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Determination of the imaginary normal octic number fields with class number one which are not CM-fields

by

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1. Introduction. It is known that there exist only finitely many normal CM-fields with class number one ([14]). K. Uchida [22] proved that for each finite group G which is the direct product of a 2-group with a finite group of odd order, there exist only finitely many imaginary G-extension of $\mathbb{Q}$ (in $\mathbb{C}$) with class number one. The purpose of this paper is to determine the imaginary normal octic number fields with class number one which are not CM-fields. There exist exactly 67 such fields with class number one. All of them are dihedral extensions of $\mathbb{Q}$, 20 fields are Hilbert class fields of imaginary quadratic number fields with class number four, the other 47 fields are ramified cyclic quartic extensions of imaginary quadratic number fields with class number one or two. We note that the normal octic CM-fields with class number one have been already determined (the abelian ones by K. Uchida [21] and the nonabelian ones by S. Louboutin and R. Okazaki [12]).

A nonabelian group of order eight is isomorphic to the quaternion group, or D_4, the dihedral group of order eight. If a quaternionic field is imaginary, then it is a CM-field. For then complex conjugation must be the unique element of order two of the Galois group, and therefore the biquadratic subfield is its fixed field and totally real. Thus, if an imaginary normal octic number field is not a CM-field, then it is dihedral. In the following, we call a dihedral octic number field a D_4-*extension* of $\mathbb{Q}$.

In the rest of this paper, we use the following notations. K always denotes a D_4-extension of $\mathbb{Q}$, M its biquadratic bicyclic subfield. We denote by F the unique quadratic subfield of K such that K/F is cyclic (quartic). We denote by M_1, M_1' and M_2, M_2' the pairs of isomorphic nonnormal quartic subfields of K, and by F_1 and F_2 the quadratic subfields of M_1 and M_2,

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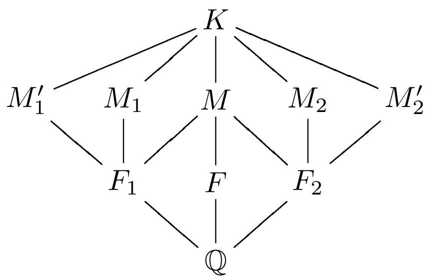
respectively. Thus, we have the following lattice of subfields of K:



We also use the following conventions. As usual, for an algebraic number field k of finite degree, $h(k), \mathrm{Cl}(k)$ and $d(k)$ denote the class number, class group, and discriminant of k, respectively.

Assume that K is imaginary but not a CM-field. Then M is also imaginary and therefore so is F. In fact, if F were real, then complex conjugation would be an element of order two of $\mathrm{Gal}(K/F)$ and therefore M would be real, which is a contradiction. Assume moreover that K has class number one. Then M has class number one or two, and F has class number one, two, or four. All such quartic fields and quadratic fields have already been determined [1, 2, 3, 13, 18, 21]. (See also [25].) We can easily see that if an odd prime number is ramified in K, then it is ramified in M (§2, Lemma 1(i)). Hence we can easily get the possibilities for such K. For each possible D_4-extension K of $\mathbb{Q}$, if the oddness of $h(K)$ is verified, then we obtain its class number $h(K)$ by calculating the class numbers $h(M_1)$ and $h(M_2)$ and using class number relation for D_4-extensions of $\mathbb{Q}$: $h(K) = h(M_1)h(M_2)$ (Proposition 3). We note that most of necessary arguments work also for fields with odd class number. Therefore for most cases to be considered we characterize K with odd class number and then determine K with class number one.

The organization of this paper is as follows: In Section 2, we discuss imaginary D_4-extensions of $\mathbb{Q}$ with odd class number. In particular, we give a proof of the finiteness of imaginary D_4-extensions of $\mathbb{Q}$ with odd class number less than any given bound. In Sections 3, 4, and 5, we determine the imaginary D_4-extensions K of $\mathbb{Q}$ with class number one which are not CM-fields. In Section 3, we treat the case $h(F) = 4$ (more precisely, $\mathrm{Cl}(F) \cong C_4$, the cyclic group of order four). In Section 4, we treat the case $h(F) = 2$. The case $h(F) = 1$ is divided into two subcases: $F = \mathbb{Q}(\sqrt{-1})$ and $F \neq \mathbb{Q}(\sqrt{-1})$. In the former case, K is the normal closure of a pure quartic number field and this case is treated in Section 5. The latter case $F \neq \mathbb{Q}(\sqrt{-1})$ is treated in Section 6. In Section 7 we summarize our results.

2. Imaginary D_4-extensions of $\mathbb{Q}$ with odd class number. First we give a proof of the finiteness of imaginary D_4-extensions of $\mathbb{Q}$ with odd class number less than any given bound, which is essentially the same as Uchida's proof in [21] in a more general situation ([1]). The first assertion of the following lemma is the key to the finiteness theorem.

LEMMA 1. *Let K be an imaginary D_4-extension of $\mathbb{Q}$ with odd class number.*

(i) *If p is an odd prime number ramified in K, then p is ramified in the biquadratic bicyclic subfield M of K.*

(ii) *The genus field M_{gen} of M equals M. Hence by genus theory two of the discriminants of the three quadratic subfields of M are prime discriminants, and except for the case $M = \mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$, exactly two discriminants are prime and the other one is the product of these two.*

Proof. (i) Let p be an odd prime number ramified in K. Suppose to the contrary that p is unramified in M. Put $p^* = (-1)^{(p-1)/2}p \equiv 1 \pmod 4$. Then the quadratic extensions $M(\sqrt{p^*})/M$ and $K(\sqrt{p^*})/K$ are clearly unramified outside p, but since p is unramified in M, the extension $M(\sqrt{p^*})/M$ is ramified at all the prime ideals of M lying above p. We also note that all the prime ideals of M lying above p are ramified in the quadratic extension K. Since the extension $K(\sqrt{p^*})/M$ are biquadratic bicyclic, the prime ideals of M lying above the odd prime p are not totally ramified in $K(\sqrt{p^*})/M$. Therefore $K(\sqrt{p^*})/K$ is unramified also at p, that is, $K(\sqrt{p^*})/K$ is unramified. Hence the class number of K is divisible by 2. This is a contradiction. Thus p is ramified in M.

