

Integer solutions of a sequence of decomposable form inequalities

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1. Introduction. Let $F(\mathbf{X}) = F(X_0, \dots, X_m) \in \mathbb{Z}[\mathbf{X}]$ be a *decomposable form*, i.e. a homogeneous polynomial which factorizes into linear forms over $\overline{\mathbb{Q}}$. Assume that $q = \deg F > 2m$, and consider the decomposable form inequality

$$(1.1) \quad 0 < |F(\mathbf{x})| < |\mathbf{x}|^\nu \quad \text{in } \mathbf{x} = (x_0, \dots, x_m) \in \mathbb{Z}^{m+1},$$

where $|\mathbf{x}| = \max_{0 \leq i \leq m} |x_i|$ and $\nu < q - 2m$. For $m = 1$, it follows from Roth's approximation theorem (cf. [Sch2], p. 120) that if the linear factors of F are pairwise nonproportional, then (1.1) has only finitely many solutions. Using his subspace theorem, W. M. Schmidt ([Sch1], [Sch2]) generalized this to arbitrary m , under the assumptions that (i) any $m + 1$ of the linear factors of F are linearly independent over $\overline{\mathbb{Q}}$, and that (ii) F is not divisible by a form with rational coefficients of degree less than $m + 1$. Later H. P. Schlickewei [Schl] extended this theorem to the case when the ground ring is an arbitrary finitely generated subring of \mathbb{Q} . These results have obvious applications to decomposable form equations of the form

$$(1.2) \quad F(\mathbf{x}) = G(\mathbf{x}) \quad \text{in } \mathbf{x} \in \mathbb{Z}^{m+1},$$

where $G \in \mathbb{Z}[\mathbf{X}]$ is a nonzero polynomial of degree $< q - 2m$ (cf. [Sch2]). In the important special case when G is a constant, the first author [Gy1] proved the finiteness of the number of solutions of (1.2) under the assumption (i) only, which is necessary in general. For the case when G is a constant, there are also more general finiteness results: see [Sch2], [EGy2], [EGy4] and the references given there. The above mentioned results have further appli-

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cations to resultant inequalities and resultant equations (cf. [Sch1], [Sch2], [Schl], [Gy1], [Gy3]).

The purpose of this paper is to improve and generalize the above results of [Sch1], [Sch2], [Schl] to a sequence of decomposable form inequalities over number fields (cf. Theorem 2). For a single decomposable form inequality, Theorem 2 implies the finiteness of the number of solutions of (1.1) without assuming (ii) (cf. Theorem 1). Our Theorems 1 and 2 do not remain valid in general for $\nu = q - 2m$. As a consequence of Theorem 2, finiteness results are established for decomposable form equations of the form (1.2) over number fields (cf. Theorems 3, 4). Some applications are also given to resultant inequalities (Theorems 5, 6). Finally, we generalize our results to the case where $q > 2m - l + 1$ with $1 \leq l \leq m + 1$, by showing that in this situation the set of solutions is contained in a finite union of subspaces of dimension at most l (cf. Theorems 7–9).

2. Notation and statement of the main results. Let \mathbb{K} be an algebraic number field. Denote by $\mathbb{M}(\mathbb{K})$ the set of places of \mathbb{K} and write $\mathbb{M}_\infty(\mathbb{K})$ for the set of archimedean places of \mathbb{K} . For $v \in \mathbb{M}(\mathbb{K})$ denote by $|\cdot|_v$ the associated absolute value, normalized so that $|\cdot|_v = |\cdot|$ (standard absolute value) on \mathbb{Q} if v is archimedean, whereas for v nonarchimedean $|p|_v = p^{-1}$ if v lies above the rational prime p . Denote by \mathbb{K}_v the completion of \mathbb{K} with respect to v and by $d_v = [\mathbb{K}_v : \mathbb{Q}_v]$ the local degree. We put $\|\cdot\|_v = |\cdot|_v^{d_v}$.

For $\mathbf{x} = (x_0, \dots, x_m) \in \mathbb{K}^{m+1}$, we put $\|\mathbf{x}\|_v = \max_{0 \leq i \leq m} \|x_i\|_v$ and we denote by

$$h(\mathbf{x}) = \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in \mathbb{M}(\mathbb{K})} \log \|\mathbf{x}\|_v$$

the *absolute logarithmic height* of \mathbf{x} . Given a polynomial P with coefficient in \mathbb{K} , we define $\|P\|_v$ and $h(P)$ as the $\|\cdot\|_v$ -value and absolute logarithmic height, respectively, of the point whose coordinates are the coefficients of P . As is known, $h(\mathbf{x})$ and $h(P)$ are independent of the choice of the field \mathbb{K} . Further, $h(\lambda\mathbf{x}) = h(\mathbf{x})$ and $h(\lambda P) = h(P)$ for all $\lambda \in \overline{\mathbb{Q}}^*$.

