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Lattice points in bodies of revolution

by

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1. Introduction and main results. Let $\mathbb{E} \subset \mathbb{R}^d$, $d > 1$, be a compact convex body such that $\vec{0} \in \mathrm{Int}(\mathbb{E})$ and its boundary, $\partial\mathbb{E}$, is a smooth compact $(d-1)$-submanifold with positive Gaussian curvature. We shall abbreviate these conditions saying that $\mathbb{E}$ is a *smooth convex body.* The main problem in lattice point theory consists in estimating the difference between the number of lattice points in large homothetic smooth convex bodies and their volume. Namely, defining for $R \ge 1$,

$$\mathcal{N}(R) = \#\{\vec{n} \in \mathbb{Z}^d : \vec{n} \in R\mathbb{E}\}$$

with $R\mathbb{E} = \{\vec{x} \in \mathbb{R}^d : R^{-1}\vec{x} \in \mathbb{E}\}$, we want to estimate

$$\mathcal{E}(R) = \mathcal{N}(R) - R^d|\mathbb{E}|$$

where $|\mathbb{E}|$ is the volume of $\mathbb{E}$ (note that $R^d|\mathbb{E}| = |R\mathbb{E}|$).

Estimates for $\mathcal{E}(R)$ are typically written (with Landau's notation) as

$$\mathcal{E}(R) = O(R^\alpha) \quad \text{as } R \to \infty.$$

Let α_d be the minimal exponent for $\mathbb{E}$, i.e.

$$\alpha_d = \inf\{\alpha > 0 : \mathcal{E}(R) = O(R^\alpha)\}.$$

An easy geometrical argument, first used by C. F. Gauss for the circle problem (see [Ga]), proves $\alpha_d \le d - 1$. Employing Fourier analysis techniques (essentially an appropriate application of the Poisson summation formula combined with the stationary phase method) it is possible to prove $\alpha_d \le d(d-1)/(d+1)$. This result was first stated in 1950 by E. Hlawka (see [Hl]). Improvements over this bound depend on the estimation of exponential sums. Using a method (sometimes known as the Discrete Hardy–Littlewood Method) originally due to E. Bombieri and H. Iwaniec and developed by several authors, M. N. Huxley has proved $\alpha_2 \le 46/73$ (see [Hu]). In higher dimensions the best known general results are due to E. Krätzel and

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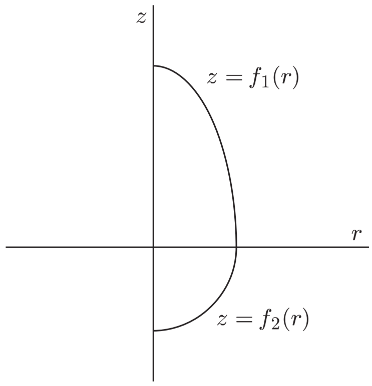
W. G. Nowak [Kr-No] who, using a refinement of a two-dimensional method invented by E. C. Titchmarsh [Ti], have proved $\alpha_d \le d-2+8/(5d+2)$ for $3 \le d \le 6$ and $\alpha_d \le d-2+3/(2d)$ for $d \ge 7$. On the other hand, if $\mathbb{E}$ is the d-dimensional ball, these results are substantially improved for $d > 2$. In fact, it is known that $\alpha_d = d-2$ for $d > 3$ and, in the case $d = 3$, D. R. Heath-Brown has recently proved $\alpha_3 \le 21/16$ (see [H-B]).

The previous results show that there is a substantial difference between lattice point problems in general smooth convex bodies and in spheres. The purpose of this paper is to show that it is possible to obtain intermediate results, even from the simplest van der Corput's estimate, if we assume the existence of an axis of rotational symmetry. Namely, we shall consider the set $\mathcal{R}_d$ of smooth convex bodies of revolution in $\mathbb{R}^d$, i.e. smooth convex bodies $\mathbb{E} \subset \mathbb{R}^d$ such that $(x_1, \ldots, x_{d-1}, z) \in \mathbb{E} \Leftrightarrow (r, 0, \ldots, 0, z) \in \mathbb{E}$ where $r^2 = \sum x_i^2$.



The boundary of $\mathbb{E}$ can be divided into its upper and lower halves, S_1, S_2, where S_i, $i = 1, 2$, is given by $z = f_i(r)$, $0 \le r \le r_0$. Under our assumptions $f_1''(r) > 0 > f_2''(r)$ for $0 < r < r_0$, $f_i^{(2k+1)}(0) = 0$ and $f_i^{(k)}(r_0) = \infty$. We write $\mathbb{E} \in \mathcal{R}_d^*$ if $\frac{1}{r} f_i'''(r)$ does not vanish for $0 \le r < r_0$ (note that this function is well defined by continuity at $r = 0$).

Our main result is

Theorem 1.1. *If* $\mathbb{E} \in \mathcal{R}_3^*$ *then* $\alpha_3 \le 11/8$.