(ii) Since the extension KM_{gen}/K is unramified abelian and its degree is a nonnegative power of 2, this extension must be trivial by the assumption, that is, M_{gen} must be contained in K. Since M_{gen} is abelian, $M_{\mathrm{gen}} = M$. ∎

By (i) we immediately obtain the following finiteness theorem.

THEOREM 1. *For any natural number N, there exist only finitely many imaginary D_4-extensions of $\mathbb{Q}$ (in $\mathbb{C}$) with odd class number $\leq N$.*

Proof. Let K be an imaginary D_4-extension of $\mathbb{Q}$ with odd class number $h(K) \leq N$. The finiteness of (normal) CM-fields is well known. So we assume that K is not a CM-field. Then its biquadratic bicyclic subfield M is imaginary, and obviously $h(M) \leq 2N$ (see Proposition 1(i) below). We know that there exist only finitely many such M [20, Theorem 2]. Therefore there exist only finitely many prime numbers which are ramified in some

([1]) Uchida refers only to fields with class number one, however, his argument works also for fields with odd class numbers. His proof is written in Japanese.

imaginary D_4-extensions of $\mathbb{Q}$ with odd class number $\le N$ by Lemma 1(i). Hence possible K with $h(K) \le N$ are finite. ■

Next, we consider relations among class numbers of subfields of K. First we review class number relations for D_4-extensions of $\mathbb{Q}$.

Lemma 2 ([4, 5]). *We have the following relation among the class numbers of subfields of a D_4-extension K of $\mathbb{Q}$:*

$$h(K) = 2^{-v}[E_K : E_{M_1}E_{M_2}E_F]h(M_1)h(M_2)h(F).$$

Here E_* denotes the group of units in $*$ and v is $4, 3,$ or 2 according as K is totally real, CM, or otherwise. Moreover, the unit index $[E_K : E_{M_1}E_{M_2}E_F]$ is a nonnegative power of 2.

By this relation, we obtain $h(K)$ (up to a power of 2) from $h(M_1)$, $h(M_2)$, and $h(F)$.

Secondly we describe divisibilities of class numbers:

Proposition 1. *Let K be a D_4-extension of $\mathbb{Q}$.*

(i) *We have $h(F) \mid 2h(M)$ and $h(M) \mid 2h(K)$. In particular, if $h(K)$ is odd, then $(h^{(2)}(F), h^{(2)}(M)) = (4,2)$, $(2,1)$, or $(1,1)$, where $h^{(2)}(F)$ (resp. $h^{(2)}(M)$) is the 2-class number, that is, the 2-part of the class number of F (resp. M), and the Hilbert 2-class field of F is K, M, or F, according as $h^{(2)}(F) = 4, 2,$ or 1. In particular, if $h(K) = 1$, then $(h(F), h(M)) = (4,2)$, $(2,1)$, or $(1,1)$, and the Hilbert class field of F is K, M, or F, according as $h(F) = 4$, 2, or 1.*

(ii) *We have $h(F_1) \mid h(M_1)$ and $h(F_2) \mid h(M_2)$.*

(iii) *For $i = 1$ and 2, we have $h(M_i) \mid h(K)$ unless M/F_i is unramified at all finite primes and M_i is totally real or totally imaginary.*

This can be easily proved by using the following two lemmas.

Lemma 3. *Let L/k be a finite extension of algebraic number fields.*

(i) *We have $h(k) \mid [L : k]h(L)$.*

(ii) ([10]) *If there exists a prime P (finite or infinite) of k which is totally ramified in L/k, then $h(k) \mid h(L)$. If, moreover, L/k is cyclic of p-power degree, where p is any prime number, and there is no prime which is ramified in L/k other than P, then $p \mid h(L)$ implies $p \mid h(k)$, or equivalently $p \nmid h(k)$ implies $p \nmid h(L)$.*

Note 1. (i) is easily proved, as is (ii). For the first assertion of (ii), it is assumed in [10] that L/k is normal, but this assumption is not necessary.

Lemma 4 ([11, Theorem A(c)]). *We have the following relation among discriminants of subfields of a D_4-extension K of $\mathbb{Q}$:*

$$d(K)/d(M) = d(M_1)^2/d(F_1)^2 = d(M_2)^2/d(F_2)^2.$$

From this we obtain

$$N_{M/\mathbb{Q}}d(K/M) = d(K)/d(M)^2 = d(M_i)^2/(d(F_i)^2 d(M)),$$

and

$$N_{M_i/\mathbb{Q}}d(K/M_i) = d(K)/d(M_i)^2 = d(M)/d(F_i)^2 = N_{F_i/\mathbb{Q}}d(M/F_i),$$

where $d_{K/M}$ *(resp.* $d(K/M_i)$*) is the relative discriminant of* K/M *(resp.* K/M_i*).*

Proof of Proposition 1. (i) The first assertion immediately follows by Lemma 3(i). Assume $h(K)$ is odd. Then $h^{(2)}(F) = 1, 2$, or 4 by $h(F) \mid 4h(K)$. Obviously if $h^{(2)}(F) = 4$, then $h^{(2)}(M) = 2$. If $h^{(2)}(F) = 2$, then K/M is ramified at some prime and therefore $h(M)$ is odd by Lemma 3(ii). Also when $h^{(2)}(F) = 1$, K/M is ramified at some prime, for K/F is cyclic and M/F is ramified at some prime. Therefore also in this case $h(M)$ is odd by Lemma 3(ii).

(ii) By genus theory any quadratic extension unramified at all finite primes of a quadratic number field is biquadratic bicyclic. Therefore both M_1/F_1 and M_2/F_2 are ramified at some finite prime. Hence the assertion follows by Lemma 3(ii).

