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Squares in products with terms in an arithmetic progression

by

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1. Introduction. Let $d \geq 1$, $k \geq 2$, $l \geq 2$, $n \geq 1$, $y \geq 1$ be integers with $\gcd(n,d) = 1$. Erdős [4] and Rigge [12] independently proved that a product of two or more consecutive positive integers is never a square. Further Erdős and Selfridge [5] showed that a product of k consecutive integers is never a power, i.e.,

$$n(n+1)\ldots(n+k-1) = y^l \quad \text{with integers } k \geq 2,\ l \geq 2,\ n \geq 1,\ y \geq 1$$

never holds. In [14, Corollary 1] the author extended this result by showing that

$$n(n+d)\ldots(n+(k-1)d) = y^l \quad \text{with integers } k \geq 3,\ l \geq 2,\ n \geq 1,\ y \geq 1$$

never holds for $1 < d \leq 6$. In this paper we extend the range of d for the preceding equation with $l = 2$.

Theorem 1. *The only solution of the equation*

$$(1)\quad n(n+d)\ldots(n+(k-1)d) = y^2 \quad \textit{in integers } k \geq 3,\ n \geq 1,\ y \geq 1 \textit{ and } 1 < d \leq 22$$

is $(n,d,k) = (18,7,3)$.

Theorem 1 is a consequence of the following more general result.

Theorem 2. *Let* $k \geq 3$, $n \geq 1$ *and* $1 < d \leq 22$. *Then there exists a prime exceeding* k *which divides* $n(n+d)\ldots(n+(k-1)d)$ *to an odd power except when* $(n,d,k) \in \{(2,7,3),(18,7,3),(64,17,3)\}$.

Equation (1) implies that every prime exceeding k divides the product $n(n+d)\ldots(n+(k-1)d)$ to an even power. This contradicts Theorem 2 except when $(n,d,k) \in \{(2,7,3),(18,7,3),(64,17,3)\}$. But in these three cases we find that $n(n+d)\ldots(n+(k-1)d)$ equals $2 \cdot 12^2$, 120^2, $2 \cdot 504^2$,

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respectively. Hence equation (1) holds only when $(n, d, k) = (18, 7, 3)$. Thus Theorem 1 follows from Theorem 2. Let $P(m)$ denote the greatest prime factor of m for any integer $m > 1$ and we write $P(1) = 1$. Then it follows from Theorem 2 that the equation

$$(2)\quad n(n+d)\dots(n+(k-1)d) = By^2 \quad \text{in positive integers } k \geq 3, n, y, B \text{ with } P(B) \leq k$$

never holds for $1 < d \leq 22$ except when $(n, d, k) \in \{(2, 7, 3), (18, 7, 3), (64, 17, 3)\}$.

Marszałek [7] proved that equation (2) with $d > 1$ and $B = 1$ implies that

$$(3)\qquad k < 2\exp(d(d+1)^{1/2}).$$

Shorey and Tijdeman [16] proved that equation (2) with $d > 1$ implies that

$$(4)\qquad k < d^{C/\log\log d}$$

where C is an effectively computable absolute constant. We prove

THEOREM 3. *Equation* (2) *with* $d \geq 23$ *implies that*

$$(5)\qquad k < \begin{cases} 4d(\log d)^2 & \text{if } d \text{ is odd,} \\ 1.3d(\log d)^2 & \text{if } d \text{ is even.} \end{cases}$$

In Theorem 3 we need to consider only $d \geq 23$ in view of Theorem 2. The estimate (5) is a considerable improvement of (3). The estimate (4) involves an unspecified constant which turns out to be large. Therefore the estimate (5) is better than (4) for small values of d.

Now we exhibit infinitely many solutions in relatively prime integers $n \geq 1$ and $d > 1$ of equation (2) with $k = 3$ and square-free integer B satisfying $P(B) \leq 3$. We observe that $B \in \{1, 2, 3, 6\}$. For $B = 1$, the existence of infinitely many solutions follows from a well known result that there are infinitely many triples of relatively prime squares in arithmetic progression. For $B > 1$, we prove

THEOREM 4. *Let* $B \in \{2, 3, 6\}$. *There are infinitely many triples* (n, d, y) *with integers* $n \geq 1$, $d > 1$, $y \geq 1$ *and* $\gcd(n, d) = 1$ *satisfying*

$$(6)\qquad n(n+d)(n+2d) = By^2.$$

Let $d = 1, k \geq 3$ and $n(n+1)\dots(n+k-1)$ be divisible by a prime greater than k. Then Erdős and Selfridge [5] proved that there exists a prime $p \geq k$ dividing $n(n+1)\dots(n+k-1)$ to an odd power. The author [14] showed that the above assertion is valid with $p > k$ whenever $k \geq 4$. If $d = 1$ and $k = 3$, we prove

THEOREM 5. *There is a prime exceeding* 3 *which divides* $n(n+1)(n+2)$ *to an odd power except when* $n \in \{1, 2, 48\}$.

When $n = 1, 2, 48$, we see that $n(n+1)(n+2)$ equals $6, 6\cdot 2^2, 6\cdot 140^2$ and the assertion of Theorem 5 is false. For the proof of Theorem 5, it suffices to show that the equation

$$n(n+1)(n+2) = By^2 \quad \text{with } B \in \{1, 2, 3, 6\} \tag{7}$$

has no solution other than $B = 6, (n, y) \in \{(1, 1), (2, 2), (48, 140)\}$. If $B = 1$, the above assertion is a particular case of the result of Erdős and Rigge mentioned at the beginning of this section. If $B = 6$ and n odd, then the assertion was proved by Meyl [8] whereas Watson [17] and Ljunggren [6] proved the case of n even.

The Algorithm in Section 3 was programmed and checkings and computations for the proof of Theorem 2 were carried out using Mathematica. I thank Professor T. N. Shorey for many helpful discussions. I also thank the referee for his valuable comments on an earlier draft of the paper.