Remark. Note that this result improves in $\mathcal{R}_3^*$ the bound $\alpha_3 \le 25/17$ deduced from [Kr-No]. Some extensions and improvements in particular cases will be considered in Section 6.

Although our methods also apply to higher dimensional cases, if $d = 4$ the techniques introduced in [Ch-Iw] and recently improved in [H-B], involving exponential sums and character sums, can be used to get a better result. On the other hand, for $d > 4$ a sharp result can be obtained by more elementary means, namely

THEOREM 1.2. *If* $\mathbb{E} \in \mathcal{R}_d$ *with* $d > 4$ *then* $\alpha_d \le d - 2$.

REMARK. Note that this is the best possible upper bound valid for every $\mathbb{E} \in \mathcal{R}_d$, because equality is reached, for instance, for the ball.

NOTATION. Throughout this paper we use Landau's notation $f = O(g)$ with the usual meaning $|f| \le Cg$ where C is a constant. We also employ Vinogradov's notations $f \ll g$ and $f \gg g$ meaning $f = O(|g|)$ and $g = O(|f|)$. When both of these conditions hold we simply write $f \asymp g$. As usual, we abbreviate $e^{2\pi i x}$ by $e(x)$ and denote the Fourier transform of f, $\int f(\vec{x}) e(-\vec{x} \cdot \vec{\xi})\, dx_1 \ldots dx_n$, by $\widehat{f}(\vec{\xi})$.

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2. A truncated Hardy–Voronoï formula. The purpose of this section is to express $\mathcal{E}(R)$ in terms of a trigonometric sum when $\mathbb{E}$ is a smooth convex body in $\mathbb{R}^3$. The resulting formula (see Proposition 2.1 below) can be considered as a generalized three-dimensional version of the classical Hardy and Voronoï's truncated formulas for the circle and divisor problems (see [Iv]).

Before stating the result we consider the following smooth functions in $\mathbb{R}^3 - \{\vec{0}\}$:

$$g(\vec{n}) = \sup\{\vec{x} \cdot \vec{n} : \vec{x} \in \mathbb{E}\}, \qquad \kappa(\vec{n}) = K(N^{-1}(\vec{n}/\|\vec{n}\|))$$

where K denotes the Gaussian curvature and $N : \partial\mathbb{E} \to S^2$ is the normal map. We also consider $\eta \in C_0^\infty((-1,1))$ with $\eta(0) = 1$ and such that the Fourier transform of $\eta(\|\vec{x}\|)$ is positive (this latter condition can be easily fulfilled by considering the convolution of a radial function with itself).

PROPOSITION 2.1. *Let* $\mathbb{E} \subset \mathbb{R}^3$ *be a smooth convex body and* g, κ, η *as before. Then given* $R > 2$ *and* $\delta = R^{-c}$, $0 < c < 1$, *there exists* $R' \in (R-2, R+2)$ *such that*

$$\mathcal{E}(R) = -\frac{R'}{\pi} \sum_{\vec{n} \in \mathbb{Z}^3 - \{\vec{0}\}} \eta(\delta\|\vec{n}\|) \frac{\cos(2\pi R' g(\vec{n}))}{\|\vec{n}\|^2 \sqrt{\kappa(\vec{n})}} + O(R^{2+\varepsilon}\delta)$$

for every $\varepsilon > 0$.

REMARK. Note that choosing $\delta = R^{-1/2}$ it follows immediately that $\alpha_3 \le 3/2$, which is the result of E. Hlawka mentioned in the introduction for $d = 3$.

Proof (of Proposition 2.1). Of course, we can assume that ε is arbitrarily small, in particular $0 < \varepsilon < 1$. Let ϕ_δ be the Fourier transform of $\eta(\delta\|\cdot\|)$. As $\eta \in C_0^\infty$ and $\eta(0) = 1$, for every $k \ge 1$ we have

$$\int\limits_{\|\vec{t}\|\le\delta^{1-\varepsilon}} \phi_\delta(\vec{t})\, d\vec{t} = 1 + O(\delta^k) \quad \text{and} \quad \int\limits_{\|\vec{t}\|\ge\delta^{1-\varepsilon}} \phi_\delta(\vec{t})\, d\vec{t} = O(\delta^k).$$

Then, if χ_R denotes the characteristic function of $R\mathbb{E}$,

$$(\phi_\delta * \chi_{R_1})(\vec{x}) = \int\limits_{\mathbb{R}^3} \phi_\delta(\vec{t})\chi_{R_1}(\vec{x}-\vec{t})\, d\vec{t} \le \chi_R(\vec{x}) + O(\delta^k)$$

where $R_1 = R - 2\delta^{1-\varepsilon}$. In the same way,

$$(\phi_\delta * \chi_{R_2})(\vec{x}) \ge \chi_R(\vec{x}) + O(\delta^k)$$

where $R_2 = R + 2\delta^{1-\varepsilon}$.

