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Factors of sums of powers of binomial coefficients

by

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Dedicated to the memory of Paul Erdős

1. Introduction. It is well known that if

$$f_{n,a} = \sum_{k=0}^{n} \binom{n}{k}^a$$

then $f_{n,0} = n+1$, $f_{n,1} = 2^n$, $f_{n,2} = \binom{2n}{n}$, and it is possible to show (Wilf, personal communication, using techniques in [8]) that for $3 \le a \le 9$, there is no closed form for $f_{n,a}$ as a sum of a fixed number of hypergeometric terms. Similarly, using asymptotic techniques, de Bruijn has shown [1] that if $a \ge 4$, then $h_{2n,a}$ has no closed form, where

$$h_{n,a} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a$$

(clearly, $h_{2n+1,a} = 0$). In this paper we will prove that while no closed form may exist, there are interesting divisibility properties of $f_{n,2a}$ and $h_{2n,a}$. We will illustrate some of the techniques which may be applied to prove these sorts of results.

Our main results are:

THEOREM 1. *For all positive n and a,*

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^a$$

is divisible by $\binom{2n}{n}$.

THEOREM 2. *For all positive integers a, m, j,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a (-1)^k q^{jk}$$

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is divisible by $t(n,q)$, *where the* $\left[{n \atop k}\right]_q$ *are the* q*-binomial coefficients*, *and* $t(n,q)$ *is a* q*-analogue of the odd part of* $\binom{n}{k}$.

2. Background. In attempting to extend the results of previous work [2], we were led to consider factorizations of sums of powers of binomial coefficients. It quickly became clear that for even exponents, small primes occurred as divisors in a regular fashion (Proposition 3), and that this result could be extended (Proposition 7) to odd exponents and alternating sums. Further investigation revealed (Proposition 8) that for all alternating sums, the primes dividing $h_{2n,a}$ coincided with those dividing $\binom{2n}{n}$. This led us to conjecture, and subsequently to prove, Theorem 1; as part of our proof we obtain (Theorem 2) a corresponding result for q-binomial coefficients.

3. Non-alternating sums

PROPOSITION 3. *For every integer* $m \ge 1$, *if* p *is a prime in the interval*

$$\frac{n}{m} < p < \frac{2a(n+1)-1}{2ma-1} = \frac{n+1}{m} + \frac{n+1-m}{m(2ma-1)}$$

then $p \mid f_{n,2a}$. *In particular*, $f_{n,2a}$ *is divisible by all primes* p *for which*

$$n < p < \frac{2a(n+1)-1}{2a-1} = n+1+\frac{n}{2a-1}.$$

The following lemma will enable us to convert information about divisors of $f_{n,a}$ which are greater than n into information about divisors less than n.

LEMMA 4. *Let* $n = n_s n_{s-1} \ldots n_2 n_1 n_0$ *be the expansion of* n *in base* p (*and similarly for* $k = k_s k_{s-1} \ldots k_2 k_1 k_0$). *Then*

$$f_{n,a} \equiv \prod_{i=0}^{s} f_{n_i,a} \pmod p.$$

Proof. By Lucas' Theorem (see for example Granville [6]),

$$\binom{n}{k} \equiv \prod_{i=0}^{s} \binom{n_i}{k_i} \pmod p$$

where as usual, $\binom{n_i}{k_i} \equiv 0 \pmod p$ if $k_i > n_i$. Hence all the terms in the sum over k for which $k_i > n_i$ for some i disappear, giving

$$\begin{aligned} f_{n,a} = \sum_{k=0}^{n} \binom{n}{k}^a &\equiv \sum_{k_s=0}^{n_s} \sum_{k_{s-1}=0}^{n_{s-1}} \cdots \sum_{k_0=0}^{n_0} \prod_{i=0}^{s} \binom{n_i}{k_i}^a \pmod p \\ &\equiv \prod_{i=0}^{s} \sum_{k_i=0}^{n_i} \binom{n_i}{k_i}^a \pmod p \equiv \prod_{i=0}^{s} f_{n_i,a} \pmod p \end{aligned}$$

as claimed. ∎

COROLLARY 5. *If $l < p$ and $p \mid f_{l,a}$ then $p \mid f_{l+jp,a}$ for all positive integers j.*

We are now in a position to prove Proposition 3. We proceed in two stages: first, the case when $n < p$.