Let S be a finite subset of $\mathbb{M}(\mathbb{K})$ containing $\mathbb{M}_\infty(\mathbb{K})$. An element $x \in \mathbb{K}$ is said to be *S-integer* if $\|x\|_v \leq 1$ for each $v \in \mathbb{M}(\mathbb{K}) - S$. Denote by O_S the set of *S-integers*. The units of O_S are called *S-units*. They form a multiplicative group which is denoted by O_S^* . For $\alpha \in \mathbb{K} - \{0\}$, let $N_S(\alpha) = \prod_{v \in S} \|\alpha\|_v$ denote the *S-norm* of α . If $\alpha \in O_S - \{0\}$ then $N_S(\alpha)$ is a positive integer and $N_S(\alpha) = 1$ for $\alpha \in O_S^*$. For $\mathbf{x} = (x_0, \dots, x_m) \in \mathbb{K}^{m+1}$, define the *S-height* as $H_S(\mathbf{x}) = \prod_{v \in S} \|\mathbf{x}\|_v$. If $\mathbf{x} \in O_S^{m+1} - \{0\}$, then $H_S(\mathbf{x}) \geq 1$ and $H_S(\alpha\mathbf{x}) = H_S(\mathbf{x})$ for $\alpha \in O_S^*$. For a polynomial P with coefficients in \mathbb{K} , let $H_S(P)$ denote the *S-height* of that point whose coordinates are the coefficients of P .

Linear forms $L_1, \dots, L_q \in \overline{\mathbb{Q}}[x_0, \dots, x_m]$ are said to be in *general position* if any $m + 1$ of them are linearly independent over $\overline{\mathbb{Q}}$.

Let q and m denote positive integers such that $q > 2m$, and let $F(\mathbf{X}) = F(X_0, \dots, X_m) \in O_S[\mathbf{X}]$ be a decomposable form of degree q . For given real numbers c and ν with $c > 0$, consider the solutions of the inequality

$$(2.1) \quad 0 < N_S(F(\mathbf{x})) \leq cH_S(\mathbf{x})^\nu \quad \text{in } \mathbf{x} \in O_S^{m+1}.$$

If \mathbf{x} is a solution of (2.1), then so is $\mathbf{x}' = \eta\mathbf{x}$ for every $\eta \in O_S^*$. Such solutions \mathbf{x}, \mathbf{x}' are called *O_S^* -proportional*.

THEOREM 1. *Suppose that $\nu < q - 2m$ and that the linear factors of F over $\overline{\mathbb{Q}}$ are in general position. Then (2.1) has only finitely many O_S^* -nonproportional solutions.*

REMARK 1. Theorem 1 does not remain valid in general for $\nu = q - 2m$ (cf. Schmidt [Sch1]). In the important case when $\mathbb{K} = \mathbb{Q}$, Theorem 1 provides an improvement of the previous results of Schmidt [Sch1, Sch2] and Schlickewei [Schl]. Further, for $\mathbb{K} = \mathbb{Q}$, $S = \mathbb{M}_\infty(\mathbb{Q})$ and $m = 1$, Theorem 1 gives Theorem 3B in Chapter V of Schmidt [Sch2].

REMARK 2. For $\nu = 0$, Theorem 1 was proved in a quantitative form in Gyóry [Gy3]. For a generalization considered over an arbitrary finitely generated ground ring over \mathbb{Z} , see Gyóry [Gy1].

The main result of this paper is Theorem 2. Theorem 1 is an immediate consequence.

THEOREM 2. *Let q, m be positive integers with $q > 2m$. Let c, ν be real numbers with $c > 0$, $\nu < q - 2m$ and \mathbb{G} a finite extension of \mathbb{K} . For $n = 1, 2, \dots$, let $F_n(\mathbf{X}) = F_n(X_0, \dots, X_m) \in O_S[\mathbf{X}]$ denote a decomposable form of degree q which factorizes into linear factors over \mathbb{G} , and suppose that these factors are in general position for each n . Then there does not exist an infinite sequence of O_S^* -nonproportional $\mathbf{x}_n \in O_S^{m+1}$, $n = 1, 2, \dots$, for which*

$$(2.2) \quad 0 < N_S(F_n(\mathbf{x}_n)) \leq cH_S(\mathbf{x}_n)^\nu \quad \text{for } n = 1, 2, \dots,$$

and

$$(2.3) \quad h(F_n) = o(h(\mathbf{x}_n)) \quad \text{if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

3. Proof of Theorem 2. We keep the notation of Section 2 and recall Schmidt's subspace theorem with moving targets proved by Min Ru and P. Vojta (see [RV]). The fixed target case, i.e. the case when $L_{j,n} = L_j$ for each j and n , is due to Min Ru and P. M. Wong (see Theorem 4.1 in [RW]).