(iii) Consider the ramification in K/M_i. By the equality $N_{M_i/\mathbb{Q}}d(K/M_i) = N_{F_i/\mathbb{Q}}d(M/F_i)$ of Lemma 4, K/M_i is unramified at all finite primes if and only if M/F_i is unramified at all finite primes. Therefore, unless M/F_i is unramified at all finite primes and M_i is totally real or totally imaginary, K/M_i is ramified at some prime. Hence the assertion follows by Lemma 3(ii). ∎

Of course, in Proposition 1(i), when $h^{(2)}(F) = 4$, the 2-class group $\mathrm{Cl}^{(2)}(F)$ of F is cyclic. We easily obtain the converse.

PROPOSITION 2. *The Hilbert* 2*-class field of a quadratic number field whose* 2*-class group is the cyclic group of order four is a* D_4*-extension of* $\mathbb{Q}$ *with odd class number.*

It is well known that such a field is a D_4-extension $\mathbb{Q}$, and the oddness of the class number immediately follows by applying the following.

LEMMA 5 (see [19, Theorem I]). *Let* k *be an algebraic number field of finite degree and* p *any prime number. If the* p*-class group*, *i.e.*, *the* p*-part of the class group of* k *is cyclic*, *then the* p*-class group of the Hilbert* p*-class field of* k *is trivial.*

PROPOSITION 3. *Let* K *be an imaginary* D_4*-extension with odd class number which is not a CM-field. Then all the class numbers* $h(M_1)$, $h(M_2)$, $h(F_1)$, *and* $h(F_2)$ *are odd and* $h(K) = h(M_1)h(M_2)h_{\mathrm{odd}}(F)$, *where* $h_{\mathrm{odd}}(F)$ *denotes the odd part of* $h(F)$. *In particular*, *if* $h(K) = 1$, *then all of them are* 1.

Proof. If $h(K)$, $h(M_1)$, and $h(M_2)$ are all odd, then by Lemma 4 we have $h(K) = h(M_1)h(M_2)h_{\mathrm{odd}}(F)$. Therefore by Proposition 1 it suffices to show that both K/M_1 and K/M_2 are ramified at some prime. We may assume that F_1 is imaginary and F_2 is real. Then since M_2 is neither totally real nor totally imaginary, K/M_2 is ramified at two infinite primes.

Suppose that K/M_1 is unramified. Then M/F_1 is unramified and therefore $h(F_1) \equiv 2 \pmod 4$ and M is its Hilbert 2-class field. (Note that since M_1/F_1 is ramified, $h(F_1)$ is not divisible by four.) Hence by Lemma 1(ii) we have $d(F_1) = d(F)d(F_2)$. Since K is cyclic over F, the equation $X^2 - d(F_1)Y^2 - d(F_2)Z^2 = 0$ has a nontrivial primitive solution in $\mathbb{Z}$ (cf. [23, Corollary 1.7(B)]). This implies that $d(F)$ is a quadratic residue modulo $d(F_2)$. Thus, by Rédei–Reichardt theory unless $d(F) = -4$ and $d(F_2) \equiv 5 \pmod 8$, the 4-rank of $\mathrm{Cl}(F_1)$ does not vanish, which contradicts $h(F_1) \equiv 2 \pmod 4$. Hence $d(F) = -4$ and $d(F_2) := p \equiv 5 \pmod 8$. Then $F = \mathbb{Q}(\sqrt{-1})$ and $M = \mathbb{Q}(\sqrt{-1}, \sqrt{p})$. Therefore M/F is ramified at the prime divisors of p, and so is K/M since K/F is cyclic. Since $p \equiv 5 \pmod 8$, there exists a unique cyclic quartic extension L of $\mathbb{Q}$ of conductor p, and its quadratic subextension is $\mathbb{Q}(\sqrt{p})$. Obviously the compositum KL is a quadratic extension of K, and KL/K is unramified outside p. Also the unramifiedness of the prime divisors of p in KL/K follows by Abhyankar's lemma [8, Lemma 9]. Hence $h(K)$ is even and this is a contradiction. Thus, K/M_1 is ramified at some finite prime. ∎

Thus, for our purpose, the knowledge of nonnormal quartic number fields with odd class number is useful. A detailed study of such fields is given in [6, Chapter III].

Finally, note that if K is an imaginary D_4-extension with odd class number which is not a CM-field, we may assume that F_1 is imaginary and F_2 are real, and that this implies that M_1 and M_1' are totally imaginary and M_2 and M_2' are neither totally real nor totally imaginary. (We know that the normal closure of a nonnormal quartic CM-field is a CM-field ([11, Theorem A(d)]).)

3. Hilbert class fields of the imaginary quadratic number fields with cyclic class group of order four. Now we turn to the determination of K with $h(K) = 1$. We treat here the case $\mathrm{Cl}(F) \cong C_4$. S. Arno [1] has confirmed that the known list of the imaginary quadratic number fields with class number four is complete. There exist exactly 30 imaginary quadratic number fields with cyclic class group of order four, and their Hilbert class fields are D_4-extensions of $\mathbb{Q}$ with odd class number by Proposition 2. Let K be the Hilbert class field of an imaginary quadratic number field with cyclic class group of order four. Then $h(K) = h(M_1)h(M_2)$ by Proposition 3.

By Lemma 1(ii), we have $d(F) = d(F_1)d(F_2)$. We know how to construct the Hilbert class field K of an imaginary quadratic number field F with cyclic class group of order four (see for example, [24]). We tabulate the factorization $d(F) = d(F_1)d(F_2)$, algebraic integers θ_i generating M_i of the form $\sqrt{a + b\sqrt{d(F_i)}}$ $(a, b \in \mathbb{Q})$ for $i = 1, 2$, and $h_1 = h(M_1)$, $h_2 = h(M_2)$, $H = h(K)$ (Table 1 below).