2. Lemmas. We suppose throughout this section that $n \geq 1, d > 1$ and $k \geq 3$ with $(n, d, k) \neq (2, 7, 3)$. Then by a result of Shorey and Tijdeman [15], we have

$$P(n(n+d)\dots(n+(k-1)d)) > k. \tag{8}$$

Further we suppose that

$$\operatorname{ord}_p(n(n+d)\dots(n+(k-1)d)) \equiv 0 \pmod 2 \quad \text{for all primes } p > k. \tag{9}$$

We write

$$n + id = a_i x_i^2, \quad a_i \text{ square-free}, \ P(a_i) \leq k, x_i > 0 \text{ for } 0 \leq i \leq k-1. \tag{10}$$

We observe that $\gcd(a_i, d) = 1$ for $0 \leq i \leq k-1$ since $\gcd(n, d) = 1$. We denote by $\{a'_1, \dots, a'_{t'}\}$ the set of all the distinct elements from $\{a_0, \dots, a_{k-1}\}$. By (8), we have

$$n + (k-1)d \geq (k+1)^2. \tag{11}$$

Let $m \geq 1$ be an integer and $2 \leq p_1^{(d)} < p_2^{(d)} < \dots$ be all the primes which are coprime to d. We define $B_{m,d} = \{a'_r \mid P(a'_r) \leq p_m^{(d)}\}$ and $g(k, m, d) = |B_{m,d}|$. We observe that

$$g(k, m, d) \geq t' - \sum_{i \geq m+1} \left(\left[\frac{k}{p_i^{(d)}} \right] + \varepsilon_i^{(d)} \right) := g_0(k, m, d) \tag{12}$$

where $\varepsilon_i^{(d)} = 0$ if $p_i^{(d)} > k$ and for $p_i^{(d)} \leq k$, $\varepsilon_i^{(d)} = 0$ or 1 according as $p_i^{(d)} \mid k$ or not for $i \geq m+1$. We note that $g(k, m, d)$ and $g_0(k, m, d)$ are the same as $g(k, m)$ and $g_0(k, m)$ of [14].

Throughout this section we assume without reference that h is a positive integer with h even whenever d is even. Further, let $\varrho > 0$. Define $V_h = \{\alpha \mid \alpha \text{ is a positive integer with } \alpha h^2 < \varrho \text{ and } \gcd(\alpha, d) = 1\}$. We write

$V_h = \bigcup_{i\ge 1} V_{hi}$ such that for every $i \ge 1$, positive integers α and β are in V_{hi} if and only if $\alpha \equiv \beta \pmod{\frac{d}{\varepsilon_h \gcd(d,h)}}$ where

$$\varepsilon_h = 1 \text{ if } 2 \nmid \frac{d}{\gcd(d,h)} \quad \text{and} \quad \varepsilon_h = 2 \text{ if } 2 \mid \frac{d}{\gcd(d,h)}. \tag{13}$$

Further, let $\delta_h = \max\{|V_{hi}|\}$ and $\delta(d) = \sum^{*}_{h<\sqrt{\varrho}} \delta_h$. Here we recall that the summation in $\sum^*$ is taken over even values of h whenever d is even. We note that $\delta(d)$ can be computed for every d and ϱ and that the values of $\delta(d)$ for $7 \le d \le 22$ and $\varrho = \frac{1}{3}d^2$ can be found in Table 1. We begin with the following lemma which gives a lower bound for the number of distinct a_i, viz., t'.

LEMMA 1. *Let $n \ge (k-1)^2 d^2/(4\varrho)$. If (9) holds, then $t' \ge k - \delta(d)$.*

Proof. Let $b_1, \ldots, b_r, \ldots$ be the a_j's which occur more than once with $n + i_r d = b_r x_{i_r}^2$ for $r \ge 1$ and such that x_{i_r} is minimal, i.e., if $a_i = b_r$ with $i \ne i_r$, then $x_i > x_{i_r}$. For any b_r with $r \ge 1$, we say that b_r is *repeated at the hth place* if there exists some j, $0 \le j \le k-1$, such that $a_j = b_r$, $j \ne i_r$, $x_j = x_{i_r} + h$ with $h \ge 1$. We observe that j is uniquely determined. We set $W_h = \{a_j \mid 0 \le j \le k-1,\ a_j = b_r,\ j \ne i_r \text{ and } x_j = x_{i_r} + h \text{ for some } r\}$. In order to get a lower bound for the number of distinct a_j's, we need to get an upper bound for $\sum_{h\ge 1} |W_h|$. We observe that $|W_h|$ is equal to the number of b_r which are repeated at the hth place. We proceed to find an upper bound for this number.

Suppose b_r is repeated at the hth place. Then by its definition, we obtain for some $j, 0 \le j \le k-1, j \ne i_r$,

$$\begin{aligned}(k-1)d &\ge (j - i_r)d = b_r(x_j^2 - x_{i_r}^2) = b_r(2hx_{i_r} + h^2) \\ &> 2hb_r^{1/2}(b_r x_{i_r}^2)^{1/2} \ge 2hb_r^{1/2} n^{1/2} > hb_r^{1/2}\frac{(k-1)d}{\sqrt{\varrho}}.\end{aligned} \tag{14}$$

Thus

$$b_r h^2 < \varrho. \tag{15}$$

Hence $h < \sqrt{\varrho}$, i.e., the number of places at which b_r can be repeated is at most $[\sqrt{\varrho}]$. Further, we note from (14), (10) and (15) that h is even whenever d is even and that $b_r \in V_h$. Also we observe that $h(2x_{i_r} + h) \equiv 0 \pmod{d}$ from which it follows that $x_{i_r} \equiv c \pmod{\frac{d}{\varepsilon_h \gcd(d,h)}}$ where c depends only on h and d with ε_h as in (13). Thus $n \equiv b_r c^2 \pmod{\frac{d}{\varepsilon_h \gcd(d,h)}}$. Further, we observe that $\gcd\big(c, \frac{d}{\varepsilon_h \gcd(d,h)}\big) = 1$ since $\gcd(n,d) = 1$.

If $b_s \ne b_r$ is such that b_s is repeated at the hth place, then by the foregoing argument, we have $b_s h^2 < \varrho$ and $n \equiv b_s c^2 \pmod{\frac{d}{\varepsilon_h \gcd(d,h)}}$. Thus $b_r \equiv b_s \pmod{\frac{d}{\varepsilon_h \gcd(d,h)}}$ since $\gcd\big(c, \frac{d}{\varepsilon_h \gcd(d,h)}\big) = 1$. Hence b_r, b_s belong to

V_{hi} for some i. Thus the number of b_r which are repeated at the hth place is $\le \delta_h$. Since $h < \sqrt{\varrho}$, we have

$$\sum_{h\ge 1} |W_h| \le \sum_{h<\sqrt{\varrho}} \delta_h = \delta(d).$$

Hence the number of distinct a_j's is at least $k - \delta(d)$. ∎

As a consequence of Lemma 1, we have

COROLLARY 1. *Let* $k \ge 2(2d-7)$. *If* (9) *holds*, *then* $t' \ge k - \delta(d)$ *where* $\delta(d)$ *is computed with* $\varrho = \frac{1}{3}d^2$.

Proof. By (11) and $k \ge 2(2d-7)$, we see that

$$n \ge (k+1)^2 - (k-1)d > (k+1)^2 - \frac{(k-1)(k+14)}{4} > \frac{3}{4}(k-1)^2.$$

Now the result follows immediately from Lemma 1. ∎

Let $1 = s_1 < s_2 < \ldots$ be the sequence of all square-free integers and $1 = s'_1 < s'_2 < \ldots$ be the sequence of all odd square-free integers.