LEMMA 6. *Let p be a prime in the interval $n < p < (2a(n+1)-1)/(2a-1)$. Then $p \mid f_{n,2a}$.*

Proof. Let $p = n + r$ where $r > 0$. Then we have

$$\begin{aligned} f_{n,2a} = \sum_{k=0}^{n} \binom{n}{k}^{2a} &\equiv \sum_{k=0}^{p-r} \binom{p-r}{k}^{2a} \pmod p \\ &\equiv \sum_{k=0}^{p-r} \binom{r+k-1}{k}^{2a} (-1)^{2ka} \pmod p \\ &\equiv \sum_{k=0}^{p-r} \binom{r+k-1}{k}^{2a} \pmod p \\ &\equiv \sum_{k=0}^{p-r} \left(\frac{(k+1)(k+2)\ldots(k+r-1)}{(r-1)!} \right)^{2a} \pmod p. \end{aligned}$$

If we write $x_{(0)} = 1$ and $x_{(r)}$ for the polynomial $x(x+1)\ldots(x+r-1)$, then this last sum becomes

$$\sum_{k=0}^{p-r} \left(\frac{(k+1)_{(r-1)}}{(r-1)!} \right)^{2a}.$$

We now observe that the polynomials $x_{(0)}, x_{(1)}, \ldots, x_{(d)}$ form an integer basis for the space of all integer polynomials of degree at most d. Hence there exist integers $c_0, c_1, \ldots, c_{(r-1)(2a-1)}$ so that

$$((k+1)_{(r-1)})^{2a-1} = \sum_{j=0}^{(r-1)(2a-1)} c_j (k+r)_{(j)}.$$

Thus

$$\begin{aligned} f_{n,2a} &\equiv \frac{1}{(r-1)!^{2a}} \sum_{k=0}^{p-r} \sum_{j=0}^{(r-1)(2a-1)} c_j (k+1)_{(r-1)} (k+r)_{(j)} \\ &\equiv \frac{1}{(r-1)!^{2a}} \sum_{j=0}^{(r-1)(2a-1)} \sum_{k=0}^{p-r} c_j (k+1)_{(r+j-1)} \\ &\equiv \frac{1}{(r-1)!^{2a}} \sum_{j=0}^{(r-1)(2a-1)} c_j \frac{(p-r+1)_{(r+j)}}{r+j}. \end{aligned}$$

Now, if $r+(r-1)(2a-1) < p$, then each of the terms in the sum is divisible by p, and $(r-1)!$ is not divisible by p; hence $f_{n,2a}$ is divisible by p. But

$$r+(r-1)(2a-1) = 2ra-2a+1 = 2pa-2na-2a+1$$

and

$$2pa-2na-2a+1 < p$$

if and only if

$$p < \frac{2a(n+1)-1}{2a-1}$$

completing the proof of the lemma. ∎

Now, suppose that $n=(m-1)p+l$ with $l>0$ and

$$l < p < \frac{2a(l+1)-1}{2ma-1}.$$

Then, by Lemma 6, p divides $f_{l,2a}$ and hence by Corollary 5, p divides $f_{n,2a}$. But $l<p$ if and only if $n<mp$, and

$$p < \frac{2a(l+1)-1}{2a-1}$$

if and only if

$$p < \frac{2a(n-(m-1)p)-1}{2a-1},$$

that is, if

$$p < \frac{2a(n+1)-1}{2ma-1}.$$

Thus, if

$$\frac{n}{m} < p < \frac{2a(n+1)-1}{2ma-1}$$

then p divides $f_{n,2a}$, completing the proof of Proposition 3. ∎

4. Alternating sums. We note that no similar result holds for the case of odd powers of binomial coefficients (with the trivial exception of $a=1$). Indeed, except for the power of 2 dividing $f_{n,2a+1}$ (which we discuss in Lemma 12), the factorizations of sums of odd powers seem to exhibit no structure; for example,

$$f_{28,3} = 2^6 \cdot 661 \cdot 3671 \cdot 5153 \cdot 313527009031.$$

However, for alternating sums of odd powers, we have

PROPOSITION 7. *p divides $h_{2n,2a+1}$ for primes in the intervals*

$$\frac{n}{m} < p < \frac{(2a+1)(n+1)-1}{m(2a+1)-1} = \frac{n+1}{m} + \frac{n+1-m}{m(2a+1)-1}.$$

P r o o f. Indeed, by examining the proof of Proposition 3, we see that if we define

$$g_{n,a} = \sum_{k=0}^{n} \left((-1)^k \binom{n}{k} \right)^a$$

so that $g_{n,2a} = f_{n,2a}$ and $g_{n,2a+1} = h_{n,2a+1}$, then $g_{n,a}$ is divisible by all primes in each of the intervals

$$\frac{n}{m} < p < \frac{(n+1)a-1}{ma-1}$$

so Propositions 3 and 7 are really the same result. ∎

For all alternating sums we have

PROPOSITION 8. *If* $p \mid \binom{2n}{n}$ *then* $p \mid h_{2n,a}$.