From this table, we see that there exist exactly 20 imaginary D_4-extensions of $\mathbb{Q}$ with class number one which are Hilbert class fields of imaginary quadratic number fields with cyclic class group of order four. Of course, we can exclude the other 10 Hilbert class fields with class numbers larger than one by the condition $h(M) = 1$ without calculations of $h(M_1)$ and $h(M_2)$. However, it is worthwhile to give the table of the class numbers and generators of the Hilbert class fields of imaginary quadratic number fields with cyclic class group of order four. We note that the class numbers of the other 10 Hilbert class fields are all prime and if we denote by p the class number, we can easily verify that the Galois group of L/F is isomorphic to the generalized quaternion group of order $4p$, where L is the second Hilbert class field of F, that is, the Hilbert class field of K (cf. [26]).

Table 1

$d(F) = d(F_1)d(F_2)$	θ_1	θ_2	h_1	h_2	H
$-39 = (-3) \cdot 13$	$\sqrt{-1+2\sqrt{-3}}$	$\sqrt{(-1+\sqrt{13})/2}$	1	1	1
$-55 = (-11) \cdot 5$	$\sqrt{(3+\sqrt{-11})/2}$	$\sqrt{3+2\sqrt{5}}$	1	1	1
$-56 = (-7) \cdot 8$	$\sqrt{(-1+\sqrt{-7})/2}$	$\sqrt{-1+2\sqrt{2}}$	1	1	1
$-68 = (-4) \cdot 17$	$\sqrt{1+4\sqrt{-1}}$	$\sqrt{(1+\sqrt{17})/2}$	1	1	1
$-136 = (-8) \cdot 17$	$\sqrt{3+2\sqrt{-2}}$	$\sqrt{(3+\sqrt{17})/2}$	1	1	1
$-155 = (-31) \cdot 5$	$\sqrt{(-7+\sqrt{-31})/2}$	$\sqrt{-7+4\sqrt{5}}$	3	1	3
$-184 = (-23) \cdot 8$	$\sqrt{(-3+\sqrt{-23})/2}$	$\sqrt{-3+4\sqrt{2}}$	3	1	3
$-203 = (-7) \cdot 29$	$\sqrt{-1+2\sqrt{-7}}$	$\sqrt{(-1+\sqrt{29})/2}$	1	1	1
$-219 = (-3) \cdot 73$	$\sqrt{(-17+\sqrt{-3})/2}$	$\sqrt{-17+2\sqrt{73}}$	1	1	1
$-259 = (-7) \cdot 37$	$\sqrt{3+2\sqrt{-7}}$	$\sqrt{(3+\sqrt{37})/2}$	1	1	1
$-291 = (-3) \cdot 97$	$\sqrt{-7+4\sqrt{-3}}$	$\sqrt{(-7+\sqrt{97})/2}$	1	1	1
$-292 = (-4) \cdot 73$	$\sqrt{3+8\sqrt{-1}}$	$\sqrt{(3+\sqrt{73})/2}$	1	1	1
$-323 = (-19) \cdot 17$	$\sqrt{(7+\sqrt{-19})/2}$	$\sqrt{7+2\sqrt{17}}$	1	1	1

Table 1 (cont.)

$d(F) = d(F_1)d(F_2)$	θ_1	θ_2	h_1	h_2	H
$-328 = (-8)\cdot 41$	$\sqrt{-3+4\sqrt{-2}}$	$\sqrt{(-3+\sqrt{41})/2}$	1	1	1
$-355 = (-71)\cdot 5$	$\sqrt{(-3+\sqrt{-71})/2}$	$\sqrt{-3+4\sqrt{5}}$	7	1	7
$-388 = (-4)\cdot 97$	$\sqrt{9+4\sqrt{-1}}$	$\sqrt{(9+\sqrt{97})/2}$	1	1	1
$-568 = (-71)\cdot 8$	$\sqrt{(-1+\sqrt{-71})/2}$	$\sqrt{-1+6\sqrt{2}}$	7	1	7
$-667 = (-23)\cdot 29$	$\sqrt{-13+2\sqrt{-23}}$	$\sqrt{(-13+3\sqrt{29})/2}$	3	1	3
$-723 = (-3)\cdot 241$	$\sqrt{-7+8\sqrt{-3}}$	$\sqrt{(-7+\sqrt{241})/2}$	1	1	1
$-763 = (-7)\cdot 109$	$\sqrt{-9+2\sqrt{-7}}$	$\sqrt{(-9+\sqrt{109})/2}$	1	1	1
$-772 = (-4)\cdot 193$	$\sqrt{7+12\sqrt{-1}}$	$\sqrt{(7+\sqrt{193})/2}$	1	1	1
$-955 = (-191)\cdot 5$	$\sqrt{-9+2\sqrt{-191}}$	$\sqrt{(-9+13\sqrt{5})/2}$	13	1	13
$-1003 = (-59)\cdot 17$	$\sqrt{(3+\sqrt{-59})/2}$	$\sqrt{3+2\sqrt{17}}$	3	1	3
$-1027 = (-79)\cdot 13$	$\sqrt{3+2\sqrt{-79}}$	$\sqrt{(3+5\sqrt{13})/2}$	5	1	5
$-1227 = (-3)\cdot 409$	$\sqrt{-19+4\sqrt{-3}}$	$\sqrt{(-19+\sqrt{409})/2}$	1	1	1
$-1243 = (-11)\cdot 113$	$\sqrt{(-21+\sqrt{-11})/2}$	$\sqrt{-21+2\sqrt{113}}$	1	1	1
$-1387 = (-19)\cdot 73$	$\sqrt{(11+3\sqrt{-19})/2}$	$\sqrt{11+2\sqrt{73}}$	1	1	1
$-1411 = (-83)\cdot 17$	$\sqrt{(23+\sqrt{-83})/2}$	$\sqrt{23+6\sqrt{17}}$	3	1	3
$-1507 = (-11)\cdot 137$	$\sqrt{-23+8\sqrt{-11}}$	$\sqrt{(-23+3\sqrt{137})/2}$	1	1	1
$-1555 = (-311)\cdot 5$	$\sqrt{(-3+\sqrt{-311})/2}$	$\sqrt{-3+8\sqrt{5}}$	19	1	19

4. D_4**-extensions of** $\mathbb{Q}$ **which are cyclic over imaginary quadratic number fields with class number two.** We first characterize imaginary D_4-extensions $K$of $\mathbb{Q}$ such that $h(K)$ is odd and $h(F) \equiv 2 \pmod 4$.