LEMMA 2. *We have*

(i) $s_i \ge (1.5)i$ *for* $i \ge 39$.
(ii) $s_i \ge (1.6)i$ *for* $286 \le i \le 570$.
(iii) $s'_i \ge (2.25)i$ *for* $i \ge 12$.

Proof. (i) We first check that $s_i \ge (1.5)i$ for $39 \le i \le 70$. Further, we check that for $0 \le r < 36$, $r \notin S_0 = \{0, 4, 8, 9, 12, 16, 18, 20, 24, 27, 28, 32\}$ we can choose an s_{i_r} with $39 \le i_r \le 70$ such that $s_{i_r} \equiv r \pmod{36}$. Now we consider any s_i with $i > 70$. Then $s_i \equiv r \pmod{36}$ for some r with $0 \le r < 36$, $r \notin S_0$. Thus $s_i \equiv s_{i_r} \pmod{36}$ with $39 \le i_r \le 70$. Hence

$$s_i - s_{i_r} = 36f \tag{16}$$

for some positive integer f. We know that in any set of 36 consecutive integers, the number of square-free integers is ≤ 24. Thus the number of square-free integers $\le 36f$ is at most $24f$. Also we observe from (16) that this number is equal to $i - i_r$. Therefore $i - i_r \le 24f \le \frac{2}{3}(s_i - s_{i_r})$ by (16). Hence $s_i \ge \frac{3}{2}(i - i_r) + s_{i_r} \ge (1.5)i$ since $s_{i_r} \ge (1.5)i_r$.

(ii) The inequality follows by direct checking.

(iii) We check that $s'_i \ge (2.25)i$ for $12 \le i \le 35$. Also we check for $0 \le r < 36$ with $r \equiv 1 \pmod 2$ and $r \notin S_0$ that we can choose an s'_{i_r} with $12 \le i_r \le 35$ such that $s'_{i_r} \equiv r \pmod{36}$. Further, we observe that the number of odd square-free integers in any set of 36 consecutive integers is ≤ 16. Now we repeat the argument in (i) for any s'_i with $i > 35$ to obtain (iii). ∎

It can be checked that

$$\prod_{i=1}^{63} s_i \ge (1.5)^{63}(63)!, \quad \prod_{i=1}^{286} s_i \ge (1.6)^{286}(286)!, \quad \prod_{i=1}^{51} s'_i \ge (2.25)^{51}(51)!.$$

By Lemma 2 and from an induction argument we derive

COROLLARY 2. *We have*

(i) $\prod_{i=1}^{\nu} s_i \ge (1.5)^{\nu}\nu!$ *for* $\nu \ge 63$.
(ii) $\prod_{i=1}^{\nu} s_i \ge (1.6)^{\nu}\nu!$ *for* $286 \le \nu \le 570$.
(iii) $\prod_{i=1}^{\nu} s'_i \ge (2.25)^{\nu}\nu!$ *for* $\nu \ge 51$.

The inequality in (i) of the above corollary has already appeared in [5].

LEMMA 3. *Let* $7 \le d \le 22$. *Suppose* (9) *holds. Then* $k \le k_0(d) := k_0$ *where* k_0 *is as given in Table* 1.

Proof. Suppose $k > k_0$. Then $k > 2(2d-7)$. Hence Corollary 1 is valid. Thus $t' \ge k - \delta(d)$ where $\delta(d)$ is computed with $\varrho = \frac{1}{3}d^2$. We note from Table 1 that $\delta(d) \le 20$. Thus $t' \ge k-20$. From now onwards we shall assume that $k \ge 83$. Since a'_i for $1 \le i \le t'$ are square-free integers, we use Corollary 2(i) to obtain

$$\prod_{i=1}^{t'} a'_i \ge \prod_{i=1}^{k-20} a'_i \ge \prod_{i=1}^{k-20} s_i \ge (1.5)^{k-20}(k-20)!. \tag{17}$$

On the other hand, by (10), we have

$$a'_1 \dots a'_{t'} \mid (k-1)! \prod_{p \le k} p. \tag{18}$$

We put $g_q = \mathrm{ord}_q(a'_1 \dots a'_{t'})$ and $h_q = \mathrm{ord}_q((k-1)! \prod_{p \le k} p)$ for any prime $q \le k$. Then it follows from Marszałek [7, p. 221] that

$$g_q \le \frac{k}{q+1} + \frac{\log k}{\log q} + 1 \quad \text{and} \quad h_q \ge \frac{k-1}{q-1} - \frac{\log k}{\log q}.$$

Thus

$$g_q - h_q \le \frac{-2k}{q^2-1} + \frac{q}{q-1} + \frac{2\log k}{\log q}. \tag{19}$$

Further, from (18) we get

$$a'_1 \dots a'_{t'} \mid (k-1)! \Big(\prod_{p \le k} p\Big)\Big(\prod_{q \le 19} q^{g_q - h_q}\Big) \tag{20}$$

where in the product signs p, q run over primes. Now by (20) and (19), we have

$$a'_1 \dots a'_{t'} \le (k-1)! \Big(\prod_{p \le k} p\Big) k^{16} \Big(\prod_{q \le 19} q^{q/(q-1)}\Big)\Big(\prod_{q \le 19} q^{2/(q^2-1)}\Big)^{-k}. \tag{21}$$

We find that

$$(22)\qquad \begin{cases} \prod_{q\le 19} q^{q/(q-1)} \le 153819970, \quad \prod_{q\le 19} q^{2/(q^2-1)} \ge 2.8819, \\ \prod_{p\le k} p \le (2.78)^k \quad \text{(see [13, p. 71])}. \end{cases}$$

Using (22) in (21) and comparing with (17), we get

$$(1.5549)^k \le (153819970)(1.5)^{20}k^{35}.$$

This inequality is not valid for $k \ge 570$. Thus we obtain $k < 570$. Now let $k \ge 485$. We use Corollary 2(ii) to get

$$\prod_{i=1}^{t'} a_i' \ge (1.6)^{k-20}(k-20)!.$$

Comparing this lower bound with the upper bound in (21), we get

$$(1.6586)^k \le (153819970)(1.6)^{20}k^{35}.$$

This inequality is not valid for $k \ge 485$. Thus we conclude that $k < 485$.

We shall bring down the value of k to k_0 in all cases except $d = 19$ by a counting argument which will be presented in the next paragraph. When $d = 19$ the counting argument fails. But a refinement of the above argument itself enables us to bring $k < 315$. When $d = 19$ we observe that $g_{19} = 0$ and we rewrite (20) as

$$a_1' \dots a_{t'}' \le (k-1)!\Big(\prod_{p\le k} p\Big)\Big(\prod_{q\le 17} q^{g_q-h_q}\Big)(19)^{-h_q}.$$

On the other hand, by Corollary 2(ii), we have for $485 > k \ge 303$, $a_1' \dots a_{t'}' \ge (1.6)^{k-17}(k-17)!$ since $\delta(d) = 17$. Now we combine the preceding estimates for $a_1' \dots a_{t'}'$ to conclude that $k < 315$ whenever $d = 19$.

