mathcal{E}(L)^{\text{univ}})\otimes\mathbb{Z}_p).$$

This proves the theorem.

Next we consider the cohomology group $H^1(\text{Gal}(K_\infty/F),\mathcal{E}(K_\infty))$. We first prove some results about $H^1(\text{Gal}(B/F),\overline{\mathcal{E}}(B))$, where B is a finite, cyclic extension of F in K_∞. Since $\pi_{\mathcal{E}}$ is dependent on F, we can write $\pi_{\mathcal{E}(F)}$ for $\pi_{\mathcal{E}}$ to indicate this dependence.

Proposition 4.3. *Suppose B is a finite, cyclic extension of F in K_∞. Let π' be the natural map*

$$\pi' : \overline{\mathcal{E}}(B)/I(\text{Gal}(K_\infty/F))\overline{\mathcal{E}}(B) \to \overline{\mathcal{E}}(F),$$

which is induced by the norm map. Then there exists an ideal $\mathcal{A}$ of height at least two in Λ, independent of B and F, such that $I_{T(K)}^{d+4}\mathcal{A}\ker(\pi') = 0$.

Proof. If we let ϕ be the natural projection

$$\phi : \overline{\mathcal{E}}(B) \to \overline{\mathcal{E}}(B)/I(\text{Gal}(K_\infty/F))\overline{\mathcal{E}}(B),$$

then $\pi_{\mathcal{E}(F)} = \pi'\circ\phi\circ\pi_{\mathcal{E}(B)}$ and for any $\xi\in\ker(\pi')$, there exists $\eta\in\mathcal{E}(B)$ such that $\phi(\eta)=\xi$. Now from Theorem 4.1, there exists an ideal $\mathcal{B}$ of height at least two in Λ such that $I_{T(K)}^3\mathcal{B}\,\text{coker}(\pi_{\mathcal{E}(B)}) = 0$. This means for any $\alpha\in I_{T(K)}^3\mathcal{B}$, there exists $\zeta\in\overline{\mathcal{E}}(K_\infty)/I(B)\overline{\mathcal{E}}(K_\infty)$ such that $\alpha\eta = \pi_{\mathcal{E}(B)}(\zeta)$. From this, we get $\alpha\xi = \phi(\alpha\eta) = \phi(\pi_{\mathcal{E}(B)}(\zeta))$, which implies $\pi'\circ\phi\circ\pi_{\mathcal{E}(B)}(\zeta) = 0$, from which we get $\pi_{\mathcal{E}(F)}(\zeta)=0$. From Theorem 4.1 again, $I_{T(K)}^{d+1}\mathcal{B}\ker(\pi_{\mathcal{E}(F)}) = 0$. This means $\beta\zeta = 0$ for any $\beta\in I_{T(K)}^{d+1}\mathcal{B}$, which implies $\alpha\beta\eta = \pi_{\mathcal{E}(B)}(\beta\zeta) = 0$. This yields $\alpha\beta\xi = \phi(\alpha\beta\eta) = 0$. The proof is complete.

Proposition 4.4. *Let B, F and π' be as in Proposition 4.3. Then*

$$\ker(\pi') = H^1(\text{Gal}(B/F),\overline{\mathcal{E}}(B)).$$

Proof. By the definition of π' and by the definition before Theorem 3 in Chapter IV of [1], we get $\ker(\pi') = \widehat{H}^{-1}(\text{Gal}(B/F),\overline{\mathcal{E}}(B))$ and

$$\widehat{H}^1(\text{Gal}(B/F),\overline{\mathcal{E}}(B)) = H^1(\text{Gal}(B/F),\overline{\mathcal{E}}(B)).$$

Since $\text{Gal}(B/F)$ is cyclic, by Theorem 5 in Chapter IV of [1], we get

$$\widehat{H}^{-1}(\text{Gal}(B/F),\overline{\mathcal{E}}(B)) = \widehat{H}^1(\text{Gal}(B/F),\overline{\mathcal{E}}(B)).$$

Combining the above gives $\ker(\pi') = H^1(\mathrm{Gal}(B/F), \overline{\mathcal{E}}(B))$. This completes the proof.