Proposition 4. *If* $h(K)$ *is odd and* $h(F) \equiv 2 \pmod 4$, *then there exists a prime number* $p \equiv 5 \pmod 8$ *such that* $F = \mathbb{Q}(\sqrt{-p})$ *and* $K = \mathbb{Q}(\sqrt{p}, \sqrt{\alpha})$, *where* α *is a generator of a prime divisor of* p *in* $\mathbb{Q}(\sqrt{-1})$. *More precisely*, *in this case* p *splits in* $\mathbb{Q}(\sqrt{-1})$ *as* $p = \pi\overline{\pi}$, *where* $\overline{\pi}$ *is the complex conjugate of* π, *and we can take* $\alpha = \pi$ *or* $\sqrt{-1}\pi$. *Conversely*, *for any prime number* $p \equiv 5 \pmod 8$, *if* α *is any generator of a prime divisor of* p *in* $\mathbb{Q}(\sqrt{-1})$, *then the field* $\mathbb{Q}(\sqrt{p}, \sqrt{\alpha})$ *is an imaginary* D_4*-extension of* $\mathbb{Q}$ *with odd class number which is not a CM-field*, *and this field is cyclic over* $\mathbb{Q}(\sqrt{-p})$.

Proof. Assume that $h(K)$ is odd and $h(F) \equiv 2 \pmod 4$. Since the 4-rank of $\mathrm{Cl}(F)$ is zero, by Rédei–Reichardt theory ([17]) we have the following four cases: (a) $d(F) = (-q) \cdot p, (-q/p) \neq 1$; (b) $d(F) = (-q) \cdot 8$, $q \equiv 3 \pmod 8$; (c) $d(F) = (-8) \cdot p$, $p \equiv 5 \pmod 8$; (d) $d(F) = (-4) \cdot p$, $p \equiv 5 \pmod 8$. Here, p and q are prime numbers with $p \equiv -q \equiv 1 \pmod 4$. Since $h(K)$ is odd, M must be the Hilbert 2-class field of F, and F must have a cyclic quartic extension containing it. Hence the equation $X^2 - d(F_1)Y^2 - d(F_2)Z^2 = 0$ must have a nontrivial solution in $\mathbb{Z}$ and only (d) is the case. Thus, $F = \mathbb{Q}(\sqrt{-p}), p \equiv 5 \pmod 8$. Then we may assume $F_1 = \mathbb{Q}(\sqrt{-1})$. Since 2 is ramified in F_1 as $(2) = (1 + \sqrt{-1})^2$, by Lemma 1(i), M_1 is of the form $F_1(\sqrt{\alpha})$, where α is a generator of a prime divisor of p in F_1. Hence $K = \mathbb{Q}(\sqrt{p}, \sqrt{\alpha})$.

Now let p be a prime number $\equiv 5 \pmod 8$ and α any generator of a prime divisor of p in $\mathbb{Q}(\sqrt{-1})$. Then by Rédei–Reichardt theory ([17]) $h(\mathbb{Q}(\sqrt{-p})) \equiv 2 \pmod 4$ and it is easily seen that the field $\mathbb{Q}(\sqrt{p}, \sqrt{\alpha})$ is an imaginary D_4-extension of $\mathbb{Q}$ which is not a CM-field and cyclic over $\mathbb{Q}(\sqrt{-p})$, and that its biquadratic bicyclic subfield is $\mathbb{Q}(\sqrt{-1}, \sqrt{p})$. Since $\mathbb{Q}(\sqrt{-1}, \sqrt{p})$ is the Hilbert 2-class field of $\mathbb{Q}(\sqrt{-p})$, its class number is odd by Lemma 5. We easily see also that there exists only one prime divisor of (2) in $\mathbb{Q}(\sqrt{-1}, \sqrt{p})$ and that only this prime is ramified in $\mathbb{Q}(\sqrt{p}, \sqrt{\alpha})$. Therefore this field has odd class number by Lemma 3(ii). ■

Now we determine K with $h(K) = 1$ and $h(F) = 2$. The imaginary quadratic number fields with class number two are $F = \mathbb{Q}(\sqrt{d})$, $-d = 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403$, and 427. Hence by the proposition above the possible F are $\mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-13})$, and $\mathbb{Q}(\sqrt{-37})$. For these fields, $p = 5, 13$, and 37, respectively, and we can take $\pi = 1 + 2\sqrt{-1}, 3 + 2\sqrt{-1}$, and $1 + 6\sqrt{-1}$, respectively. For all the possibilities, by computer calculations we obtain $h(M_1) = h(M_2) = 1$ and therefore $h(K) = h(M_1)h(M_2) = 1$ by Proposition 3. Thus, there exist exactly 6 D_4-extensions of $\mathbb{Q}$ with class number one which are cyclic over an imaginary quadratic number field with class number two.

5. Normal closures of pure quartic number fields. A D_4-extension of $\mathbb{Q}$ which is a cyclic quartic extension of the Gaussian field $\mathbb{Q}(\sqrt{-1})$ is the normal closure of a pure quartic number field (see below). The parity of the class number of the normal closure of a pure quartic number field has already been determined by C. J. Parry [15]. Our formulation is as follows:

PROPOSITION 5. *If* $F = \mathbb{Q}(\sqrt{-1})$, *then* K *is the normal closure* $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{m})$ *of a real pure quartic number field* $\mathbb{Q}(\sqrt[4]{m})$, *where* m *is a fourth power free positive integer*, *not a perfect square. If*, *moreover*, $h(K)$

is odd, then $m = 2$, p, *or* $4q$, *where* p *and* q *are prime numbers with* $p \equiv 3 \pmod 8$ *and* $q \equiv 3 \pmod 4$. *Conversely, if* $m = 2$, p, *or* $4q$ *with* p *and* q *as above, then the field* $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{m})$ *is an imaginary* D_4*-extension of* $\mathbb{Q}$ *with odd class number which is not a CM-field, and this field is cyclic over* $\mathbb{Q}(\sqrt{-1})$.