Let $7 \le d \le 22$, $d \ne 19$ and $k < 485$. Since a_i' for $1 \le i \le t'$ are distinct and square-free we have

$$(23)\qquad g(k,m,d) \le 1 + \binom{m}{1} + \dots + \binom{m}{m} = 2^m.$$

Thus if $g_0(k,m,d) \ge 2^m + 1$, we get a contradiction by (12). Since $t' \ge k - \delta(d)$, we replace t' in (12) by $k - \delta(d)$ and using Table 1 for the values of $\delta(d)$ we check that $g_0(k,m,d) \ge 2^m + 1$ for a proper choice of m whenever $k_0 < k < 485$. For example, when $d = 13$ we observe from $p_6^{(d)} = 17$ and the definition of $g_0(k,m,d)$ that $g_0(k,5,13) \ge 33$ for $120 \le k < 485$. The other cases are checked similarly. See Table 1 for the choices of m when different values of d and k are considered. This completes the proof. ∎

Table 1

d	$\delta(d)$	Range for k	m	k_0	d	$\delta(d)$	Range for k	m	k_0
7	6	27–310	3	26	14	5	43–349	2	42
		311–484	4				350–484	3	
8	4	19–20	2	18	15	14	47–484	3	46
		21–106	3		16	11	51–238	3	50
		107–318	4				239–484	4	
		319–484	5		17	15	255–484	6	254
9	7	40–285	3	39	18	6	59–102	1	58
		286–484	4				103–348	2	
10	2	27–136	1	28			349–484	3	
		137–383	2		19	17	–	–	314
		384–484	3		20	8	67–318	2	66
11	9	55–484	4	54			319–484	3	
12	4	35–372	2	34	21	20	100–484	3	99
		373–484	3		22	7	75–310	3	74
13	12	120–484	5	119			311–484	4	

We see from Table 1 that $\delta(d) \le 20$ for $7 \le d \le 22$. In the following lemma we give an upper bound for $\delta(d)$ whenever $d \ge 23$.

LEMMA 4. *For $d \ge 23$ and $\varrho = \frac{1}{3}d^2$, we have*

$$\delta(d) \le \begin{cases} \frac{1}{4} d \log d + (.8323)d & \text{if } d \text{ is odd,} \\ \frac{1}{3} d \log d + (.118)d & \text{if } d \text{ is even.} \end{cases}$$

Proof. By the definition of $\delta(d)$, we obtain

$$\delta(d) \le \sum_{h<d/\sqrt{3}} \left(\left[\frac{d^2}{3h^2} \cdot \frac{\varepsilon_h \gcd(d,h)}{d} \right] + 1 \right) \tag{24}$$

where the sum is taken over even values of h whenever d is even. We observe from (13) that $\varepsilon_h \le 2$. Thus from (24) we get

$$\delta(d) \le \frac{d}{3} \sum_{h<d/(2\sqrt{3})} \frac{1}{h} + \frac{d}{2\sqrt{3}} \quad \text{if } d \text{ is even.} \tag{25}$$

Let d be odd. Then by (13), $\varepsilon_h = 1$. Further, $\gcd(d,h) \le h/2$ whenever h is even. Hence from (24) we get

$$\begin{aligned} \delta(d) &\le \frac{d}{3} \sum_{h<d/\sqrt{3},\, h \text{ odd}} \frac{1}{h} + \frac{d}{3} \sum_{h<d/\sqrt{3},\, h \text{ even}} \frac{1}{2h} + \frac{d}{\sqrt{3}} \\ &\le \frac{d}{3} \sum_{h<d/\sqrt{3}} \frac{1}{h} - \frac{d}{6} \sum_{h<d/\sqrt{3},\, h \text{ even}} \frac{1}{h} + \frac{d}{\sqrt{3}} \end{aligned}$$

$$\le \frac{d}{3}\sum_{h<d/\sqrt{3}}\frac{1}{h}-\frac{d}{12}\sum_{h<d/(2\sqrt{3})}\frac{1}{h}+\frac{d}{\sqrt{3}}$$
$$\le \frac{d}{4}\sum_{h<d/(2\sqrt{3})}\frac{1}{h}+\frac{d}{3}\sum_{d/(2\sqrt{3})<h<d/\sqrt{3}}\frac{1}{h}+\frac{d}{\sqrt{3}}.$$

Thus

$$\delta(d)\le \frac{d}{4}\sum_{h<d/(2\sqrt{3})}\frac{1}{h}+\frac{d}{3}+\frac{d}{\sqrt{3}}+\frac{2}{\sqrt{3}}\quad \text{if } d \text{ is odd.} \tag{26}$$

We use $\sum_{h<x} 1/h < \log x + \gamma + 1/x$ where $x > 1$ and γ is Euler's constant whose value is $< .5773$ (see [1, p. 55] and [13, pp. 65–66]) and $d \ge 23$ in the estimates (25) and (26) to prove the assertion of the lemma. ∎

LEMMA 5. *Let* $k = 3$. *Suppose* (9) *holds. Then* $(a_0, a_1, a_2) \in S$ *where* $S = S_1 \cup S_2 \cup S_3 \cup S_4$ *with* $S_1 = \{(1,1,1)\}$, $S_2 = \{(2,1,2)\}$, $S_3 = \{(1,2,3), (2,3,1), (3,1,2), (6,1,2), (1,1,2)\}$ *and* $S_4 = \{(1,3,2), (2,1,3), (3,2,1), (2,1,6), (2,1,1)\}$. *Further, we have*

$$d \equiv \begin{cases} 0 \pmod 8 & \text{if } (a_0,a_1,a_2)\in S_1;\\ \pm 1 \pmod 8 & \text{if } (a_0,a_1,a_2)\in S_2;\\ 1 \pmod 8 & \text{if } (a_0,a_1,a_2)\in S_3;\\ -1 \pmod 8 & \text{if } (a_0,a_1,a_2)\in S_4.\end{cases}$$

Proof. From (9), we see that (10) holds and therefore $\{a_0, a_1, a_2\} \subset \{1, 2, 3, 6\}$. Also $\gcd(n, n+d) = \gcd(n+d, n+2d) = 1$ and $\gcd(n, n+2d) = 1$ or 2 since $\gcd(n, d) = 1$. Thus we find that there are 20 possible values for the triple (a_0, a_1, a_2), viz., $(1,1,1)$, $(1,1,2)$, $(1,1,3)$, $(1,1,6)$, $(1,2,1)$, $(1,2,3)$, $(1,3,1)$, $(1,3,2)$, $(1,6,1)$, $(2,1,1)$, $(2,1,2)$, $(2,1,3)$, $(2,1,6)$, $(2,3,1)$, $(2,3,2)$, $(3,1,1)$, $(3,1,2)$, $(3,2,1)$, $(6,1,1)$, $(6,1,2)$.