THEOREM A [RV]. *Given linear forms $L_{1,n}, \dots, L_{q,n} \in \mathbb{K}[X_0, \dots, X_m]$, $n = 1, 2, \dots$, in general position for each n and a sequence $\mathbf{x}_n \in \mathbb{K}^{m+1}$ such*

that for $j = 1, \dots, q$, $L_{j,n}(\mathbf{x}_n) \neq 0$ and $h(L_{j,n}) = o(h(\mathbf{x}_n))$ as $n \rightarrow \infty$. Then, for any $\varepsilon > 0$, there exists an infinite subsequence $\{n_k\} \subseteq \mathbb{N}$ such that

$$\frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in S} \sum_{j=1}^q \log \frac{\|\mathbf{x}_{n_k}\|_v \cdot \|L_{j,n_k}\|_v}{\|L_{j,n_k}(\mathbf{x}_{n_k})\|_v} \leq (2m + \varepsilon)h(\mathbf{x}_{n_k}) \quad \text{for all } k.$$

The above statement is contained in the second part of Theorem 3.1 in [RV]. Note that, due to a printing mistake, the term $(2m + \varepsilon)h(\mathbf{x}_{n_k})$ on the right-hand side of the above inequality was incorrectly stated as $(2m + 1 + \varepsilon)h(\mathbf{x}_{n_k})$ in [RV].

Proof of Theorem 2. We shall prove Theorem 2 by using the above theorem of [RV]. Assume that there is an infinite sequence $\mathbf{x}_n = (x_{0,n}, \dots, x_{m,n}) \in O_S^{m+1}$ which satisfies (2.2). First consider the case when the values $h(\mathbf{x}_n)$ are bounded. We may assume without loss of generality that $x_{0,n} \neq 0$ for each n . Then the $h(\mathbf{x}_n/x_{0,n})$ are bounded and this implies that $\mathbf{x}_n/x_{0,n}$ may assume only finitely many values in \mathbb{K}^{m+1} . Hence there are infinitely many n such that $\mathbf{x}_n = x_{0,n}\mathbf{x}_0$ for some $\mathbf{x}_0 \in \mathbb{K}^{m+1}$. For these n we deduce from (2.2) that

$$0 < N_S(x_{0,n})^q N_S(F_n(\mathbf{x}_0)) \leq cN_S(x_{0,n})^\nu H_S(\mathbf{x}_0)^\nu$$

and hence $N_S(x_{0,n})$ are bounded. Since $x_{0,n} \in O_S$, it follows (see e.g. [EGY3]) that there are infinitely many n for which $x_{0,n} = \eta_n x'_0$ with some $\eta_n \in O_S^*$ and fixed $x'_0 \in O_S$. This implies that for these n the \mathbf{x}_n considered above are O_S^* -proportional, which is a contradiction.

Next consider the case when $h(\mathbf{x}_n)$ are not bounded. We may assume that $h(\mathbf{x}_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, by assumption, (2.3) also holds. Further, it follows that $H_S(\mathbf{x}_n) \rightarrow \infty$ as $n \rightarrow \infty$. For $n = 1, 2, \dots$, let $F_n = L_{1,n} \cdots L_{q,n}$ be a factorization of F_n over \mathbb{G} into linear factors. Then by Proposition 2.4 of Chapter 3 in [L] it follows that

$$\max_j h(L_{j,n}) \leq h(F_n) + c_1$$

where c_1 is a positive constant which depends only on q, m and \mathbb{G} . Together with (2.3) this gives

$$(3.1) \quad \max_j h(L_{j,n}) = o(h(\mathbf{x}_n)) \quad \text{as } n \rightarrow \infty.$$