Proof. Assume that $F = \mathbb{Q}(\sqrt{-1})$. Then by Kummer theory, K is of the form $F(\sqrt[4]{m})$, where m is a fourth power free integer in F. The normality of K implies $\sqrt[4]{\overline{m}} \in K$ and therefore we can write

$$\sqrt[4]{\overline{m}} = \alpha + \beta\sqrt[4]{m} + \gamma\sqrt[4]{m^2} + \delta\sqrt[4]{m^3} \quad (\alpha, \beta, \gamma, \delta \in F).$$

Since we may assume $\sqrt[4]{m}\sqrt[4]{\overline{m}} = \sqrt[4]{N_{F/\mathbb{Q}}m} \in M$, this implies $\alpha = \gamma = 0$. Hence $\overline{m} = m(\beta + \delta\sqrt[4]{m^2})^4 \in F$. This implies $\beta = 0$, or $\delta = 0$. If $\beta = 0$, then $\sqrt[4]{\overline{m}} = \delta\sqrt[4]{m^3}$, which yields that K is abelian. Thus, we obtain $\overline{m} = \beta^4 m$. Since m is fourth power free, we have $\beta^4 = 1$ and therefore $m \in \mathbb{Q}$. Since $\mathbb{Q}((1+i)\sqrt[4]{m}) = \mathbb{Q}(\sqrt[4]{-4m})$ is contained in $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{m})$, we may take m as a positive integer.

For the other assertions, we explain here only that if $h(\mathbb{Q}(\sqrt{-1}, \sqrt[4]{m}))$ is odd, then $m = 2$, p, or $4p$, where p is a prime number with $p \equiv 3 \pmod 4$. For this we need not to consider units (cf. [15]).

Assume $K = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{m})$ and $h(K)$ is odd. We may assume that F_1 is imaginary and F_2 is real. Then by Lemma 1(ii) $d(F_2) = 8$, l, or $(-4)\cdot(-p)$, where p and l are prime numbers with $-p \equiv l \equiv 1 \pmod 4$. If $d(F_2) = l \equiv 1 \pmod 4$, then as shown in the proof of Proposition 3, $h(K)$ would be even. Hence $d(F_2) = 8$ or $d(F_2) = 4p$. Therefore by Lemma 1(i) except for the infinite prime only 2 is ramified in K if $d(F_2) = 8$, and only 2 and p are ramified in K if $d(F_2) = 4p$. The prime numbers ramified in $\mathbb{Q}(\sqrt[4]{m})$ are the prime factors of $2m$. In fact, the prime factors of $d(\mathbb{Q}(\sqrt[4]{m}))$ are those of $2m$ (see, for example, [9, Corollary 1]). Therefore since $F_2 = \mathbb{Q}(\sqrt{m})$, we have $m = 2$ if $d(F_2) = 8$, and $m = p$ or $4p$ if $d(F_2) = 4p$. ∎

By Proposition 5 we can easily obtain all $K = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{m})$ with class number one. We know that the class number of $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$ is one ([16]). Therefore we need to consider only fields of the forms $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{p}) = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{-4p})$, $p \equiv 3 \pmod 8$, and $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{4q}) = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{-q})$, $q \equiv 3 \pmod 4$. If such a K has class number one, its imaginary quadratic subfield $\mathbb{Q}(\sqrt{-p})$ or $\mathbb{Q}(\sqrt{-q})$ must have class number one, and therefore $p = 3, 11, 19, 43, 67, 163$ and $q = 3, 7, 11, 19, 43, 67, 163$. These values give the possibilities of fields with class number one. For these fields, we have $h(K) = h(\mathbb{Q}(\sqrt[4]{p}))h(\mathbb{Q}(\sqrt[4]{-4p}))$ and $h(K) = h(\mathbb{Q}(\sqrt[4]{4q}))h(\mathbb{Q}(\sqrt[4]{-q}))$ by Proposition 3. By computer calculations, we get the following tables. (Some of the values are already given in [16].)

Table 2

p	3	11	19	43	67	163
$h(\mathbb{Q}(\sqrt[4]{p}))$	1	1	1	3	3	1
$h(\mathbb{Q}(\sqrt[4]{-4p}))$	1	1	1	3	3	1
$h(\mathbb{Q}(\sqrt{-1}, \sqrt[4]{p}))$	1	1	1	9	9	1

Table 3

q	3	7	11	19	43	67	163
$h(\mathbb{Q}(\sqrt[4]{4q}))$	1	1	1	1	1	1	1
$h(\mathbb{Q}(\sqrt[4]{-q}))$	1	1	1	1	1	1	1
$h(\mathbb{Q}(\sqrt{-1}, \sqrt[4]{-q}))$	1	1	1	1	1	1	1

From these tables, we obtain 12 D_4-extensions of $\mathbb{Q}$ with class number one which are normal closures of pure quartic number fields. Note that $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2}) = \mathbb{Q}(\sqrt{-1}, \sqrt[4]{-2})$, whereas $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{p}) \neq \mathbb{Q}(\sqrt{-1}, \sqrt[4]{-p})$ for an odd prime p.

6. D_4-extensions of $\mathbb{Q}$ which are cyclic over imaginary quadratic number fields with class number one not equal to the Gaussian field. We treat here the case $h(F) = 1$ and $F \neq \mathbb{Q}(\sqrt{-1})$. That is, $F = \mathbb{Q}(\sqrt{d})$, $-d = 3, 7, 8, 11, 19, 43, 67$, or 163.