We use without mentioning that x_0, x_1, x_2 are pairwise coprime and $a_0x_0^2 + a_2x_2^2 = 2a_1x_1^2$. We exclude the possibilities $(1,1,3)$ and $(1,1,6)$ since $2x_1^2 - x_0^2 \not\equiv 0 \pmod 3$; $(1,3,1), (1,6,1), (2,3,2)$ since $x_0^2 + x_2^2 \not\equiv 0 \pmod 3$; $(3,1,1)$ and $(6,1,1)$ since $2x_1^2 - x_2^2 \not\equiv 0 \pmod 3$; $(1,2,1)$ since $x_0^2 + x_2^2 \not\equiv 0 \pmod 4$. The remaining 12 possibilities are given by S.

We take $(a_0, a_1, a_2) = (1,1,1) \in S_1$. Then $n = x_0^2$, $n+d = x_1^2$, $n+2d = x_2^2$, implying x_0 and x_2 are odd since n and $n + 2d$ are both odd or both even and $\gcd(n, n + 2d) = 1$ or 2. Thus $2d = x_2^2 - x_0^2 \equiv 0 \pmod 8$, yielding $d \equiv 0, 4 \pmod 8$. If $d \equiv 4 \pmod 8$, then $x_1^2 = n + d \equiv 5 \pmod 8$, which is not possible. Thus $d \equiv 0 \pmod 8$.

Next we consider $(a_0, a_1, a_2) = (2,1,2) \in S_2$. Then x_1 is odd and $n \equiv 0$ or $2 \pmod 8$ according as x_0 is even or odd. Hence $1 \equiv x_1^2 = n + d \equiv d$ or $d + 2 \pmod 8$, which implies that $d \equiv 1$ or $7 \pmod 8$.

Now we take $(a_0, a_1, a_2) = (1, 2, 3) \in S_3$. Then x_0, x_2 are odd and $d = 2x_1^2 - x_0^2 \equiv -1$ or $1 \pmod 8$ according as x_1 is even or odd. If $d \equiv -1 \pmod 8$, then $d = 3x_2^2 - 2x_1^2 \equiv 3 - 0 \equiv 3 \pmod 8$, a contradiction. Thus $d \equiv 1 \pmod 8$. Similarly we prove for other possibilities in S_3 that $d \equiv 1 \pmod 8$.

Lastly, we consider $(a_0, a_1, a_2) = (1, 3, 2) \in S_4$. Then x_0 is even and hence x_1, x_2 are odd. Thus $d = 2x_2^2 - 3x_1^2 \equiv 7 \pmod 8$. Likewise we prove for other possibilities in S_4 that $d \equiv 7 \pmod 8$. ∎

LEMMA 6. *Let $k = 3$. Suppose* (9) *holds. Then one of the following possibilities holds*: (i) $d = 1$, (ii) $d \geq 23$, (iii) $(n, d) \in \{(2, 7), (18, 7), (64, 17)\}$.

Proof. Let $1 < d < 23$. We shall show that (iii) holds. By Lemma 5, we need to consider $(a_0, a_1, a_2) \in S_1$ with $d = 8, 16$; $(a_0, a_1, a_2) \in S_2 \cup S_3$ with $d = 9, 17$ and $(a_0, a_1, a_2) \in S_2 \cup S_4$ with $d = 7, 15$.

Let $(a_0, a_1, a_2) \in S_1$ with $d = 8, 16$. Since $x_1^2 - x_0^2 = d$ we find that $x_0 = 1, x_1 = 3$ and $x_0 = 3, x_1 = 5$ and hence $x_2^2 = 17$ and 41, respectively. This is not possible.

Let $(a_0, a_1, a_2) \in S_2$ with $d = 7, 9, 15, 17$. Then $x_2^2 - x_0^2 = d$ implies that $(n, d) = (18, 7)$.

Let $(a_0, a_1, a_2) \in S_3$ with $d = 9, 17$. Let $d = 9$. In the first 4 possibilities in S_3 we observe that 3 divides one of $n, n+d, n+2d$. Hence $3 \mid n$, which is a contradiction since $\gcd(n, d) = 1$. Let $(a_0, a_1, a_2) = (1, 1, 2)$. Then $x_1^2 - x_0^2 = 9$ gives $x_0 = 4$ and hence $n + 2d = 2x_2^2 = 34$, which is impossible. Let $d = 17$ and $(a_0, a_1, a_2) = (1, 2, 3)$. Then $n + 34 \equiv 0, 3 \pmod 9$, implying $x_0^2 = n \equiv 2, 5 \pmod 9$, a contradiction. The next three possibilities are excluded similarly. Let $(a_0, a_1, a_2) = (1, 1, 2)$. Then $x_1^2 - x_0^2 = 17$ implies that $(x_0, x_1, x_2) = (8, 9, 7)$. Thus $(n, d) = (64, 17)$.

Let $(a_0, a_1, a_2) \in S_4$ with $d = 7, 15$. Let $d = 7$ and $(a_0, a_1, a_2) = (1, 3, 2)$. Then $n + 7 \equiv 0, 3 \pmod 9$. Hence $x_0^2 = n \equiv 2, 5 \pmod 9$, a contradiction. The next three possibilities are excluded similarly. Let $(a_0, a_1, a_2) = (2, 1, 1)$. Then $x_2^2 - x_1^2 = 7$ gives $(x_0, x_1, x_2) = (1, 3, 4)$. Thus $(n, d) = (2, 7)$. Let $d = 15$. The first four possibilities in S_4 are excluded since 3 divides one of $n, n + d, n + 2d$. Let $(a_0, a_1, a_2) = (2, 1, 1)$. Then $x_2^2 - x_1^2 = 15$ implies that $x_1 = 7$ or 1, giving $2x_0^2 = n = 34$ or -14, which are impossible. ∎

LEMMA 7. *Let $7 \leq d \leq 22$ and $(n, d) \notin \{(2, 7), (18, 7), (64, 17)\}$. Assume that $t' = k$. Then* (9) *does not hold.*

Proof. Suppose (9) holds. Then by Lemmas 3, 6 and Table 1, we have $4 \leq k \leq k_0 \leq 314$. We observe that (10) holds and $a_0, \ldots, a_{k-1}$ are all distinct since $t' = k$. We often use these facts and the property that

$\gcd(a_i, d) = 1$ for $0 \le i < k$ without any reference. We check that

$$(27)\qquad \begin{cases} g_0(k,1,d) \ge 3 & \text{for } 4 \le k \le 8 \text{ if 2 or 3 divides } d;\\ g_0(k,2,7) \ge 5 & \text{for } k = 7, 8;\\ g_0(k,2,d) \ge 5 & \text{for } 9 \le k \le 22;\\ g_0(k,3,d) \ge 9 & \text{for } 23 \le k \le 78;\\ g_0(k,4,d) \ge 17 & \text{for } 79 \le k \le 276;\\ g_0(k,5,d) \ge 33 & \text{for } 277 \le k \le 314. \end{cases}$$

But (27) contradicts (23), by (12). Thus we may assume that $4 \le k \le 6$ if $d = 7$ and $4 \le k \le 8$ if $d \in \{11, 13, 17, 19\}$.