COROLLARY 4.5. *Suppose F is a finite extension over K and suppose $B \subset K_\infty$ is finite and cyclic over F. Then there exists an ideal $\mathcal{A}$ of height at least two in Λ, independent of B and F, such that*

$$I_{T(K)}^{d+4}\mathcal{A}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0.$$

Proof. Combining Propositions 4.3 and 4.4, we get

$$I_{T(K)}^{d+4}\mathcal{A}H^1(\mathrm{Gal}(B/F), \overline{\mathcal{E}}(B)) = 0.$$

Since F/K is a finite extension, the extension B/K is also finite. This implies $H^1(\mathrm{Gal}(B/F), \overline{\mathcal{E}}(B)) = H^1(\mathrm{Gal}(B/F), \mathcal{E}(B) \otimes \mathbb{Z}_p)$ by our assumption of Leopoldt's conjecture. Now since $\mathrm{Gal}(B/F)$ is a p-group, we get

$$H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) \cong H^1(\mathrm{Gal}(B/F), \mathcal{E}(B) \otimes \mathbb{Z}_p),$$

as Λ-modules. This shows that

$$I_{T(K)}^{d+4}\mathcal{A}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0.$$

THEOREM 4.6. *Assume that the Iwasawa-μ-conjecture is true for K. Also assume that for any field F between K and K_∞ such that F is finite over K, Leopoldt's conjecture holds for F. Suppose F is a field such that $K \subset F \subset K_\infty$. There exists an ideal $\mathcal{A}$ of height at least two in Λ, independent of F, such that*

$$I_{T(K)}^{d(d+4)}\mathcal{A}H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)) = 0.$$

Proof. When $d = 1$, this result is due to Iwasawa ([5]).

First we assume that F/K is a finite extension. Since

$$H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)) = \varinjlim_{B\subset K_\infty,\, B/F \text{ finite}} H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)),$$

we only need to show $I_{T(K)}^{d(d+4)}\mathcal{A}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0$ when $B \subset K_\infty$ and B/F is a finite extension.

Since K_∞/K is a $\mathbb{Z}_p^d$-extension, $\mathrm{Gal}(B/F)$ is a product of m cyclic factors, where m is an integer, $m \le d$. If $m = 0$, $\mathrm{Gal}(B/F)$ is trivial, so we can assume $1 \le m \le d$.

We use induction on m to prove $I_{T(F)}^{m(d+4)}\mathcal{A}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0$.

If $m = 1$, then B/F is a cyclic extension. From Corollary 4.5, there exists an ideal $\mathcal{A}$ of height at least two in Λ, independent of F and B, such that $I_{T(K)}^{d+4}\mathcal{A}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0$.

Suppose the conclusion is true for $m-1$, that is, if $\mathrm{Gal}(B/F)$ is a product of $m-1$ cyclic factors, then there exists an ideal $\mathcal{B}$ of height at least two in Λ, independent of F and B, such that $I_{T(K)}^{(m-1)(d+4)}\mathcal{B}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0$.

Now if $\mathrm{Gal}(B/F)$ is a product of m cyclic factors, we let H be a subgroup of $\mathrm{Gal}(B/F)$ such that H is a product of $m-1$ cyclic factors, and $\mathrm{Gal}(B/F)/H$ is cyclic. Let C be the fixed field of H. Then the restriction-inflation sequence gives us the exact sequence

$$0 \to H^1(\mathrm{Gal}(C/F), \mathcal{E}(C)) \to H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) \\ \to H^1(\mathrm{Gal}(B/C), \mathcal{E}(B)).$$

Since $\mathrm{Gal}(C/F)$ is cyclic, we have an ideal $\mathcal{C}$ of height at least two in Λ, independent of F and C, such that $I_{T(K)}^{d+4}\mathcal{C}H^1(\mathrm{Gal}(C/F), \mathcal{E}(C)) = 0$. As for $H^1(\mathrm{Gal}(B/C), \mathcal{E}(B))$, the induction hypothesis implies

$$I_{T(K)}^{(m-1)(d+4)}\mathcal{B}H^1(\mathrm{Gal}(B/C), \mathcal{E}(B)) = 0.$$

Combining these we get

$$I_{T(K)}^{m(d+4)}\mathcal{B}\mathcal{C}H^1(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0.$$

This completes the proof of the theorem for F/K finite.

We now consider the case when F/K is an infinite extension. Let L be any subextension of F/K such that L/K is finite. Consider the inflation-restriction exact sequence

$$H^1(\mathrm{Gal}(F/L), \mathcal{E}(F)) \to H^1(\mathrm{Gal}(K_\infty/L), \mathcal{E}(K_\infty)) \\ \to H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty))^{\mathrm{Gal}(F/L)} \to H^2(\mathrm{Gal}(F/L), \mathcal{E}(F)),$$

which implies, after taking direct limits,

$$\varinjlim_{L\subset F,\, L/K \text{ finite}} H^1(\mathrm{Gal}(K_\infty/L), \mathcal{E}(K_\infty)) \cong H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)),$$

since

$$\varinjlim_{L\subset F,\, L/K \text{ finite}} H^i(\mathrm{Gal}(F/L), \mathcal{E}(F)) = 0 \quad \text{for } i = 1, 2,$$

and

$$\varinjlim_{L\subset F,\, L/K \text{ finite}} H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty))^{\mathrm{Gal}(F/L)} = H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)).$$

Now we have $I_{T(K)}^{d(d+4)}\mathcal{A}H^1(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)) = 0$. This completes the proof of the theorem.