As in the previous sections, we first consider general D_4-extensions of $\mathbb{Q}$ with odd class number that are cyclic over imaginary quadratic number fields with odd class number. Assume that K is imaginary but not a CM-field. Then we may assume that F_1 is imaginary. Assume, moreover, that $h(K)$ and $h(F)$ are odd. If $M = \mathbb{Q}(\zeta_8)$, then by using Lemma 1(i) we have $K = \mathbb{Q}(\sqrt{-2}, \sqrt{1+\sqrt{-1}})$. This field was treated by J. Cougnard in detail [7] and $h(K) = 1$. Thus, in the following, we assume that $d(F)$ and $d(F_1)$ are relatively prime and $d(F_2) = d(F)d(F_1)$ (Lemma 1(ii)). Since K is cyclic over F, the equation $X^2 - d(F_1)Y^2 - d(F_2)Z^2 = 0$ has a nontrivial primitive solution in $\mathbb{Z}$. This implies that $d(F_1)$ is a quadratic residue modulo $|d(F_1)|$. Therefore, if we denote by p the unique (rational) prime divisor of $d(F)$ ($p = 2$ if $d(F) = -8$, and $p = -d(F)$ otherwise), p splits in F_1 ($(d(F_1)/p) = 1$). This implies $F_1 \neq \mathbb{Q}(\sqrt{-1})$, because $p \equiv 3 \pmod 4$ if $d(F) \neq -8$. We consider also the ramification in K/M. For this we consider the one in M_1/F_1, because we have $N_{M/\mathbb{Q}} d(K/M) = d(M_1)^2/(d(F_1)^2 d(M))$ by Lemma 4. Since $d(M_1) = d(F_1)^2 N_{F_1/\mathbb{Q}} d(M_1/F_1), d(M) = d(F)d(F_1)d(F_2)$ in general, and $d(F_2) = d(F)d(F_1)$ now, we have $N_{M/\mathbb{Q}} d(K/M) = (N_{F_1/\mathbb{Q}} d(M_1/F_1))^2/d(F)^2$.

In the present case, $N_{F_1/\mathbb{Q}} d(M_1/F_1) =_2 N_{F_1/\mathbb{Q}}(\alpha) = pp_1 =_2 d(F_1)d(F)$, where "$=_2$" means "equals up to a power of 2". Thus, we have $N_{M/\mathbb{Q}} d(K/M) =_2 d(F_1)^2$. This implies that only the prime divisors of 2 and p_1 can be ramified in K/M. We also note that there is only one prime divisor of M above p_1. In fact, $(d(F_1)/p) = 1$ implies $(d(F)/p_1) = -1$, that is, the rational prime p_1 remains prime in F, and obviously this is ramified in M/F. Therefore, in particular, if $d(F_1) = -8$, then the oddness of $h(K)$ follows by Lemma 3(ii).

From now on we assume $h(K) = 1$. Then $h(F_1) = 1$ by Proposition 3. That is, $F_1 = \mathbb{Q}(\sqrt{d_1}), -d_1 = 3, 7, 8, 11, 19, 43, 67$, or 163. Therefore M_1 is of the form $F_1(\sqrt{\alpha})$, where α is a squarefree integer of F_1 whose absolute norm is $d(F_2)$ up to the square of a rational integer. More precisely, $\alpha = \pi\sqrt{-p_1}$, where π is a generator of a prime divisor of p in F_1, and p_1 is the unique (rational) prime divisor of $d(F_1)$. ($p_1 = 2$ if $d(F_1) = -8$, and $p_1 = -d(F_1)$ otherwise. Note that p_1 is ramified in F_1 as $(p_1) = (\sqrt{-p_1})^2$.) Thus, if we let $p = \pi\pi'$ be the factorization of p in F_1, then M_1 is isomorphic to $\mathbb{Q}(\sqrt{\pi\sqrt{-p_1}})$ or $\mathbb{Q}(\sqrt{-\pi\sqrt{-p_1}})$. Since $F_1 \neq \mathbb{Q}(\sqrt{-1})$, these two fields are not conjugate. Note that this is valid also when $F_1 = \mathbb{Q}(\sqrt{-3})$. We also have $h(M) = 1$ by Proposition 1(i). The imaginary biquadratic bicyclic number fields with class number one have been determined by Uchida [21], and E. Brown and C. J. Parry [2], independently. Also the list of these fields restricts the possibilities for K.

Now we turn to determination of K with $h(K) = 1$. We first treat the case $F_1 = \mathbb{Q}(\sqrt{-2})$. Then by the conditions $h(M) = 1$ and $(-8/p) = 1$, we have $d(F) = -3, -11, -19, -43$, or -67. By the argument above, we have $h(K) = h(M_1)h(M_2)$ by Proposition 3. For $p = -d = 3, 11, 19, 43$, and 67, we can take $\pi = 1 + \sqrt{-2}, 3 + \sqrt{-2}, 1 + 3\sqrt{-2}, 5 + 3\sqrt{-2}$, and $7 + 3\sqrt{-2}$, respectively. By computer calculations we obtain $h(M_1) = h(M_2) = 1$ in both cases $M_1 = \mathbb{Q}(\sqrt{\pi\sqrt{-2}})$ and $M_1 = \mathbb{Q}(\sqrt{-\pi\sqrt{-2}})$ for all possibilities. Thus, we obtain 10 fields with class number one.

Next, we treat the cases $d(F_1) = -p_1$ with $p_1 \equiv 3 \pmod 4$. We first treat the case $F = \mathbb{Q}(\sqrt{-2})$ and then $F = \mathbb{Q}(\sqrt{-p})$ with $p \equiv 3 \pmod 4$.

Assume $F = \mathbb{Q}(\sqrt{-2})$. Then from the condition $(d(F_1)/2) = 1$, we have $d(F_1) = -7$. The class number of $\mathbb{Q}(\sqrt{-2}, \sqrt{-7})$ is one. We can take $\pi = (1 + \sqrt{-7})/2$. By computer calculations, $h(\mathbb{Q}(\sqrt{\pi\sqrt{-2}})) = 2$ but $h(\mathbb{Q}(\sqrt{-\pi\sqrt{-2}})) = 1$. Let $M_1 = \mathbb{Q}(\sqrt{-\pi\sqrt{-2}})$. Then $d(M_1) = 2744 = 2^3 \cdot 7^3$ and we have $N_{M/\mathbb{Q}} d(K/M) = 7^2$. Hence only the unique prime divisor of 7 is ramified in K/M. Therefore K has odd class number by Lemma 3(ii) and $h(K) = h(M_1)h(M_2)$ by Proposition 3. By computer calculations, $h(M_2) = 1$ and therefore the field $\mathbb{Q}(\sqrt{-2}, \sqrt{(7 - \sqrt{-7})/2})$ has class number one.