Let $\mathbb{M}(\mathbb{G})$ denote the set of places of \mathbb{G} . For $v \in \mathbb{M}(\mathbb{G})$, define and normalize $\|\cdot\|_v$ as over \mathbb{K} above. Further, let T denote the set of extensions to \mathbb{G} of the places in S . Then we deduce from (2.2) that

$$(3.2) \quad 0 < N_T(F_n(\mathbf{x}_n)) = N_S(F_n(\mathbf{x}_n))^{[\mathbb{G}:\mathbb{K}]} \leq (cH_S(\mathbf{x}_n)^\nu)^{[\mathbb{G}:\mathbb{K}]} = c_2 H_T(\mathbf{x}_n)^\nu,$$

where $c_2 = c^{[\mathbb{G}:\mathbb{K}]}$. Here $N_T(\cdot)$, $H_T(\cdot)$ are defined over \mathbb{G} in the same way as $N_S(\cdot)$, $H_S(\cdot)$ over \mathbb{K} .

Let $\varepsilon > 0$ with $0 < \varepsilon < q - 2m - \nu$. Then by the above Theorem A of [RV], there is an infinite subsequence $\mathbf{x}_{n_k} \in O_S^{m+1}$, $k = 1, 2, \dots$, of $\{\mathbf{x}_n\}$, which, without loss of generality, we assume to be $\{\mathbf{x}_n\}$ itself, such that

$$\frac{1}{[\mathbb{G} : \mathbb{Q}]} \sum_{v \in T} \sum_{j=1}^q \log \frac{\|\mathbf{x}_n\|_v \cdot \|L_{j,n}\|_v}{\|L_{j,n}(\mathbf{x}_n)\|_v} \leq (2m + \varepsilon)h(\mathbf{x}_n).$$

On the other hand, $F_n(\mathbf{x}_n) = \prod_{j=1}^q L_{j,n}(\mathbf{x}_n)$. Furthermore, in view of $\mathbf{x}_n \in O_S^{m+1}$, we have

$$h(\mathbf{x}_n) \leq \frac{1}{[\mathbb{G} : \mathbb{Q}]} \log H_T(\mathbf{x}_n).$$

Hence

$$(3.3) \quad \prod_{v \in T} \frac{\|\mathbf{x}_n\|_v^q \cdot \prod_{j=1}^q \|L_{j,n}\|_v}{\|F_n(\mathbf{x}_n)\|_v} \leq H_T(\mathbf{x}_n)^{2m+\varepsilon},$$

whence

$$(3.4) \quad \frac{H_T^q(\mathbf{x}_n) \cdot \prod_{v \in T} \prod_{j=1}^q \|L_{j,n}\|_v}{N_T(F_n(\mathbf{x}_n))} \leq H_T(\mathbf{x}_n)^{2m+\varepsilon}.$$

Since the coefficients of $L_{j,n}$ are T -integers,

$$(3.5) \quad \prod_{v \in T} \prod_{j=1}^q \|L_{j,n}\|_v \geq 1 \quad \text{for } n = 1, 2, \dots$$

Furthermore, it follows from (3.2) that

$$(3.6) \quad N_T(F_n(\mathbf{x}_n)) \leq c_2 H_T(\mathbf{x}_n)^\nu \quad \text{for } n = 1, 2, \dots$$

Combining (3.4)–(3.6) gives

$$H_T(\mathbf{x}_n)^q \leq c_2 H_T(\mathbf{x}_n)^{\nu+2m+\varepsilon}.$$

Since $H_T(\mathbf{x}_n) \rightarrow \infty$ as $n \rightarrow \infty$, and $q > \nu + 2m + \varepsilon$, this is a contradiction. ■

4. Consequences of Theorem 2. In this section, we give four further consequences of our Theorem 2.

First, it is easy to see that Theorem 2 implies the following result concerning the S -integer solutions of a sequence of decomposable form equations of the form (1.2).

THEOREM 3. *Given positive integers q, m with $q > 2m$, a finite extension \mathbb{G} of \mathbb{K} , and a sequence of polynomials $G_n(\mathbf{X}) \in O_S[\mathbf{X}]$ in $\mathbf{X} = (X_0, \dots, X_m)$ such that $\deg(G_n(\mathbf{X})) < q - 2m$ for $n = 1, 2, \dots$. Let $F_n(\mathbf{X}) = F_n(X_0, \dots, X_m) \in O_S[\mathbf{X}]$ be a sequence of decomposable forms of degree q*

such that F_n factorizes into linear forms over \mathbb{G} which are in general position for each n . Then there does not exist an infinite sequence of O_S^* -non-proportional $\mathbf{x}_n \in O_S^{m+1}$ for which

$$(4.1) \quad F_n(\mathbf{x}_n) = G_n(\mathbf{x}_n) \neq 0, \quad n = 1, 2, \dots,$$

$$(4.2) \quad \log H_S(G_n) = o(\log H_S(\mathbf{x}_n)) \quad \text{if } H_S(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty$$

and

$$(4.3) \quad h(F_n) = o(h(\mathbf{x}_n)) \quad \text{if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