Let $k = 4$ and $d \in \{7, 11, 13, 17, 19\}$. We know that $P(a_i) \le 3$ and hence $a_i \in \{1, 2, 3, 6\}$. Thus $n(n + d)(n + 2d)(n + 3d)$ is a square. But this is impossible by a well known result of Euler (see Dickson [3, p. 635] and Mordell [9, p. 21, Corollary]). We also use this fact without reference when we deal with other values of k.

Let $k = 5$. Since a_i's are distinct, we need only consider the case when 5 divides one and only one of $n, n + d, n + 2d, n + 3d, n + 4d$ and hence at most one a_i. The values of the other a_i's belong to $\{1, 2, 3, 6\}$. We may assume that 5 divides one of $n + d, n + 2d, n + 3d$. Suppose $5 \mid n + d$. Then $\{n, n + 2d, n + 3d, n + 4d\} \in \{y_1^2, 2y_2^2, 3y_3^2, 6y_4^2\}$ for some positive integers y_1, y_2, y_3, y_4. We explain the case $d = 7$. Then $n \equiv 3 \pmod 5$. Hence $n = 2y_2^2$ or $3y_3^2$. Let $n = 2y_2^2$. Then $n+14 = 3y_3^2$, $n+21 = y_1^2$ and hence $n+28 = 6y_4^2$, which gives $3 \mid 14$, a contradiction. When $n = 3y_3^2$, we get $n + 14 = 2y_2^2$, $n + 21 = y_1^2$ and hence $n + 28 = 6y_4^2$, implying $3 \mid 28$, a contradiction. As another example, we take $d = 11$. Then $n \equiv 4 \pmod 5$. We find that $n = 6y_4^2$, $n + 22 = y_1^2$, $n + 33 = 3y_3^2$, $n + 44 = 2y_2^2$. Here we observe that y_1 is even, y_2, y_3, y_4 are odd. Hence $n \equiv 6 \pmod 8$ and $n + 33 \equiv 7 \pmod 8$. But $n + 33 = 3y_3^2 \equiv 3 \pmod 8$, a contradiction. By a similar argument, we exclude all the cases $5 \mid n + d$, $5 \mid n + 2d$, $5 \mid n + 3d$ for $d \in \{7, 11, 13, 17, 19\}$. Thus $k \ne 5$.

Let $k = 6$. Then $P(a_i) \le 5$ and we may assume that $5 \nmid n$. Hence 5 divides only one of $\{n + d, n + 2d, n + 3d, n + 4d\}$. Therefore five of the a_i's belong to $\{1, 2, 3, 6\}$. This is not possible since a_i's are all distinct. Thus $k \ne 6$.

Let $k = 7$ and $d \in \{11, 13, 17, 19\}$. Then $P(a_i) \le 7$ and we may assume that there exist distinct i_1, i_2 and i_3 between 0 and 6 such that $7 \mid n + i_1 d$, $5 \mid n+i_2 d$, $5 \mid n+i_3 d$ since otherwise $g_0(k, 2, d) \ge 5$ leading to a contradiction. There are 8 possibilities for (i_1, i_2, i_3) for each d. We check the case $7 \mid n+d$, $5 \mid n$, $5 \mid n+5d$ for $d = 17$. Then $n+2d = 6y_4^2$, $n+3d = y_1^2$, $n+4d = 2y_2^2$ and hence $n + 6d = 3y_3^2$, which implies $3 \mid 4d$, a contradiction. The other cases are excluded similarly.

Finally, let $k = 8$ and $d \in \{11, 13, 17, 19\}$. Then $P(a_i) \le 7$ and we may assume that $7 \mid n$, $7 \mid n + 7d$, $5 \mid n + d$, $5 \mid n + 6d$ for otherwise $g_0(k, 2, d) \ge 5$, which is a contradiction. Then $(n+2d)(n+3d)(n+4d)(n+5d)$ is a square, which is impossible. ∎

The following lemma deals with the integral solutions of certain Diophantine equations.

LEMMA 8. (i) *There are infinitely many integral solutions in x and y of the equation $x^2 - 2y^2 = 1$ with x odd and of the equation $x^2 - 3y^2 = 1$ with x odd as well as with x even.*

(ii) *All solutions of the equation $3x^2 + y^2 = z^2$ in integers x, y, and z are given by*

$$x = \varrho_0 us, \quad y = \tfrac{1}{2}\varrho_0(\alpha u^2 - \beta s^2), \quad z = \tfrac{1}{2}\varrho_0(\alpha u^2 + \beta s^2)$$

where $\alpha\beta = 3$, u and s are positive integers with $\gcd(u,s) = 1$ and ϱ_0 is any integer when u and s are odd but ϱ_0 is even when one of u and s is even and the other is odd.

(iii) *The only solutions in non-zero integers of $x^4 + y^4 = 2z^2$ with $\gcd(x,y) = 1$ are $x^2 = 1$, $y^2 = 1$ and $z^2 = 1$. There is no solution in non-zero integers of the equation $x^4 - y^4 = 2z^2$ with $\gcd(x,y) = 1$.*

Lemma 8(i) is a well known result in continued fraction theory. We refer to [10, Theorem 7.25, pp. 173–174] from where the result in Lemma 8(i) can be derived easily using the facts that $\sqrt{2} = \langle 1, \overline{2}\rangle$ and $\sqrt{3} = \langle 1, \overline{1,2}\rangle$. Lemma 8(ii) can be found in [2, pp. 40–41]. The first assertion in Lemma 8(iii) is proved in [11, p. 38]. It also follows from A14.4 of [11, p. 171]. In fact, the statement given therein should be corrected as: *If $m \geq 0$ and $x^4 + y^4 = 2^m z^2$ with $\gcd(x,y) = 1$, then $m = 1$ and $x^2 = y^2 = z^2 = 1$.* The second assertion in Lemma 8(iii) follows from A14.5 of [11, p. 172].