Assume $F=\mathbb{Q}(\sqrt{-p}), p=3,7,11,19,43,67$, or 163 (and $F_1\neq\mathbb{Q}(\sqrt{-2})$). Then one of $\mathbb{Q}(\sqrt{\pi\sqrt{-p_1}})$ and $\mathbb{Q}(\sqrt{-\pi\sqrt{-p_1}})$ has odd class number, and the other has even class number ([6, Exercise (23.10)]). We take π such that $\mathbb{Q}(\sqrt{-\pi\sqrt{-p_1}})$ has odd class number, and put $M_1=\mathbb{Q}(\sqrt{-\pi\sqrt{-p_1}})$. Then M_1/F_1 is ramified only at π and $\sqrt{-p_1}$ and therefore $d(M_1/F_1)=N_{F_1/\mathbb{Q}}(-\pi\sqrt{-p_1})=pp_1=d(F)d(F_1)$. From this we have $N_{M/\mathbb{Q}}d(K/M)=p_1^2$. Hence only the unique prime divisor of p_1 is ramified in K/M. Therefore K has odd class number by Lemma 3(ii) and we have $h(K)=h(M_1)h(M_2)$ by Proposition 3. For all possibilities except $p=163$ and $d(F_1)=-67$, by computer calculations we have $h(M_1)=h(M_2)=1$. (For $p=163$ and $d(F_1)=-67$, we have $h(M_1)=h(M_2)=3$.) Thus, we obtain 17 fields with class number one. We give a table of $d(F_1)$ and π such that $h(\mathbb{Q}(\sqrt{-\pi\sqrt{-p_1}}))=1$ for each p below.

Table 4

p	$d(F_1)$	π	$-\pi\sqrt{-p_1}$
3	-11	$(1+\sqrt{-11})/2$	$(11-\sqrt{-11})/2$
7	-3	$2+\sqrt{-3}$	$3-2\sqrt{-3}$
	-19	$(3+\sqrt{-19})/2$	$(19-3\sqrt{-19})/2$
11	-7	$2+\sqrt{-7}$	$7-2\sqrt{-7}$
	-19	$(5+\sqrt{-19})/2$	$(19-5\sqrt{-19})/2$
19	-3	$4-\sqrt{-3}$	$-3-4\sqrt{-3}$
	-67	$(3+\sqrt{-67})/2$	$(67-3\sqrt{-67})/2$
43	-3	$(13+\sqrt{-3})/2$	$(3-13\sqrt{-3})/2$
	-7	$6+\sqrt{-7}$	$7-6\sqrt{-7}$
	-163	$(3+\sqrt{-163})/2$	$(163-3\sqrt{-163})/2$
67	-3	$8-\sqrt{-3}$	$-3-8\sqrt{-3}$
	-11	$(13+3\sqrt{-11})/2$	$(-33-13\sqrt{-11})/2$
	-43	$(15+\sqrt{-43})/2$	$(43-15\sqrt{-43})/2$
163	-3	$4+7\sqrt{-3}$	$21-4\sqrt{-3}$
	-7	$10-3\sqrt{-7}$	$-21-10\sqrt{-11}$
	-11	$8+3\sqrt{-11}$	$33-8\sqrt{-11}$
	-19	$12-\sqrt{-19}$	$-19-12\sqrt{-19}$

7. Conclusion. Now we summarize our results.

THEOREM 2. *There exist exactly* 67 *imaginary normal octic number fields with class number one which are not CM-fields. All of them are dihedral extensions of* $\mathbb{Q}$. 20 *fields of them are Hilbert class fields of imag-*

inary quadratic number fields with class number four: $\mathbb{Q}(\sqrt{d(F_1)},\theta_2)$ *with* $H=1$ *in Table* 1. 6 *of them are the ramified cyclic quartic extensions of imaginary quadratic number fields with class number two*, *containing their Hilbert class fields*: $\mathbb{Q}(\sqrt{-p},\sqrt{\pi})$ *and* $\mathbb{Q}(\sqrt{-p},\sqrt{\sqrt{-1}\pi})$ *with* $(p,\pi)=(5,1+2\sqrt{-1}),(13,3+2\sqrt{-1}),(37,1+6\sqrt{-1})$. 12 *of them are the normal closures of pure quartic number fields*: $\mathbb{Q}(\sqrt{-1},\sqrt[4]{2})$, *and* $\mathbb{Q}(\sqrt{-1},\sqrt[4]{p})$, $p=3,11,19,163$, *and* $\mathbb{Q}(\sqrt{-1},\sqrt[4]{-q})$, $q=3,7,11,19,43,67,163$. *The other* 29 *are ramified cyclic quartic extensions of imaginary quadratic number fields with class number one not equal to* $\mathbb{Q}(\sqrt{-1})$: $\mathbb{Q}(\sqrt{-2},\sqrt{1+\sqrt{-1}})$, $\mathbb{Q}(\sqrt{-2},\sqrt{(7-\sqrt{-7})/2}),\mathbb{Q}(\sqrt{-p},\sqrt{\pm\pi\sqrt{-2}})$ *with* $(p,\pi)=(3,1+\sqrt{-2}),(11,3+\sqrt{-2}),(19,1+3\sqrt{-2}),(43,5+3\sqrt{-2}),(67,7+3\sqrt{-2})$, *and* $\mathbb{Q}(\sqrt{-p},\sqrt{-\pi\sqrt{-p_1}})$ *for* p,π,p_1 *given in Table* 4.

We have used KANT for class number calculations.

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