Zero density estimates of $L$-functions associated with cusp forms

by

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1. Introduction. Let $k$ be a positive even integer, and $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi iz}$ a holomorphic cusp form of weight $k$ with respect to $\Gamma = SL_2(\mathbb{Z})$. We denote by $S_k(\Gamma)$ the space of those functions. Let $q$ be a positive integer, and $\chi$ a Dirichlet character mod $q$. Let $s = \sigma + it$ be a complex variable. We define the $L$-function by

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s}$$

for $\sigma > (k+1)/2$. Denote by $\chi^*$ the primitive character mod $q_1$ inducing $\chi$. It is known that the function $L_f(s, \chi^*)$ has an analytic continuation to the whole complex plane and satisfies the functional equation (see [5])

$$\left(\frac{2\pi}{q_1}\right)^{-s} \Gamma(s)L_f(s, \chi^*) = i^k \left(\frac{W(\chi^*)}{|W(\chi^*)|}\right)^2 \left(\frac{2\pi}{q_1}\right)^{s-k} \Gamma(k-s)L_f(k-s, \overline{\chi^*}),$$

where $W(\chi^*)$ is Gaussian sum and $\Gamma(s)$ is the gamma function. Moreover, if the cusp form $f$ is the normalized eigenform, that is, the eigenfunction of all Hecke operators with $a(1) = 1$, then $a(n)$'s are real numbers and $L_f(s, \chi)$ has the Euler product expansion

$$L_f(s, \chi) = \prod_p (1 - \chi(p)a(p)p^{-s} + \chi(p)^2 p^{k-1-2s})^{-1}$$

for $\sigma > (k+1)/2$, where the product runs over all prime numbers. Therefore, $L_f(s, \chi)$ has the representation

$$L_f(s, \chi) = L_f(s, \chi^*) \prod_{p|q} (1 - \chi^*(p)a(p)p^{-s} + \chi^*(p)^2 p^{k-1-2s}),$$

and (1) gives the analytic continuation of $L_f(s, \chi)$ to the whole complex plane for every $\chi$. We can also see that $L_f(s, \chi)$ has no zeros for $\sigma > (k+1)/2$.  

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has simple zeros at non-positive integers, and has no zeros for \( \sigma < (k - 1)/2 \) except non-positive integers. We call zeros at non-positive integers trivial, and those lying in \((k - 1)/2 \leq \sigma \leq (k + 1)/2\) non-trivial. Since \( a(n) \)'s are real, we have the relation \( L_f(s, \overline{\chi}) = L_f(s, \chi) \) for any \( s \). If \( \chi \) is a primitive character, from this relation and the functional equation, non-trivial zeros of \( L_f(s, \chi) \) are distributed symmetrically with respect to the line \( \sigma = k/2 \). In case \( \chi \) is an imprimitive character, non-trivial zeros of \( L_f(s, \chi) \) are those of \( L_f(s, \chi^*) \) and infinite zeros on \( \sigma = (k - 1)/2 \) which are coming from the finite products in (1).

For the purpose of counting the number of non-trivial zeros, we define

\[
N_f(T, \chi) = \# \{ \eta = \beta + i\gamma \mid L_f(\eta, \chi) = 0, (k - 1)/2 \leq \beta \leq (k + 1)/2, -T \leq \gamma \leq T \},
\]

for \( \sigma_0 \geq k/2 \). We can show the following results by modifying the proof for the case of Dirichlet \( L \)-functions in an obvious way (see [1]). We have

\[
N_f(T + 1, \chi) - N_f(T - 1, \chi) \leq C \log(q(T + 1)),
\]

for any \( T \geq 1 \) and some positive constant \( C \). We also have

\[
N_f(T, \chi) = \frac{2T}{\pi} \log \frac{qT}{2\pi} + O(T \log(q + 1)), \quad T \to \infty,
\]

uniformly in \( q \). In particular, for a primitive character \( \chi \),

\[
N_f(T, \chi) = \frac{2T}{\pi} \log \frac{qT}{2\pi} - \frac{2T}{\pi} + O(\log(qT)), \quad T \to \infty,
\]

uniformly in \( q \).

The purpose of this paper is to show the following theorem.

**Theorem 1.** Let \( f \in S_k(\Gamma) \) be the normalized eigenform and \( \chi \) a Dirichlet character mod \( q \). If \( q \ll T \), then

\[
\sum_{\chi} N_f(\sigma_0, T, \chi) \ll (qT)^{k+1-2\sigma_0/2} (\log(qT))^{69}, \quad T \to \infty,
\]

uniformly in \( \sigma_0 \) and \( q \) for \( k/2 + 1/\log(qT) \leq \sigma_0 \leq k/2 + 1/3 \), and

\[
\sum_{\chi} N_f(\sigma_0, T, \chi) \ll (qT)^{3(k+1-2\sigma_0)/2} (\log(qT))^{100}, \quad T \to \infty,
\]

uniformly in \( \sigma_0 \) and \( q \) for \( k/2 + 1/3 \leq \sigma_0 \leq (k + 1)/2 \), where \( \sum_{\chi} \) means a sum running over all Dirichlet characters mod \( q \).

Specialising \( q = 1 \) in Theorem 1, we have

\[
N_f(\sigma_0, T, \chi_0) \ll T^{k+1-2\sigma_0/2} (\log T)^{69}, \quad T \to \infty,
\]
uniformly for \( k/2 + 1/\log T \leq \sigma_0 \leq k/2 + 1/3 \),

\[ N_f(\sigma_0, T, \chi_0) \ll T^{3(k+1-2\sigma_0)/2}(\log T)^{100}, \quad T \to \infty, \]

uniformly for \( k/2 + 1/3 \leq \sigma_0 \leq (k+1)/2 \), where \( \chi_0 \) is the trivial character.