REMARK 3. We note that under the assumptions (4.2) and (4.3), equations (4.1) may have infinitely many O_S^* -proportional solutions \mathbf{x}_n . Indeed, this is the case if F_n is the same for each n , η_n is an infinite sequence of S -units, $\mathbf{x}_0 \in O_S^{m+1}$ with $F_n(\mathbf{x}_0) = 1, G_n = \eta_n^q$ and $\mathbf{x}_n = \eta_n \cdot \mathbf{x}_0$ for $n = 1, 2, \dots$

PROOF (of Theorem 3). Suppose that there is an infinite sequence of O_S^* -nonproportional $\mathbf{x}_n \in O_S^{m+1}$ satisfying the conditions of Theorem 3. It is easy to see that the $H_S(\mathbf{x}_n)$ are not bounded. We may assume that $H_S(\mathbf{x}_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows that

$$(4.4) \quad 0 < N_S(F_n(\mathbf{x}_n)) = N_S(G_n(\mathbf{x}_n)) = \prod_{v \in S} \|G_n(\mathbf{x}_n)\|_v \\ \leq c_3 \prod_{v \in S} (\|G_n\|_v \cdot \|\mathbf{x}_n\|_v^{\deg G_n}) = c_3 H_S(G_n) H_S(\mathbf{x}_n)^{\deg G_n}$$

where c_3 is a positive constant which depends only on q and \mathbb{G} . We choose ν such that $q - 2m - 1 < \nu < q - 2m$. Then, by (4.2), we deduce that

$$H_S(G_n) \leq H_S(\mathbf{x}_n)^{\nu - \deg G_n} \quad \text{as } n \rightarrow \infty.$$

Hence (4.4) implies that

$$0 < N_S(F_n(\mathbf{x}_n)) \leq c_3 H_S(\mathbf{x}_n)^\nu \quad \text{as } n \rightarrow \infty,$$

which contradicts Theorem 2. ■

We deduce from Theorem 3 the following.

THEOREM 4. *Given positive integers q, m with $q > 2m$, and a polynomial $G(\mathbf{X}) \in O_S[\mathbf{X}]$ in $\mathbf{X} = (X_0, \dots, X_m)$ with total degree less than $q - 2m$. Let $F(\mathbf{X}) \in O_S[\mathbf{X}]$ be a decomposable form whose linear factors are in general position. Then the equation*

$$(4.5) \quad F(\mathbf{X}) = G(\mathbf{X})$$

has only finitely many solutions $\mathbf{x} = (x_0, \dots, x_m) \in O_S^{m+1}$ with $G(\mathbf{x}) \neq 0$.

REMARK 4. For $\mathbb{K} = \mathbb{Q}, S = \mathbb{M}_\infty(\mathbb{Q})$ and $m = 1$, our Theorem 4 gives the second assertion of Theorem 3B of [Sch2, Ch. V]. When G is constant, Theorem 4 was proved over more general ground rings in K. Györy [Gy1].

PROOF (of Theorem 4). Suppose that (4.5) has infinitely many solutions \mathbf{x} with $G(\mathbf{x}) \neq 0$. Then, by Theorem 3, there are also infinitely many solutions \mathbf{x} such that $\mathbf{x} = \eta \mathbf{x}_0$ with $\eta \in O_S^*$ and with some fixed $\mathbf{x}_0 \in O_S^{m+1}$. Then it follows from (4.5) that

$$\eta^q F(\mathbf{x}_0) = G(\eta \mathbf{x}_0).$$

This can be regarded as an equation of degree q in η with leading coefficient $F(\mathbf{x}_0) \neq 0$. However, this equation has at most q solutions in η , which proves the assertion. ■

Let q, m be positive integers with $q > 2m$, and let $P \in O_S[X]$ be a polynomial of degree q without multiple zeros. For given $c > 0$ and ν , consider the solutions of the resultant inequality

$$(4.6) \quad 0 < N_S(\text{Res}(P, Q)) \leq cH_S(Q)^\nu \quad \text{in } Q \in O_S[X] \text{ of degree } m.$$

If Q is a solution then so is ηQ for each $\eta \in O_S^*$. Such solutions $Q, \eta Q$ are called O_S^* -proportional.