Hence, for some R' such that $|R' - R| < 2\delta^{1-\varepsilon}$,

$$\sum_{\vec{n}\in\mathbb{Z}^3} (\phi_\delta * \chi_{R'})(\vec{n}) = \mathcal{N}(R) + O(R^3\delta^k).$$

The Poisson summation formula applied to the first term and the definition of $\mathcal{E}(R)$ give

$$\mathcal{E}(R) = \widehat{\phi}_\delta(\vec{0})\widehat{\chi}_{R'}(\vec{0}) - R^3|\mathbb{E}| + \sum_{\vec{n}\in\mathbb{Z}^3-\{\vec{0}\}} \widehat{\phi}_\delta(\vec{n})\widehat{\chi}_{R'}(\vec{n}) + O(R^3\delta^k). \tag{2.1}$$

By our choice of ϕ_δ, χ_R and η,

$$\widehat{\phi}_\delta(\vec{0}) = 1, \quad \widehat{\chi}_{R'}(\vec{0}) = (R')^3|\mathbb{E}|, \quad \widehat{\phi}_\delta(\vec{n}) = \eta(\delta\|\vec{n}\|),$$

and by the main result of [He] (similar results are contained in [Hl])

$$\widehat{\chi}_{R'}(\vec{n}) = \frac{R}{2\pi i\|\vec{n}\|^2}\left(\frac{e(Rg(\vec{n})-1/4)}{\sqrt{\kappa(\vec{n})}} - \frac{e(-Rg(-\vec{n})+1/4)}{\sqrt{\kappa(-\vec{n})}}\right) + O\left(\frac{R}{\|\vec{n}\|^3}\right).$$

Finally, substituting in (2.1) and taking k large enough gives the result. ∎

3. Lattice error term and exponential sums. If $t \mapsto (A(t), B(t))$, $0 \le t \le T$, $A \ge 0$, is a parametrization of the generatrix of $\mathbb{E} \in \mathcal{R}_3$, then $\partial\mathbb{E}$ can be parametrized as

$$(A(t)\cos u, A(t)\sin u, B(t)), \qquad 0 \le u < 2\pi,\ 0 \le t \le T,$$

and the function g defined in the previous section is

$$g(n_1, n_2, n_3) = \sup_{\substack{0\le u<2\pi \\ 0\le t\le T}} \{A(t)(n_1\cos u + n_2\sin u) + B(t)n_3\}. \tag{3.1}$$

By rotational symmetry we have

$$g(n_1, n_2, n_3) = g(\sqrt{n_1^2 + n_2^2}, 0, n_3).$$

We denote this latter function by $h(n_1^2+n_2^2,n_3)$, i.e.

$$h(n,m)=g(\sqrt{n},0,m).$$

According to (3.1), $h(n,m)$ is implicitly defined for $n>0$ and $m\neq 0$ as

$$\begin{cases} h(n,m)=\sqrt{n}\,A(t)+mB(t),\\ \qquad\quad 0=\sqrt{n}\,A'(t)+mB'(t).\end{cases}$$

With the notation of Section 1, the generatrix of $\mathbb{E}$ is composed of the arcs $t\mapsto(t,f_1(t))$ and $t\mapsto(t,f_2(t))$, $0\le t\le r_0$. Hence

$$\begin{cases} h(n,m)=t\sqrt{n}+mf_i(t),\\ \qquad\quad 0=\sqrt{n}+mf_i'(t)\end{cases}\tag{3.2}$$

where $i=1$ if $m>0$ and $i=2$ if $m<0$ (note that $f_2'\ge 0\ge f_1'$).

The purpose of this section is to relate the size of the lattice error term, $\mathcal{E}(R)$, and the exponential sum defined for $N,M,L\ge 1$, $x\in\mathbb{R}$, by

$$T_{NML}(x)=\frac{1}{L}\sum_{l\le L}\Big|\sum_{n\asymp N}\sum_{m\asymp M}{}' e(x(h(n,m+l)-h(n,m)))\Big|,$$

where the prime indicates that if I_1 and I_2 are the (positive and negative) intervals in $\mathbb{Z}$ defined by the condition $m\asymp M$, then the summation is restricted to the values of m such that m and $m+l$ belong to the same interval I_i.

PROPOSITION 3.1. *If R,δ and R' are as in Proposition 2.1, then for every $\varepsilon>0$ and any $L\ge 1$ (perhaps depending on N,M and R),*

$$\mathcal{E}(R)\ll R^{2+\varepsilon}\delta+R^{1+\varepsilon}\sup_{N,M^2<\delta^{-2}}\frac{M^{1/2}N^{1/2}}{N+M^2}(|T_{NML}|^{1/2}+M^{1/2}N^{1/2}L^{-1/2})$$

where $T_{NML}=T_{NML}(R')$.