3. An algorithm. In this section, we modify the algorithm given in [14, §4].

ALGORITHM. *Let d and $k \geq 4$ be given. Also let $\mu > 0$.*

STEP 1. *Find all primes $q_1, \ldots, q_\theta, q_{\theta+1}, \ldots, q_{\theta+\eta}$ which are coprime to d and such that $q_1 < \ldots < q_\theta \leq k < q_{\theta+1} < \ldots < q_{\theta+\eta}$ and $q_i^2 < k^2d^2/\mu$ for $1 \leq i \leq \theta + \eta$.*

STEP 2. *Set $D = \{q_1^{\alpha_1} \ldots q_\theta^{\alpha_\theta} q_{\theta+1}^{2\beta_1} \ldots q_{\theta+\eta}^{2\beta_\eta} \mid q_1^{\alpha_1} \ldots q_\theta^{\alpha_\theta} q_{\theta+1}^{2\beta_1} \ldots q_{\theta+\eta}^{2\beta_\eta} < k^2d^2/\mu$ for non-negative integers α_i, β_j, $1 \leq i \leq \theta$, $1 \leq j \leq \eta$ and $\beta_1, \ldots, \beta_\eta$ not all zero$\}$.*

STEP 3. *For every $q \in D$, find the smallest $j_0 \geq 1$ such that $d < q/(k - j_0)$. Then find some $j = j(q)$ with $j_0 \leq j \leq k-1$ such that $P(q + jd)$ and $P(q - (k-j)d)$ are $> q_{\theta+\eta}$.*

In our application, it is always possible to find j_0 in Step 3 because $d \leq 22$ and $q \geq q_{\theta+1}^2 \geq 25$. Also $q - (k-j)d$ is positive since $d < q/(k - j_0) <$

$q/(k-j)$ as $j \geq j_0$. We derive from the above Algorithm the following result.

LEMMA 9. *Let d, k and $\mu > 0$ be given such that $n + (k-1)d < k^2d^2/\mu$. If* (9) *and Step* 3 *hold*, *then* (8) *does not hold.*

Proof. Since $n+(k-1)d < k^2d^2/\mu$, every term $n+id$ for $0 \leq i \leq k-1$ is of the form $q \in D$ or q with $P(q) \leq q_\theta$. Now we follow the proof of [14, Lemma 11] to obtain the assertion of the lemma. ∎

4. Proof of Theorem 2. Let $1 < d \leq 22$ and $(n, d, k) \notin \{(2, 7, 3), (18, 7, 3), (64, 17, 3)\}$. We assume that (9) holds and arrive at a contradiction. We apply Lemma 6 to get $k \geq 4$. Next we use Theorem 1 of [14] to derive that $d \geq 7$. Then by Lemma 3, we may assume that $k \leq k_0$ where k_0 is as given in Table 1.

Suppose $n \geq (k-1)^2d^2/4$. Then we take $\varrho = 1$ in Lemma 1 and observe that $\delta(d) = 0$. Thus from Lemma 1, we derive that $t' = k$ and hence it follows from Lemma 7 that (9) does not hold. Thus our supposition $n \geq (k-1)^2d^2/4$ is false. We assume from now onwards that $n < (k-1)^2d^2/4$ and hence

$$n + (k-1)d < \frac{k^2d^2}{4} \quad \text{for } 4 \leq k \leq k_0 \text{ and } 7 \leq d \leq 22. \tag{28}$$

Suppose $k \geq 27$. Then from Table 1 we have $d \geq 9$. Assume that $n \geq (k-1)^2d^2/36$. Then we take $\varrho = 9$ in Lemma 1. Thus $h \leq 2$. Suppose $d = 9$. Then $V_1 = \{1, 2, 4, 5, 7, 8\}$, $V_2 = \{1, 2\}$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $\delta_1 = 1$ and $\delta_2 = 1$. Hence $\delta(d) \leq 2$. Similarly, for other values of d we find that $\delta(d) \leq 2$. Hence by Lemma 1, $t' \geq k-2$. We check that $g(k, 3, d) \geq 9$ for $27 \leq k \leq 66$. This contradicts (23). Thus we derive that

$$n + (k-1)d < \frac{(k-1)^2d^2}{36} + (k-1)d < \frac{k^2d^2}{32} \quad \text{for } 27 \leq k \leq 66 \text{ and } 9 \leq d \leq 22. \tag{29}$$

Let $k > 66$. Then we see from Table 1 that $d \in \{13, 17, 19, 21, 22\}$ and the corresponding upper bound for k, viz., k_0 is large. We use the idea in the preceding paragraph in order to get a good upper bound for $n+(k-1)d$ in different ranges of k. First we assume that $n+(k-1)d \geq k^2d^2/\mu$ for some positive integer μ. Then we find a lower bound for t', say t_0, and a range of k, say $R_1 \leq k \leq R_2$, in which by (12), we check that $g(k, m, d) \geq 2^m + 1$ for a suitable choice of m. Since this is a contradiction we derive that

$$n + (k-1)d < \frac{k^2d^2}{\mu} \quad \text{for } R_1 \leq k \leq R_2. \tag{30}$$

In Table 2, we tabulate the choice of μ, values of t_0, R_1, R_2 and m when $d \in \{13, 17, 19, 21, 22\}$.

Table 2

d	μ	t_0	R_1–R_2	m
13	92	$k-5$	67–119	4
17	92	$k-5$	67–119	4
	135	$k-7$	120–159, 160–174	4, 5
	184	$k-8$	175–215, 216–233	5, 6
	240	$k-10$	234–254	6
19	32	$k-2$	67–80	4
	132	$k-6$	81–143, 144–174	4, 5
	185	$k-8$	175–302	5
21	165	$k-7$	67–99	2
22	32	$k-1$	67–74	2

For a given d, k, we use (28)–(30) with Table 2 and construct the set D mentioned in Step 2 of the Algorithm in Section 3. Next we proceed to check that Step 3 holds for the given d, k and $q \in D$. This would contradict (8) by Lemma 9. The verification of Step 3 is done as follows. First, we delete from D all the integers q for which both $P(q + jd)$ and $P(q - (k - j)d)$ exceed $q_{\theta+\eta}$ with $j = j_0$. We denote the set of remaining integers of D by D_1. Secondly if $D_1 \neq \emptyset$, we delete from D_1 those integers for which both $P(q + jd)$ and $P(q - (k - j)d)$ exceed $q_{\theta+\eta}$ with $j = j_0 + 1$. The remaining set of integers from D_1 is denoted by D_2. The above process is continued till we reach $j = k - 1$ or until D_i becomes an empty set for some integer $i \geq 1$. For the values of d and k under consideration, we find that we need only take j with $j_0 \leq j \leq \min(k - 1, 25)$.