THEOREM 5. *If $\nu < q - 2m$, then (4.6) has only finitely many O_S^* -nonproportional solutions.*

REMARK 5. For $\mathbb{K} = \mathbb{Q}$, Theorem 5 is an improvement of previous results of E. Wirsing [W], W. M. Schmidt [Sch1] and H. P. Schlickewei [Schl]. Theorem 5 does not remain valid when $\nu = q - 2m$ (cf. [Sch1]). For $\nu = 0$, Theorem 5 was proved in a quantitative form in K. Györy [Gy2]. For a generalization to polynomials over more general ground rings, see K. Györy [Gy1].

Theorem 5 is an immediate consequence of the next theorem which will be deduced from Theorem 2.

THEOREM 6. *Let q, m be positive integers with $q > 2m$, c, ν real numbers with $c > 0, \nu < q - 2m$, and \mathbb{G} a finite extension of \mathbb{K} . For every integer $n \geq 1$, let $P_n \in O_S[X]$ denote a polynomial of degree q with distinct zeros in \mathbb{G} . Then there does not exist a sequence of O_S^* -nonproportional $Q_n \in O_S[X]$ with $\deg Q_n = m$ for which*

$$(4.7) \quad 0 < N_S(\text{Res}(P_n, Q_n)) \leq cH_S(Q_n)^\nu, \quad n = 1, 2, \dots,$$

and

$$(4.8) \quad h(P_n) = o(h(Q_n)) \quad \text{if } h(Q_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This should be compared with Corollary 4 of K. Györy [Gy2] where $\nu = 0$, but O_S is replaced by a more general ground ring.

PROOF (of Theorem 6). Put

$$P_n(X) = a_{0,n}(X - \alpha_{1,n}) \dots (X - \alpha_{q,n}) \quad \text{for } n = 1, 2, \dots$$

Assume that there is an infinite sequence of

$$Q_n(X) = x_{0,n}X^m + x_{1,n}X^{m-1} + \dots + x_{m,n},$$

satisfying (4.7) and (4.8). For $n \geq 1$ set

$$F_n(\mathbf{X}) = F_n(X_0, \dots, X_m) = a_{0,n}^m \prod_{i=1}^q (X_0 \alpha_{i,n}^m + X_1 \alpha_{i,n}^{m-1} + \dots + X_m).$$

Then F_n has its coefficients in O_S . Further, it factorizes into linear factors over \mathbb{G} , and these are in general position for every n . For $\mathbf{x}_n = (x_{0,n}, \dots, x_{m,n})$ we have

$$\text{Res}(P_n, Q_n) = F_n(\mathbf{x}_n).$$

Hence (4.7) implies that

$$(4.9) \quad 0 < N_S(F_n(\mathbf{x}_n)) \leq cH_S(Q_n)^\nu = cH_S(\mathbf{x}_n)^\nu.$$

Further, by using Proposition 2.4 of Chapter 3 in [L] it is easy to see that

$$(4.10) \quad h(F_n) \leq qh(P_n) + c_4,$$

where c_4 is a constant which depends only on q, m and \mathbb{G} . Hence if $h(Q_n) \rightarrow \infty$ as $n \rightarrow \infty$, it follows from (4.10) and (4.8) that

$$h(F_n) = o(h(\mathbf{x}_n)) \quad \text{if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Together with (4.9) this contradicts our Theorem 2. ■

5. Some generalizations. In the previous sections, we assume that $q > 2m$ where q is the degree of a decomposable form F , and $m + 1$ is the number of variables in F . In this section, we consider the case $q > 2m - l + 1$, where l is an integer with $1 \leq l \leq m + 1$. As we indicated earlier, for $l > 1$, no finiteness result is expected. Rather, we show that the set of solutions is then contained in a finite union of proper subspaces.

First of all, Theorem 1 can be generalized as follows.

THEOREM 7. *Let q, m, l denote positive integers with $q > 2m - l + 1, 1 \leq l \leq m + 1$, and \mathbb{G} a finite extension of \mathbb{K} . Let $F(\mathbf{X}) = F(X_0, \dots, X_m) \in O_S[\mathbf{X}]$ be a decomposable form of degree q which factorizes into linear factors over \mathbb{G} . For given real numbers c, ν with $c > 0$, consider the solutions of the inequality*

$$(5.1) \quad 0 < N_S(F(\mathbf{x})) \leq cH_S(\mathbf{x})^\nu \quad \text{in } \mathbf{x} \in O_S^{m+1}.$$

Suppose that $\nu < q - 2m + l - 1$ and that the linear factors of F over $\overline{\mathbb{Q}}$ are in general position. Then the set of solutions of (5.1) is contained in a finite union of linear subspaces of \mathbb{K}^{m+1} of dimension at most l .