P r o o f. Clearly 2 divides $h_{2n,a}$ if and only if 2 divides the middle term, $\binom{2n}{n}^a$, as all of the other terms cancel (mod 2). Since 2 divides $\binom{2n}{n}$, 2 divides $h_{2n,a}$.

Now let p be an odd prime dividing $\binom{2n}{n}$; we will show that p divides $h_{2n,a}$. By Kummer's theorem, at least one of the "digits" of $2n$ written in base p is odd (since if all are even, then there are no carries in computing $n + n = 2n$ in base p). Let the digits of $2n$ in base p be $(2n)_s, (2n)_{s-1}, \ldots, (2n)_1, (2n)_0$. Then as in Lemma 4,

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^a \equiv \prod_{i=0}^{s} \left(\sum_{k_i=0}^{(2n)_i} (-1)^{k_i} \binom{(2n)_i}{k_i}^a \right)$$

(since p is odd, $(-1)^k = (-1)^{k_0+k_1+\ldots+k_s}$). Now, since $p \mid \binom{2n}{n}$, at least one of the digits of $2n$ in base p is odd; but then the corresponding term in the product is zero, and so $p \mid h_{2n,a}$, completing the proof of Proposition 8. ∎

After computing some examples, it is natural to conjecture (and then, of course, to prove!) Theorem 1.

5. The main theorems. We will prove Theorem 1 by considering q-binomial coefficients.

DEFINITIONS. Let n be a positive integer. Throughout we will denote the number of 1's in the binary expansion of n by $l(n)$ (so that $2^{l(n)} \parallel \binom{2n}{n}$). We further define the following polynomials in an indeterminate q:

$$\theta_n(q) = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}$$

(the q-analogue of n),

$$\phi_n(q) = \prod_{d \mid n} (1-q^d)^{\mu(n/d)}$$

(the nth cyclotomic polynomial in q),

$$n!_q = \prod_{i=1}^{n} \theta_i(q)$$

(the q-analogue of $n!$), and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q (n-k)!_q}$$

(the q-analogue of $\binom{n}{k}$).

Further, define

$$r(x,q) = \prod_{j \le x} (1-q^j) = (1-q)^n n!_q, \qquad s(x,q) = \prod_{2j+1 \le x} (1-q^{2j+1})$$

and

$$t(n,q) = \frac{s(n,q)}{s(n/2,q)s(n/4,q)s(n/8,q)\dots}.$$

Note that the apparently infinite product in the denominator is in fact finite, since $s(x,q) = 1$ if $x < 1$. We now make some useful observations about $t(n,q)$. First,

$$s(n,q) = \frac{r(n,q)}{r(n/2,q^2)},$$

so

$$\begin{aligned} t(n,q) &= \frac{s(n,q)}{s(n/2,q)^2} \cdot \frac{s(n/2,q)}{s(n/4,q)^2} \cdot \frac{s(n/4,q)}{s(n/8,q)^2} \cdot \frac{s(n/8,q)}{s(n/16,q)^2} \cdots \\ &= \frac{\dfrac{r(n,q)}{r(n/2,q)^2} \cdot \dfrac{r(n/2,q)}{r(n/4,q)^2} \cdot \dfrac{r(n/4,q)}{r(n/8,q)^2}}{\dfrac{r(n/2,q^2)}{r(n/4,q^2)^2} \cdot \dfrac{r(n/4,q^2)}{r(n/8,q^2)^2} \cdot \dfrac{r(n/8,q^2)}{r(n/16,q^2)^2}} \cdots \\ &= \frac{r(n,q)}{r(n/2,q)^2} \cdot \frac{\dfrac{r(n/2,q)}{r(n/4,q)^2}}{\dfrac{r(n/2,q^2)}{r(n/4,q^2)^2}} \cdot \frac{\dfrac{r(n/4,q)}{r(n/8,q)^2}}{\dfrac{r(n/4,q^2)}{r(n/8,q^2)^2}} \cdot \frac{\dfrac{r(n/8,q)}{r(n/16,q)^2}}{\dfrac{r(n/8,q^2)}{r(n/16,q^2)^2}} \cdots \end{aligned}$$

where again, the apparently infinite product is in fact finite.