Proof. As the Gaussian curvature K of $\partial\mathbb{E}$ is positive and it coincides with the determinant of the Weingarten map $dN:T_p(\partial\mathbb{E})\to T_p(\partial\mathbb{E})$ (see p. 104 of [Sp]), the normal map $N:\partial\mathbb{E}\to S^2$ is a diffeomorphism and $K\circ N^{-1}:S^2\to\mathbb{R}$ is a C^∞ function. Hence, for a suitably chosen η, we can apply partial summation in Proposition 2.1 (compare with Lemma 4 of [Kr-No]) to prove

$$\mathcal{E}(R)\ll R^{2+\varepsilon}\delta+R^{1+\varepsilon}\sup_{N,M^2<\delta^{-2}}\frac{1}{N+M^2}\sum_{n\asymp N}\sum_{m\asymp M}r(n)e(R'h(m,n))$$

where $r(n)$ denotes the number of representations of n as a sum of two squares (note that the contribution of the terms with $n=0$ or $m=0$ is negligible).

The result will be a consequence of the previous bound if we prove

$$\Big|\sum_{n\asymp N}\sum_{m\in I_i} r(n)e(R'h(m,n))\Big|^2 \ll MN^{1+\varepsilon}(|T_{NML}|+MNL^{-1}). \tag{3.3}$$

This inequality is nothing but a version of the Weyl–van der Corput inequality (compare with Theorem 2.21 of [Kr] and Lemma β' of [Ti]). To prove it, note first that

$$L\sum_{n\asymp N}\sum_{m\in I_i} r(n)e(R'h(n,m)) = \sum_{m}\sum_{n\asymp N} r(n)\sum_{l\le L}\phi_{m+l}$$

where $\phi_{m+l}=e(R'h(n,m+l))$ if $m+l$ belongs to I_i and $\phi_{m+l}=0$ otherwise. Applying Cauchy's inequality twice, we get

$$\begin{aligned} L^2\Big|\sum_{n\asymp N}\sum_{m\in I_i} r(n)e(R'h(n,m))\Big|^2 &\ll M\sum_{m}\Big(\sum_{n\asymp N} r(n)\Big|\sum_{l\le L}\phi_{m+l}\Big|\Big)^2 \\ &\ll M\sum_{m} N^{1+\varepsilon}\sum_{n\asymp N}\Big|\sum_{l\le L}\phi_{m+l}\Big|^2. \end{aligned}$$

Expanding the square in the latter expression, we have

$$\begin{aligned} &L^2\Big|\sum_{n\asymp N}\sum_{m\in I_i} r(n)e(R'h(n,m))\Big|^2 \\ &\qquad\ll MN^{1+\varepsilon}\sum_{m}\sum_{n\asymp N}\sum_{l_1\le L}\sum_{l_2\le L}\phi_{m+l_1}\overline{\phi}_{m+l_2}. \end{aligned}$$

Changing the variable m to $m-l_2$, extracting the diagonal contribution coming from $l_1=l_2$ and writing $l=l_1-l_2$, we conclude that the right hand side is majorized by

$$MN^{1+\varepsilon}L\sum_{l\le L}\Big|\sum_{m\in I_i}\sum_{n\asymp N}\phi_{m+l}e(-R'h(n,m))\Big| + M^2N^{2+\varepsilon}L,$$

which proves (3.3). ∎

4. Estimation of the exponential sum. In this section we proceed to estimate T_{NML}. Our arguments depend on the study of the functions F_1 and F_2 appearing in the next lemma. Note that the convexity of $-f_1$ and f_2 allows us to prove that $-f_1', f_2' : (0,r_0)\to\mathbb{R}^+$ are one-to-one. Let ϕ_1 and ϕ_2 be their inverse functions.

LEMMA 4.1. *For $n>0$ and $m\ne 0$,*

$$\frac{\partial}{\partial m}h(n,m) = \begin{cases} F_1(n/m^2) & \text{for } m>0,\\ F_2(n/m^2) & \text{for } m<0,\end{cases}$$

where $F_i(u)=f_i(\phi_i(\sqrt{u}))$, $i=1,2$.

Proof. By (3.2),

$$h(n,m)=\sqrt{n}\,\phi_i\Big(\frac{\sqrt{n}}{|m|}\Big)+mf_i\Big(\phi_i\Big(\frac{\sqrt{n}}{|m|}\Big)\Big)$$

where $i=1$ if $m>0$ and $i=2$ if $m<0$. Hence

$$\frac{\partial}{\partial m}h(n,m)$$
$$=\frac{n}{m|m|}\phi_i'\Big(\frac{\sqrt{n}}{|m|}\Big)+f_i\Big(\phi_i\Big(\frac{\sqrt{n}}{|m|}\Big)\Big)+\frac{\sqrt{n}}{|m|}\phi_i'\Big(\frac{\sqrt{n}}{|m|}\Big)f_i'\Big(\phi_i\Big(\frac{\sqrt{n}}{|m|}\Big)\Big).$$

As $f_1'(\phi_1(u))=-u$ and $f_2'(\phi_2(u))=u$, the first and last terms cancel out. ∎

A fundamental point is to take control of the second derivative of F_i in order to apply van der Corput estimates.