There are triples (d, k, q) for which the Algorithm fails, i.e., we are unable to find some j with $j_0 \leq j < k$ such that both $P(q + jd)$ and $P(q - (k - j)d)$ exceed $q_{\theta+\eta}$. In all, we find 207 triples which are not covered by the Algorithm. For each d, we give below a few examples of such triples. For a given d, we have chosen as examples those triples for which either k or q is maximum among all the triples (d, k, q): $(7, 4, 25)$, $(8, 5, 49)$, $(9, 12, 169)$, $(10, 6, 49)$, $(11, 12, 169)$, $(12, 4, 49)$, $(13, 13, 1058)$, $(14, 5, 363)$, $(14, 13, 361)$, $(15, 5, 578)$, $(16, 7, 361)$, $(16, 12, 169)$, $(17, 7, 1058)$, $(17, 25, 961)$, $(18, 5, 289)$, $(18, 9, 121)$, $(19, 9, 1681)$, $(19, 25, 961)$, $(20, 10, 867)$, $(20, 16, 289)$, $(21, 15, 361)$, $(21, 16, 289)$, $(22, 6, 637)$, $(22, 16, 289)$.

We observe that in all the 207 cases k and q are not very large. For all these triples (d, k, q), we factorize the product $n(n + d) \ldots (n + (k - 1)d)$ directly with $n \in \{q, q - d, \ldots, q - (k - 1)d\}$ to find a prime exceeding k which divides the product to an odd power. This completes the proof of Theorem 2. ∎

5. Proof of Theorem 3. Suppose that $d \geq 23$ and $k \geq 4d(\log d)^2$ if d is odd and $k \geq (1.3)d(\log d)^2$ if d is even. Then $k > 2(2d - 7)$. Also by equation (2), we may assume that (9) holds. Thus we conclude from Corollary 1 that $t' \ge k - \delta(d)$ where $\delta(d)$ is computed with $\varrho = \frac{1}{3}d^2$. By Lemma 4, $\delta(d) \le (.52)d\log d$ for d odd, $\delta(d) \le (.38)d\log d$ for d even and hence $k - \delta(d) \ge 63$. Hence by Corollary 2(i), we have

$$\prod_{i=1}^{t'} a_i' \ge \prod_{i=1}^{k-\delta(d)} a_i' \ge \prod_{i=1}^{k-\delta(d)} s_i \ge (1.5)^{k-\delta(d)}(k-\delta(d))!. \tag{31}$$

We use (31), (21) and (22) to get

$$1.5549 \le (153819970)^{1/k}(1.5)^{\delta(d)/k}k^{(15+\delta(d))/k}. \tag{32}$$

Let d be even. Then we turn to sharpening (32). Since $\gcd(a_i', d) = 1$ for $1 \le i \le t'$ we find that $a_1', \ldots, a_{t'}'$ are odd. By Corollary 2(iii), we have

$$\prod_{i=1}^{t'} a_i' \ge \left(\frac{9}{4}\right)^{k-\delta(d)}(k-\delta(d))!.$$

We note that $g_2 = 0$ and we use (20) with (19) for $q \ge 3$, $g_2 - h_2 \le -k + \frac{\log k}{\log 2}$ to estimate

$$\prod_{i=1}^{t'} a_i' \le (38454993)(.7657)^k k^{15}(k-1)!.$$

Finally, we combine the upper and lower estimates for $\prod_{i=1}^{t'} a_i'$ to conclude that

$$2.9384 \le (38454993)^{1/k}\left(\frac{9}{4}\right)^{\delta(d)/k} k^{(14+\delta(d))/k} \quad \text{for } d \text{ even}. \tag{33}$$

We observe that the right hand sides of the inequalities (32) and (33) are decreasing functions of k. Therefore, we put $k = (3.8)d(\log d)^2$ in (32) when d is odd and $k = (1.3)d(\log d)^2$ in (33) when d is even. By Lemma 4, we may replace $\delta(d)$ in (32) by $\frac{1}{4}d\log d + (.8323)d$ and $\delta(d)$ in (33) by $\frac{1}{3}d\log d + (.118)d$. The resulting inequalities do not hold for $d \ge 23$. ∎

6. Proofs of Theorems 4 and 5

Proof of Theorem 4. We choose

$$(n,d,y) = \begin{cases} (2, x_0^2 - 2, 2x_0y_0) & \text{where } x_0^2 - 2y_0^2 = 1 \text{ with } x_0 \text{ odd},\\ (2, x_0^2 - 2, 2x_0y_0) & \text{where } x_0^2 - 3y_0^2 = 1 \text{ with } x_0 \text{ odd},\\ (1, x_0^2/2 - 1, x_0y_0/2) & \text{where } x_0^2 - 3y_0^2 = 1 \text{ with } x_0 \text{ even}. \end{cases}$$

Then we observe that (n, d, y) is a solution of equation (6) with $B \in \{2, 3, 6\}$. Now by Lemma 8(i), there are infinitely many such triples (n, d, y) satisfying equation (6) with $B \in \{2, 3, 6\}$. This proves Theorem 4. ∎

Proof of Theorem 5. Let $n \notin \{1, 2, 48\}$. By the remarks following Theorem 5 in Section 1, we need to consider equation (7) with $B = 2, 3$. Thus we may assume that (9) holds and we shall arrive at a contradiction. We apply Lemma 5 to assume that $(a_0, a_1, a_2) = (1, 1, 2)$ if $B = 2$ and $(a_0, a_1, a_2) = (6, 1, 2)$ if $B = 3$. The first case implies $x_1^2 - x_0^2 = 1$, which is not possible. In the second case we have $3x_0^2 + x_2^2 = x_1^2$ with x_0 even, and x_0, x_1, x_2 pairwise coprime. Hence by Lemma 8(ii), we have $\varrho_0 = 2$ and $1 = x_1^2 - 6x_0^2 = \alpha^2 u^4 + \beta^2 s^4 - 18u^2 s^2$ with $\alpha\beta = 3$. It is no loss of generality to assume that $\alpha = 1$, $\beta = 3$ while dealing with this equation. Thus we consider $u^4 + 9s^4 - 18u^2 s^2 = 1$. This implies that $1 + 8u^4 = c_1^2$ for some integer c_1. This immediately reduces to the equation $u_1^4 - 2u_2^4 = \pm 1$ for some integers u_1 and u_2 with $u_1 u_2 = \pm u$. We apply Lemma 8(iii) to observe that $(u_1, u_2) = (\pm 1, 0)$ or $(\pm 1, \pm 1)$. Hence $u = 0$ or ± 1. Thus we have $9s^4 = 1$ or $s^2(9s^2 - 18) = 0$. The former is impossible while the latter gives $s = 0$, implying that $x_0 = 0$ by Lemma 8(ii) and this is not possible. ∎

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