Now, since

$$\frac{\dfrac{r(x,q)}{r(x/2,q)^2}}{\dfrac{r(x,q^2)}{r(x/2,q^2)^2}} \to \begin{cases} 1 & \text{if } \lfloor x \rfloor \text{ is even,} \\ \frac{1}{2} & \text{if } \lfloor x \rfloor \text{ is odd,} \end{cases}$$

as $q \to 1$, we see that

$$\lim_{q\to 1} t(2n,q) = \frac{\binom{2n}{n}}{2^l}.$$

Further, $t(2n+1,q)$ has a factor $q-1$, so

$$\lim_{q\to 1} t(2n+1,q) = 0.$$

In other words, since $2^l \parallel \binom{2n}{n}$, we may regard $t(2n,q)$ as the q-analogue of the largest odd factor of $\binom{2n}{n}$.

LEMMA 9. *We have $t(n,q) = \prod'_m \phi_m(q)$ where the product is over those odd m for which $\lfloor n/m \rfloor$ is odd.*

Proof. Clearly, if m is even then $\phi_m(q)$ does not divide $t(n,q)$. Suppose m is odd; then $\phi_m(q)$ divides $s(n,q)$ exactly $\lceil \lfloor n/m \rfloor /2 \rceil$ times, and hence $\phi_m(q)$ divides $t(n,q)$

$$\left\lceil \left\lfloor \frac{n}{m} \right\rfloor /2 \right\rceil - \left\lceil \left\lfloor \frac{n}{2m} \right\rfloor /2 \right\rceil - \left\lceil \left\lfloor \frac{n}{4m} \right\rfloor /2 \right\rceil - \ldots - \left\lceil \left\lfloor \frac{n}{2^j m} \right\rfloor /2 \right\rceil - \ldots$$

times. Now, by considering the binary expansion of $\lfloor n/m \rfloor$, it is immediate that this is 0 if $\lfloor n/m \rfloor$ is even, and 1 if $\lfloor n/m \rfloor$ is odd. ∎

LEMMA 10. *Let m, n, k be non-negative integers and write*

$$n = n''m + n', \qquad k = k''m + k',$$
$$n - k = (n-k)''m + (n-k)',$$

where n', k' are the least non-negative residues of n, k (mod m). Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \begin{bmatrix} n' \\ k' \end{bmatrix}_q \binom{n''}{k''} \pmod{\phi_m(q)}$$

where $\left[\begin{smallmatrix} n' \\ k' \end{smallmatrix}\right]_q$ is taken to be 0 if $n' < k'$.

Proof. See [3], [4] or [7].

Proof of Theorem 2. It is enough to show that if m and $n'' = \lfloor n/m \rfloor$ are odd, then

$$\phi_m(q) \mid \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a (-1)^k q^{jk}.$$

But, from Lemma 10,

$$\begin{aligned} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a (-1)^k q^{jk} &\equiv \sum_{k'=0}^{n'} \sum_{k''=0}^{n''} \begin{bmatrix} n' \\ k' \end{bmatrix}_q^a \binom{n''}{k''}^a (-1)^{k'+k''} q^{jk'} \pmod{\phi_m(q)} \\ &= \Bigg(\sum_{k'=0}^{n'} \begin{bmatrix} n' \\ k' \end{bmatrix}_q^a (-1)^{k'} q^{jk'} \Bigg) \Bigg(\sum_{k''=0}^{n''} \binom{n''}{k''}^a (-1)^{mj} \Bigg) \end{aligned}$$

and since m and n'' are odd, the second sum is zero, and we are done. ∎

We observe now that both sides of Theorem 2 are integer polynomials; thus when we evaluate them at $q = 1$, the left hand side (if non-zero) will divide the right hand side. But we have already observed that $t(2n, 1) = \binom{2n}{n}/2^l$, and hence we have proved

COROLLARY 11.
$$\binom{2n}{n} \mid 2^{l(n)} \sum_{k=0}^{2n} \binom{2n}{k}^a (-1)^k. \ \blacksquare$$

To prove Theorem 1 it remains to show that
$$2^{l(n)} \mid \sum_{k=0}^{2n} \binom{2n}{k}^a (-1)^k.$$

We prove a stronger result by induction.

LEMMA 12. *For all positive integers a and n,*
$$2^{l(n)} \mid \sum_{k=0}^{n} \binom{n}{k}^a$$
and
$$2^{l(n)} \mid \sum_{k=0}^{n} \binom{n}{k}^a (-1)^k.$$

Proof. The assertion is clearly true when $n = 1$. Assume now that it holds for all values less than n. For each $1 \le i \le l(n)$, let $m = 2^i$ and let $n', n'', k', k'', (n-k)', (n-k)''$ be defined as in Lemma 10. Writing
$$w_i(q) = \left(\sum \begin{bmatrix} n' \\ k' \end{bmatrix}_q^a \right) \left(\sum \binom{n''}{k''}^a \right)$$
we have
$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a \equiv w_i(q) \pmod{\phi_{2^i}(q)}.$$
By our induction hypothesis, since $l(n) = l(n') + l(n'')$, $2^{l(n)} \mid w_i(1)$ for each i.