LEMMA 4.2. *There exist positive constants C_1 and C_2 such that*

$$C_1<(1+u)^{5/2}|F_i''(u)|<C_2\quad \textit{for every } u>0.$$

Proof. We only consider $i=2$, the case $i=1$ is completely similar. From Lemma 4.1 (recall that ϕ_2 is the inverse function of f_2'),

$$F_2'(u)=f_2'(\phi_2(\sqrt{u}))\frac{\phi_2'(\sqrt{u})}{2\sqrt{u}}=\frac{1}{2f_2''(\phi_2(\sqrt{u}))},$$

hence

$$F_2''(u)=-\frac{f_2'''(\phi_2(\sqrt{u}))}{4\sqrt{u}(f_2''(\phi_2(\sqrt{u})))^3}.$$

By our hypothesis, $\mathbb{E}\in\mathcal{R}_3^*$, f_2''' does not vanish, thus it is enough to prove that

$$L_1=\lim_{u\to 0^+}F_2''(u)\quad\text{and}\quad L_2=\lim_{u\to\infty}u^{5/2}F_2''(u)$$

are finite and non-zero.

As $\mathbb{E}$ is a smooth convex body, the curvature $k(t)$ of $t\mapsto(t,f_2(t))$, $0\le t\le r_0$, is positive. Using the formula (see p. 11 of [Sp])

$$f_2''(t)=k(t)(1+|f_2'(t)|^2)^{3/2} \tag{4.1}$$

and $\phi_2(x)=x/k(0)+O(x^2)$, since the Taylor expansion $f_2(t)=a+k(0)t^2/2+\ldots$ (see p. 41 of [Sp]), we obtain

$$L_1=-\lim_{u\to 0^+}\frac{f_2'''(\sqrt{u}/k(0))}{4\sqrt{u}(k(0))^3}=-\frac{1}{4(k(0))^4}\lim_{r\to 0^+}\frac{f_2'''(r)}{r},$$

which is non-zero by our hypothesis ($f_2'''(r)/r$ does not vanish for $0\le r<r_0$).

Note that $\phi_2(x)\to r_0$ as $x\to\infty$ (because $f_2'(r_0^-)=\infty$). Thus changing $\sqrt{u}$ to $f_2'(t)$ we have

$$L_2=-\lim_{t\to r_0^-}\frac{(f_2'(t))^4 f_2'''(t)}{4(f_2''(t))^3}=-\frac{1}{4}\lim_{t\to r_0^-}\frac{(f_2'(t))^6}{(f_2''(t))^2}\cdot\lim_{t\to r_0^-}\frac{f_2'''(t)}{f_2''(t)(f_2'(t))^2}.$$

The first limit is $(k(r_0))^{-2}$ by (4.1), and we can apply l'Hôpital rule to the second one to obtain

$$L_2=-\frac{3}{4(k(r_0))^2}\lim_{t\to r_0^-}\frac{f_2''(t)}{(f_2'(t))^3},$$

which gives, by (4.1), $L_2=-3/(4k(r_0))$. ■

LEMMA 4.3. *If* $N,M,L\ge 1$ *then*

$$T_{NML}\ll\begin{cases}R^{1/2}L^{1/2}N^{-1/4}M^{3/2}+R^{-1/2}L^{-1/2}N^{5/4}M^{1/2} & \text{for } N\ge M^2,\\ R^{1/2}L^{1/2}NM^{-1}+R^{-1/2}L^{-1/2}M^3 & \text{for } N\le M^2.\end{cases}$$

Proof. By the mean value theorem and Lemma 4.1,

$$\frac{\partial^2}{\partial n^2}(h(n,m+l)-h(n,m))=l\frac{\partial^3}{\partial n^2\partial m}h(n,\widetilde{m})=\frac{l}{\widetilde{m}^4}F_i''\Big(\frac{n}{\widetilde{m}^2}\Big)$$

for some $\widetilde{m}\in[m,m+l]$ and $i\in\{1,2\}$. By Lemma 4.2,

$$\frac{\partial^2}{\partial n^2}(h(n,m+l)-h(n,m))\asymp lM^{-4}(1+NM^{-2})^{-5/2}.$$

Hence, by van der Corput's well known estimate (see, for instance, Theorem 2.2 of [Gr-Ko]), we have

$$\sum_{n\asymp N}e(R'(h(n,m+l)-h(n,m)))$$
$$\ll N(RlN^{-5/2}M)^{1/2}+(RlN^{-5/2}M)^{-1/2}$$

for $N\ge M^2$, and

$$\sum_{n\asymp N}e(R'(h(n,m+l)-h(n,m)))\ll N(RlM^{-4})^{1/2}+(RlM^{-4})^{-1/2}$$

for $N\le M^2$. Substituting these bounds in the definition of T_{NML} we conclude the proof. ■

5. Proof of Theorems 1.1 and 1.2. In this section we combine Proposition 3.1 and Lemma 4.3 to prove Theorem 1.1. We also give a proof of Theorem 1.2 using easier arguments.