Next, we are going to show that $H^i(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty))$ can be annihilated by similar products for all $i \geq 2$. Since

$$H^i(\mathrm{Gal}(K_\infty/F), \mathcal{E}(K_\infty)) = \varinjlim_{B\subset K_\infty,\, B/F \text{ finite}} H^i(\mathrm{Gal}(B/F), \mathcal{E}(B)),$$

we only need to prove the following:

THEOREM 4.7. *Assume that the Iwasawa-μ-conjecture is true for K. Also assume that for any field F between K and K_∞ such that F is finite over K, Leopoldt's conjecture holds for F. Let B be a finite extension of F contained in K_∞. For any integer $i \geq 1$, there exists a positive integer n and an ideal $\mathcal{A}$ of height at least two in Λ, both independent of F and B, such that*

$$I_{T(K)}^n \mathcal{A} H^i(\mathrm{Gal}(B/F), \mathcal{E}(B)) = 0.$$

Proof. Since K_∞/K is a $\mathbb{Z}_p^d$-extension, $\mathrm{Gal}(B/F)$ is an abelian group which is a product of w finite cyclic groups, where w is an integer between 1 and d. We use induction on i.

If $i = 1$, the theorem is true because of Theorem 4.6 above. Suppose it is true up to some $i \geq 1$; we need to show that it is also true for $i + 1$.

If $w = 1$, then B/F is cyclic. This means

$$H^{i+1}(\mathrm{Gal}(B/F), \mathcal{E}(B)) = H^1(\mathrm{Gal}(B/F), \mathcal{E}(B))$$

when i is even, and

$$H^{i+1}(\mathrm{Gal}(B/F), \mathcal{E}(B)) = (\mathcal{E}(F)/\mathcal{E}(F)^{\mathrm{univ}}) \otimes \mathbb{Z}_p$$

when i is odd. This and Theorems 4.2 and 4.6 imply that the conclusion is true in this case. Suppose that the conclusion of the theorem is true up to some $w \geq 1$. We need to show that it is also true for $w + 1$.

Let C be an extension of F in B such that $\mathrm{Gal}(B/C)$ is a product of w finite cyclic groups and that $\mathrm{Gal}(C/F)$ is cyclic. Then by Section 4 of Chapter 2 in [10], we have the following Hochschild–Serre spectral sequence:

$$H^p(\mathrm{Gal}(C/F), H^q(\mathrm{Gal}(B/C), \mathcal{E}(B))) \Rightarrow_p H^*(\mathrm{Gal}(B/F), \mathcal{E}(B)).$$

Using the notation in the same section of [10], we let

$$E_2^{p,q} = H^p(\mathrm{Gal}(C/F), H^q(\mathrm{Gal}(B/C), \mathcal{E}(B))).$$

Here p, q are nonnegative integers.

Since the conclusion of the theorem is true for $H^q(\mathrm{Gal}(B/C), \mathcal{E}(B))$ for any integer q between 1 and $i + 1$, there exists a positive integer m and an ideal $\mathcal{B}$ of height at least two in Λ, both independent of B and C, such that $I_{T(K)}^m \mathcal{B}$ annihilates $H^q(\mathrm{Gal}(B/C), \mathcal{E}(B))$ for all integers q between 1 and $i+1$. Since $H^0(\mathrm{Gal}(B/C), \mathcal{E}(B)) = \mathcal{E}(C)$, there exists a positive integer l and an ideal $\mathcal{C}$ of height at least two in Λ, both independent of F and C, such that

$$I_{T(K)}^l \mathcal{C} H^{i+1}(\mathrm{Gal}(C/F), H^0(\mathrm{Gal}(B/C), \mathcal{E}(B))) = 0.$$

This implies that there exists a positive integer k and an ideal $\mathcal{D}$ of height at least two in Λ, both independent of F, B or C, such that

$$I_{T(K)}^k \mathcal{D} \bigoplus_{p+q=i+1} E_2^{p,q} = 0.$$

From this we get
$$I_{T(K)}^{k}\mathcal{D}\bigoplus_{p+q=i+1}E_{\infty}^{p,q}=0.$$
This means
$$I_{T(K)}^{k(i+1)}\mathcal{D}^{(i+1)}H^{i+1}(\mathrm{Gal}(B/F),\mathcal{E}(B))=0.$$
Now we can conclude that there exists a positive integer n and an ideal $\mathcal{A}$ of height at least two in Λ, both independent of F and B, such that $I_{T(K)}^{n}\mathcal{A}H^{i+1}(\mathrm{Gal}(B/F),\mathcal{E}(B))=0$.

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Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, Ohio 43210
U.S.A.
E-mail: mingxu@math.ohio-state.edu

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