It is easy to see that for $l = 1$, Theorem 7 gives Theorem 1. For $\nu = 0$, compare Theorem 3 of [EGy1].

The proof of Theorem 7 is similar to the proof of Theorem 2. First we recall Theorem 4.1 of Ru–Wong [RW] (see also [RV], Theorem 3.1).

THEOREM B (Theorem 4.1 of [RW]). *Given linear forms $L_1, \dots, L_q \in \mathbb{K}[X_0, \dots, X_m]$ in general position. Then for any $\varepsilon > 0$, the set of points $\mathbf{x} \in \mathbb{K}^{m+1}$ such that $L_j(\mathbf{x}) \neq 0$ for $j = 1, \dots, q$ and*

$$\frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{v \in S} \sum_{j=1}^q \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \geq (2m - l + 1 + \varepsilon)h(\mathbf{x})$$

is contained in a finite union of linear subspaces of \mathbb{K}^{m+1} of dimension at most l .

Proof (of Theorem 7). Let $\mathbf{x} \neq \mathbf{0}$ be a solution of (5.1). Let $F = L_1 \dots L_q$ be a factorization of F over \mathbb{G} into linear factors. Let $\mathbb{M}(\mathbb{G})$ denote the set of places of \mathbb{G} . For $v \in \mathbb{M}(\mathbb{G})$, define and normalize $\|\cdot\|_v$ as over \mathbb{K} above. Further, let T denote the set of extensions to \mathbb{G} of the places in S . Then we deduce from (5.1) that

$$0 < N_T(F(\mathbf{x})) = N_S(F(\mathbf{x}))^{[\mathbb{G}:\mathbb{K}]} \leq (cH_S(\mathbf{x})^\nu)^{[\mathbb{G}:\mathbb{K}]} = c_2H_T(\mathbf{x})^\nu,$$

where $c_2 = c^{[\mathbb{G}:\mathbb{K}]}$. Here $N_T(\cdot), H_T(\cdot)$ are defined over \mathbb{G} in the same way as $N_S(\cdot), H_S(\cdot)$ over \mathbb{K} . We have $F(\mathbf{x}) = \prod_{j=1}^q L_j(\mathbf{x})$. It follows that

$$\begin{aligned} (5.2) \quad \prod_{v \in T} \frac{\|\mathbf{x}\|_v^q \cdot \prod_{j=1}^q \|L_j\|_v}{\prod_{j=1}^q \|L_j(\mathbf{x})\|_v} &= \frac{H_T(\mathbf{x})^q \cdot (\prod_{v \in T} \prod_{j=1}^q \|L_j\|_v)}{N_T(F(\mathbf{x}))} \\ &\geq \frac{H_T(\mathbf{x})^q \cdot (\prod_{v \in T} \prod_{j=1}^q \|L_j\|_v)}{c_2H_T(\mathbf{x})^\nu} \\ &= c_5H_T(\mathbf{x})^{q-\nu}, \end{aligned}$$

where $c_5 = \prod_{v \in T} \prod_{j=1}^q \|L_j\|_v / c_2$. In view of $\mathbf{x} \in O_T^m$, we have

$$h(\mathbf{x}) \leq \frac{1}{[\mathbb{G} : \mathbb{Q}]} \log H_T(\mathbf{x}).$$

Thus, by taking logarithms on both sides of (5.2),

$$\frac{1}{[\mathbb{G} : \mathbb{Q}]} \sum_{v \in T} \sum_{j=1}^q \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \geq (q - \nu)h(\mathbf{x}) + c_6,$$

where $c_6 = \log c_5 / [\mathbb{G} : \mathbb{Q}]$. Let $\varepsilon > 0$ with $q - \nu \geq 2m + l - 1 > 2\varepsilon$. Then

$$\frac{1}{[\mathbb{G} : \mathbb{Q}]} \sum_{v \in T} \sum_{j=1}^q \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \geq (2m - l + 1 + 2\varepsilon)h(\mathbf{x}) + c_6.$$

Since the set of points $\mathbf{x} \in \mathbb{G}^{m+1}$ with $\varepsilon h(\mathbf{x}) + c_6 < 0$ is finite, excluding

these points yields

$$\frac{1}{[\mathbb{G} : \mathbb{Q}]} \sum_{v \in T} \sum_{j=1}^q \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \geq (2m - l + 1 + \varepsilon)h(\mathbf{x}).$$