Proof of Theorem 1.1. Let S_1 and S_2 be the arguments of the supremum in Proposition 3.1 when $M^2<N$ and when $N\le M^2$, respectively. By Lemma 4.3 we have

$$S_1\ll R^{1/4}L^{1/4}M^{5/4}N^{-5/8}+R^{-1/4}L^{-1/4}M^{3/4}N^{1/8}+L^{-1/2}M$$

and

$$S_2\ll R^{1/4}L^{1/4}M^{-2}N+R^{-1/4}L^{-1/4}N^{1/2}+L^{-1/2}M^{-1}N.$$

If $R^{-1/3}M^{-1/3}N^{5/6} \geq 1$ we choose $L = R^{-1/3}M^{-1/3}N^{5/6}$ in S_1, obtaining

$$S_1 \ll R^{1/6}M^{7/6}N^{-5/12} + R^{-1/6}M^{5/6}N^{-1/12}.$$

On the other hand, if $R^{-1/3}M^{-1/3}N^{5/6} < 1$ it is easy to prove that $M < R^{1/4}$, $N < R^{1/2}$ (use the fact that $N > M^2$) and $S_1 \ll R^{1/4}$ after the choice $L = 1$. Hence we have

$$S_1 \ll R^{1/6}M^{7/6}N^{-5/12} + R^{-1/6}M^{5/6}N^{-1/12} + R^{1/4} \quad \text{for } N \geq M^2.$$

In the same way, if we choose $L = R^{-1/3}M^{4/3}$ or $L = 1$ in S_2, depending on whether $R^{-1/3}M^{4/3} \geq 1$ or $R^{-1/3}M^{4/3} < 1$, respectively, we obtain

$$S_2 \ll R^{1/6}M^{-5/3}N + R^{-1/6}M^{-1/3}N^{1/2} + R^{1/4} \quad \text{for } N < M^2.$$

Substituting in Proposition 3.1 we get

$$\mathcal{E}(R) \ll (R^2\delta + R^{7/6}\delta^{-1/3} + R^{5/6}\delta^{-2/3} + R^{5/4})R^{\varepsilon}.$$

Finally, choosing $\delta = R^{-5/8}$ yields the result. ∎

Proof of Theorem 1.2. Consider a parametrization of the generatrix of $\mathbb{E}$ of the form $t \mapsto (\psi(t), t)$, $a \leq t \leq b$. With the previous notation we have $\psi(f_i(t)) = t$ for $0 \leq t \leq r_0$ and $a = f_2(0)$, $b = f_1(0)$.

It is plain (consider horizontal sections of $\mathbb{E}$) that

$$\mathcal{N}(R) = \sum_{aR \leq n \leq bR} \#\{\vec{m} \in \mathbb{Z}^{d-1} : \|\vec{m}\| \leq R\psi(n/R)\}.$$

As mentioned in the introduction, $\alpha_d = d-3$ holds for the $(d-1)$-dimensional ball when $d > 4$, hence for every $\varepsilon > 0$,

$$\mathcal{N}(R) = c_{d-1} \sum_{aR \leq n \leq bR} (R\psi(n/R))^{d-1} + O(R^{d-2+\varepsilon}) \tag{5.1}$$

where c_{d-1} is the volume of the unit $(d-1)$-ball.

The convexity of $\mathbb{E}$ implies, for $r \to 0$, $f_i(r) = f_i(0) + (-1)^i K_i r^2 + O(r^4)$ with $K_i > 0$; thus the vanishing order of ψ at the endpoints is exactly $1/2$. Consequently, $(\psi^{d-1})''(a^+)$ and $(\psi^{d-1})''(b^-)$ are finite and ψ^{d-1} has a bounded second derivative in $[a, b]$. After these considerations, a double partial integration in the Poisson summation formula gives

$$\begin{aligned}\sum_{aR \leq n \leq bR} (\psi(n/R))^{d-1} &= \int_{aR}^{bR} (\psi(t/R))^{d-1}\,dt \\ &\quad - \int_{aR}^{bR} \sum_{n=1}^{\infty} \frac{\cos(2\pi n t)}{2\pi^2 n^2} \cdot \frac{d^2}{dt^2}(\psi(t/R))^{d-1}\,dt \\ &= R\int_a^b (\psi(t))^{d-1}\,dt + O(R^{-1}).\end{aligned}$$