We now wish to combine these equivalences modulo
$$\theta_{2^{l(n)}}(q) = \phi_2(q)\phi_4(q)\phi_8(q)\ldots\phi_{2^{l(n)}}(q)$$
and evaluate them at $q = 1$. To do this, define
$$\pi_1 = \frac{1}{2^{l-1}}$$
and
$$\pi_i = \frac{1}{2^{l-i+1}}(1-q) \quad \text{for } i = 2, 3, \ldots, l(n).$$
Then, setting
$$u_i(q) = \phi_2(q)\phi_4(q)\ldots\phi_{2^{i-1}}(q)\pi_i\phi_{2^{i+1}}(q)\ldots\phi_{2^{l(n)}}(q)$$

we have

$$u_1(q) = \frac{1}{2^{l-1}}(1+q^2)(1+q^4)\dots(1+q^{2^{l(n)-1}}) \equiv 1 \pmod{(1+q)}$$

and for $i \geq 2$,

$$\begin{aligned} u_i(q) & \\ &\equiv \frac{1}{2^{l-i+1}}(1-q^2)(1+q^2)(1+q^4)\dots(1+q^{2^{i-2}})(1+q^{2^i})\dots(1+q^{2^{l(n)-1}}) \\ &\equiv \frac{1}{2^{l-i+1}}(1-q^{2^{i-1}})(1+q^{2^i})(1+q^{2^{i+1}})\dots(1+q^{2^{l(n)-1}}) \\ &\equiv 1 \pmod{(1+q^{2^{i-1}})}. \end{aligned}$$

Further, if $i \neq j$, then $u_i(q) \equiv 0 \pmod{\phi_{2^j}(q)}$. Hence,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a \equiv \sum_{i=1}^{l(n)} w_i(q)u_i(q) \pmod{\theta_{2^{l(n)}}(q)},$$

that is,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a = P(q)\theta_{2^{l(n)}}(q) + \sum_{i=1}^{l(n)} w_i(q)u_i(q)$$

where we wish to conclude that $P(q)$ is an integer polynomial. Observe that it is sufficient to prove that each $w_i(q)u_i(q)$ is an integer polynomial, since $\theta_{2^l}(q)$ is monic.

To do this, consider $w_i(q)$. First, observe that $w_i(q)$ is divisible by $2^{l(n'')}$ by our inductive hypothesis, since $n'' < n$; further, if n is odd, so is n', and hence the q-binomial sum in $w_i(q)$ is symmetric and its coefficients are even; if n is even, then $l(n') \leq i-1$, and in each case, $2^{l-i} \mid w_i(q)$ (that is, each coefficient of $w_i(q)$ is divisible by 2^{l-i+1}). Thus, for each i, $w_i(q)u_i(q)$ is an integer polynomial.

We have thus proven that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^a = P(q)\theta_{2^{l(n)}}(q) + \sum_{i=1}^{l(n)} w_i(q)u_i(q)$$

where $P(q)$ has integer coefficients. Now, setting $q = 1$ in both sides, we observe that $u_i(1)$ is an integer for each i, $2^{l(n)} \mid w_i(1)$ for each i (indeed, $u_i(1) = 0$ for $i \geq 2$, and $u_1(1) = 1$), and that $\theta_{2^{l(n)}}(1) = 2^{l(n)}$. Hence each term on the right is divisible by $2^{l(n)}$, proving that

$$2^{l(n)} \mid \sum_{k=0}^{n} \binom{n}{k}^a.$$

To prove that

$$2^{l(n)} \mid \sum_{k=0}^{n} \binom{n}{k}^a (-1)^k$$

we proceed similarly, setting

$$v_i(q) = \Big(\sum \begin{bmatrix} n' \\ k' \end{bmatrix}_q^a (-1)^k\Big)\Big(\sum \binom{n''}{k''}^a\Big),$$

with the only major difference being in the proof that $\sum_{i=1}^{l(n)} v_i(q)u_i(q)$ is an integer polynomial: in this case, if n is even, things work as above, and if n is odd, then we have n' is odd, and $v_i(q)$ is identically equal to 0. Note that we need to have already proven the lemma for non-alternating sums to prove the alternating case. This completes the proof of Lemma 12 and thus of Theorem 1. ∎

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