Thus, the above quoted Theorem B of [RW] implies Theorem 7. ■

Similarly, by using Theorem 3.1 of [RV] Theorem 2 can be generalized as follows:

THEOREM 8. *Let q, m, l be positive integers with $q > 2m - l + 1, 1 \leq l \leq m + 1$. Let c, ν be real numbers with $c > 0, \nu < q - 2m + l - 1$ and \mathbb{G} a finite extension of \mathbb{K} . For $n = 1, 2, \dots$, let $F_n(\mathbf{X}) = F_n(X_0, \dots, X_m) \in O_S[\mathbf{X}]$ denote a decomposable form of degree q which factorizes into linear factors over \mathbb{G} , and suppose that these factors are in general position for each n . Then the points $\mathbf{x}_n \in O_S^{m+1}, n = 1, 2, \dots$, satisfying*

$$0 < N_S(F_n(\mathbf{x}_n)) \leq cH_S(\mathbf{x}_n)^\nu \quad \text{for } n = 1, 2, \dots,$$

and

$$h(F_n) = o(h(\mathbf{x}_n)) \quad \text{if } h(\mathbf{x}_n) \rightarrow \infty \text{ as } n \rightarrow \infty,$$

are l -degenerate (for the concept of l -degenerate, see [RV]).

Theorems 7 and 8 have applications to resultant inequalities of the same type as Theorems 1 and 2 above.

As was shown earlier, Theorem 1 implies Theorem 4. Similarly, our Theorem 7 has the following consequence.

THEOREM 9. *Given positive integers q, m, l with $q > 2m - l + 1, 1 \leq l \leq m + 1$, and a polynomial $G(\mathbf{X}) \in O_S[\mathbf{X}]$ in $\mathbf{X} = (X_0, \dots, X_m)$ with total degree less than $q - 2m + l - 1$. Let $F(\mathbf{X}) \in O_S[\mathbf{X}]$ be a decomposable form of degree q whose linear factors are in general position. Then the set of solutions of*

$$F(\mathbf{x}) = G(\mathbf{x}) \quad \text{in } \mathbf{x} = (x_0, \dots, x_m) \in O_S^{m+1}$$

is contained in a finite union of linear subspaces of \mathbb{K}^{m+1} of dimension at most l .

References

- [EGy1] J. H. Evertse and K. Györy, *Finiteness criteria for decomposable form equations*, Acta Arith. 50 (1988), 357–379.
- [EGy2] —, —, *Decomposable form equations*, in: New Advances in Transcendence Theory, A. Baker (ed.), Cambridge Univ. Press, 1988, 175–202.
- [EGy3] —, —, *Effective finiteness results for binary forms with given discriminant*, Compositio Math. 79 (1991), 169–204.

- [EGy4] J. H. Evertse and K. Györy, *The number of families of solutions of decomposable form equations*, Acta Arith. 80 (1997), 367–394.
- [Gy1] K. Györy, *Some applications of decomposable form equations to resultant equations*, Colloq. Math. 65 (1993), 267–275.
- [Gy2] —, *On the number of pairs of polynomials with given resultant or given semi-resultant*, Acta Sci. Math. (Szeged) 57 (1993), 515–529.
- [Gy3] —, *On the irreducibility of neighbouring polynomials*, Acta Arith. 67 (1994), 283–296.
- [L] S. Lang, *Fundamentals of Diophantine Geometry*, Springer, 1983.
- [RV] M. Ru and P. Vojta, *Schmidt's subspace theorem with moving targets*, Invent. Math. 127 (1997), 51–65.
- [RW] M. Ru and P. M. Wong, *Integral points of $\mathbf{P}^n - \{2n + 1 \text{ hyperplanes in general position}\}$* , *ibid.* 106 (1991), 195–216.
- [Sch1] H. P. Schlickewei, *Inequalities for decomposable forms*, Astérisque 41-42 (1977), 267–271.
- [Sch1] W. M. Schmidt, *Inequalities for resultants and for decomposable forms*, in: Diophantine Approximation and its Applications, Academic Press, New York, 1973, 235–253.
- [Sch2] —, *Diophantine Approximation*, Lecture Notes in Math. 785, Springer, Berlin, 1980.
- [W] E. Wirsing, *On approximations of algebraic numbers by algebraic numbers of bounded degree*, in: Proc. Sympos. Pure Math. 20, Amer. Math. Soc., Providence, R.I., 1971, 213–247.

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