Substituting in (5.1) we have

$$\mathcal{N}(R) = R^d \int_a^b c_{d-1}(\psi(t))^{d-1}\, dt + O(R^{d-2+\varepsilon}).$$

The integral equals $|\mathbb{E}|$, therefore $\alpha_d \le d-2$. ∎

6. Other results. There are linear transformations leaving $\mathcal{R}_3^*$ invariant. For instance, consider

$$\mathbb{E}_{\lambda\mu} = \{(x,y,z) \in \mathbb{R}^3 : (x,y,\lambda(z-\mu)) \in \mathbb{E}\}, \qquad \lambda \ne 0,\ \mu \in \mathbb{R};$$

if $\mathbb{E} \in \mathcal{R}_3^*$ and $\vec{0} \in \mathrm{Int}(\mathbb{E}_{\lambda\mu})$, then $\mathbb{E}_{\lambda\mu} \in \mathcal{R}_3^*$. It is also obvious that the dilations preserve $\mathcal{R}_3^*$.

The purpose of this section is to show that there are some other linear transformations not leaving $\mathcal{R}_3^*$ invariant but producing off-centered bodies with elliptic sections to which our methods can be applied. Namely, consider the translation

$$T(x,y,z) = (x-\tau_1, y-\tau_2, z), \qquad \tau_1,\tau_2 \in \mathbb{R},$$

and the action of a non-singular matrix, $A = (a_{ij}) \in \mathcal{M}_{2\times 2}(\mathbb{R})$, over the two first variables:

$$G(x,y,z) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y, z).$$

Theorem 6.1. *Let T and G be as before. If $\mathbb{E} \in \mathcal{R}_3^*$, $\vec{0} \in \mathrm{Int}(TG\mathbb{E})$ and $A \cdot A^t \in \mathcal{M}_{2\times 2}(\mathbb{Q})$, then the bound $\alpha_3 \le 11/8$ also holds for $TG\mathbb{E}$.*

Remark. Of course, $TG\mathbb{E}$ is only an abbreviation for $\{TG(x,y,z) : (x,y,z) \in \mathbb{E}\}$. Note (see the proof) that the condition $\vec{0} \in \mathrm{Int}(TG\mathbb{E})$ is superfluous and it is only imposed to preserve the original geometrical meaning of the problem.

Proof (of Theorem 6.1). After a dilation of $\mathbb{E}$ we can assume that $A \cdot A^t$ has integral coefficients, i.e.

$$Q(n_1,n_2) = \|(n_1,n_2)A\|^2$$

is an integral binary quadratic form.

For $TG\mathbb{E}$, the terms under summation in Proposition 2.1 are

$$|A|\eta(\delta\|\vec{n}\|)\frac{\cos(2\pi R'(g(T^t\vec{n}) - \tau_1 n_1 - \tau_2 n_2))}{\|T^t\vec{n}\|^2\sqrt{\kappa(T^t\vec{n})}}$$

where T^t is defined as T with A replaced by A^t, and the functions g and κ

(and h later) correspond to $\mathbb{E}$. This can be checked by direct calculations ([1]) but it is much easier to follow the steps of the proof of Proposition 2.1 noting that the Fourier transform of the characteristic function of $R'TG\mathbb{E}$, $\chi_{R'}(G^{-1}T^{-1}\vec{x})$, is

$$|A|e(-R'(\tau_1\xi_1+\tau_2\xi_2))\widehat{\chi}_{R'}(T^t\vec{\xi}).$$

As $g(x,y,z)=g(\sqrt{x^2+y^2},z)=h(x^2+y^2,z)$, we have

$$g(a_{11}n_1+a_{21}n_2,a_{12}n_1+a_{22}n_2,m)=h(n,m)$$

where $n=Q(n_1,n_2)$. Therefore the bound of Proposition 3.1 (and hence the final result $\alpha_3\le 11/8$) holds not only for $\mathbb{E}$ but for $TG\mathbb{E}$ if we are able to prove (3.3) with $r(n)$ replaced by

$$\widetilde{r}(n)=\sum_{Q(n_1,n_2)=n} e(-R'(\tau_1n_1+\tau_2n_2)).$$

But this change does not affect the arguments used there because the bound $r(n)\ll n^{\varepsilon}$ (the only property of $r(n)$ employed in the proof) is also satisfied by $\widetilde{r}(n)$. ■

The centered sphere belongs to $\mathcal{R}_3^*$ (note that if $f(x)=(1-x^2)^{1/2}$ then $f'''(x)/x=3(1-x^2)^{-5/2}\ne 0$), hence $\mathbb{E}:\alpha(x^2+y^2)+\beta(z-\gamma)^2\le 1$ also belongs to $\mathcal{R}_3^*$ if $\vec{0}\in\mathrm{Int}(\mathbb{E})$. Applying the previous theorem to $\mathbb{E}$ shows at once

COROLLARY 6.2. *Let $\mathbb{E}$ be an ellipsoid in $\mathbb{R}^3$ of the form*

$$a_{11}x^2+a_{12}xy+a_{22}y^2+a_{33}z^2+a_1x+a_2y+a_3z\le 1.$$

If $a_{22}/a_{11}, a_{12}/a_{11}\in\mathbb{Q}$, then the bound $\alpha_3\le 11/8$ holds for $\mathbb{E}$.

As far as we know, this result has not been stated before, but in this particular case it can be improved using more sophisticated methods of exponential sums. In fact, exponent pairs theory and the Discrete Hardy–Littlewood Method (see [Gr-Ko] and [Hu]) could be used to improve Theorem 1.1 (via a sharper estimation of T_{NML}) in the cases where some complicated functions involving higher derivatives of f_i do not vanish; but, as we mentioned in the introduction, our main here interest is to show how the symmetries of a body can be exploited to get general bounds from the simplest van der Corput's estimate. In order to illustrate the situation, we

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([1]) Doing these calculations requires to know the behaviour of κ when $\mathbb{E}$ is replaced by $L\mathbb{E}$ with $L\in\mathcal{M}_{3\times3}(\mathbb{R})$. The needed formula is

$$\kappa_{L\mathbb{E}}(\vec{n})\cdot\|\vec{n}\|^4|L|^2=\kappa_{\mathbb{E}}(L^t\vec{n})\cdot\|L^t\vec{n}\|^4.$$

Although this answers a natural question (how Gaussian curvature changes under general linear transformations) we have not found this formula in geometry textbooks. We are indebted to J. Gonzalo for supplying us with a proof.

shall finish this section stating a result showing that the exponential sums estimates of J.-R. Chen [Ch] and I. M. Vinogradov [Vi] (in the sharper and simplified form given in Lemma 3.1 of [Ch-Iw]) can be used to get the best known result for a certain class of ellipsoids.

THEOREM 6.3 (cf. [Ch] and [Vi]). *Let* $\mathbb{E}$ *be an ellipsoid of the form*

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz + a_1x + a_2y + a_3z \le 1.$$

If $a_{ij}/a_{11} \in \mathbb{Q}$, $1 \le i \le j \le 3$, *then the bound* $\alpha_3 \le 4/3$ *holds for* $\mathbb{E}$.

Proof. Obviously we can write $\mathbb{E}$ (or an ellipsoid homothetic to $\mathbb{E}$) as

$$\|\vec{x} - \vec{\tau}\|_* \le 1 \quad \text{where } \vec{\tau} \in \mathbb{R}^3,\ \vec{x}^t = (x, y, z) \text{ and } \|\vec{x}\|_* = \vec{x}^t A \vec{x}$$

where A is a rational positive definite matrix.

A calculation proves (use the formula for the curvature of an ellipsoid or the previous footnote)

$$g(\vec{n}) = \sqrt{Q(\vec{n})} - \vec{\tau} \cdot \vec{n} \quad \text{and} \quad \|\vec{n}\|^2 \sqrt{\kappa(\vec{n})} = Q(\vec{n})$$

where $Q(\vec{n}) = \vec{n}^t (A^{-1})^t \vec{n}$. Note that $Q(\vec{n}) = aL_1^2 + bL_2^2 + cL_3^2$, where $a, b, c \in \mathbb{Q}$ and $L_i = L_i(\vec{n})$, $i = 1, 2, 3$, are linear forms with rational coefficients, and after a dilation of $\mathbb{E}$ we can assume that a, b, c and these coefficients are actually integers.

On the other hand, it is easy to check that in Proposition 3.1, $\eta(\delta\|\vec{n}\|)$ can be replaced by $\widetilde{\eta}(\delta\|\vec{n}\|_*)$ with analogous properties (see the first steps of the proof and note that $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent norms), hence

$$\mathcal{E}(R) \ll R^{2+\varepsilon}\delta + R^{1+\varepsilon} \sup_{N<\delta^{-2}} N^{-1} V_N \tag{6.1}$$

where

$$V_N = \sum_{n \asymp N} r_*(n) e(R'\sqrt{n}) \quad \text{and} \quad r_*(n) = \sum_{aL_1^2 + bL_2^2 + cL_3^2 = n} e(-R'\vec{\tau} \cdot \vec{n}).$$

By Lemma 3.1 of [Ch-Iw] (with minor modifications)

$$V_N \ll N^{5/4+\varepsilon} + N^{\varepsilon} \min(R^{3/8} N^{15/16} + R^{1/8} N^{17/16},\ R^{7/24} N^{49/48} + R^{5/24} N^{53/48}).$$

Substituting in (6.1) and choosing $\delta = R^{-2/3}$ we obtain $\alpha_3 \le 4/3$. ∎

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