As regards the estimate of \( N_f(\sigma_0, T, \chi_0) \), Ivić has shown in [4] that

\[ N_f(\sigma_0, T, \chi_0) \ll T^{k+1-2\sigma_0 + \varepsilon}, \quad T \to \infty, \]

for \( k/2 \leq \sigma_0 \leq k/2 + 1/4 \),

\[ N_f(\sigma_0, T, \chi_0) \ll T^{k+1-2\sigma_0 + \varepsilon}, \quad T \to \infty, \]

for \( k/2 + 1/4 \leq \sigma_0 \leq (k+1)/2 \), and also has shown sharper bounds when \( \sigma_0 \) is near \((k+1)/2\). Therefore, Theorem 1 is a natural extension of Ivić’s results for \( k/2 + 1/\log T \leq \sigma_0 \leq k/2 + 1/4 \).

Theorem 1 is an analogue of zero density estimates of Dirichlet \( L \)-functions by Montgomery [6]. Montgomery used the estimate of the mean fourth power of Dirichlet \( L \)-functions on the critical line for this problem. Since the corresponding fourth power result is not known in our case, we shall use the mean square of \( L_f(s, \chi) \) to prove Theorem 1 (see Theorem 2 in Section 3).

To estimate the mean square of \( L_f(s, \chi) \), we reduce the problem to the study of the mean square of the Dirichlet polynomial by using the approximate functional equation of \( L_f(s, \chi) \), which is proved by applying the method of Good [3].

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2. The approximate functional equation. Throughout this section, we suppose \( f \) is in \( S_k(\Gamma) \) and \( \chi \) is a primitive character mod \( q \). We shall prove the approximate functional equation of \( L_f(s, \chi) \) whose implied constant is uniform in \( q \), following the method of Good [3].

Rankin has shown in [7] that

\[ \sum_{n \leq x} |a(n)|^2 = Cx^k + O(x^{k-2/5}), \quad x \to \infty, \]

where \( C \) is a positive constant depending on \( k \). By Cauchy’s inequality,

\[ \sum_{n \leq x} |a(n)| \ll x^{(k+1)/2}, \quad x \to \infty, \]

hence we obtain the following lemma by partial summation.

**Lemma 1.** Let \( \sigma \) be a real number. Then

\[ \sum_{n \leq x} |a(n)|n^{-\sigma} \ll x^{(k+1)/2-\sigma}, \quad x \to \infty, \]
uniformly for $\sigma \leq \sigma_1 < (k+1)/2$, and

\[
\sum_{n \leq x} |a(n)|^2 n^{-2\sigma} \ll \begin{cases} 
  x^{k-2\sigma} & \text{uniformly for } \sigma \leq \sigma_2 < k/2, \\
  \log x & \text{uniformly for } k/2 - 1/\log x \leq \sigma \leq k/2 + 1/\log x,
\end{cases}
\]

where $\sigma_1$ and $\sigma_2$ are constants.

Following the notation in [3], let $\varphi(q)$ be a real-valued function in $[0, \infty)$ which is infinitely differentiable and satisfies $\varphi(q) = 1$ for $0 \leq q \leq 1/2$ and $\varphi(q) = 0$ for $q \geq 2$. We denote by $\Phi$ the set of those functions. The function $\varphi_0(q) = 1 - \varphi(1/q)$ is also an element of $\Phi$. For $\varphi$ in $\Phi$ and for a complex variable $w = u + iv$ with $u > 0$, let

\[
K_\varphi(w) = w \int_0^\infty \varphi(q) q^{w-1} dq.
\]

The function $K_\varphi(w)$ has an analytic continuation to the whole complex $w$-plane, because the relation

\[
K_\varphi(w) = -2 \int_{1/2}^\infty \varphi'(q) q^w dq
\]

can be verified by integration by parts. Let $\varphi^{(j)}$ denote the $j$th derivative of $\varphi$ and define

\[
\|\varphi^{(j)}\|_1 = \int_0^\infty |\varphi^{(j)}(q)| dq.
\]

For $\tau > 0$, $t \neq 0$, and $j = 0, 1, \ldots$, let

\[
\gamma_j(s, \tau) = \frac{1}{2\pi i \Gamma(s)} \int_{\mathcal{F}} \Gamma(s + w) \left( \tau \exp\left(-i \frac{\pi}{2} \text{sgn}(t) \right) \right)^w \frac{dw}{w(w+1) \cdots (w+j)},
\]

where $\text{sgn}(t) = t/|t|$ and $\mathcal{F}$ means that integration is taken over the curve which encircles $w = 0, -1, \ldots, -j$. If $j = 0$, it is easy to see that $\gamma_0(s, \tau) = 1$ for any $s$. In case $j \neq 0$, it was shown in [3] that

\[
\gamma_j(s, |t|^{-1}) \ll \begin{cases} 
  |t|^{-(j+1)/2} & \text{for odd } j, \\
  |t|^{-j/2} & \text{for even } j,
\end{cases}
\]

uniformly for $\sigma$ which is in a fixed strip. For $x > 0$ and $\varphi$ in $\Phi$, let

\[
G_f(s, x; \varphi, \chi) = \frac{1}{2\pi i \Gamma(s)} \prod_{k=2}^{k/2+1-\sigma} \Gamma(s + w) L_{ij}(s + w, \chi) \frac{K_\varphi(w)}{w} \times \left( \frac{q x}{2\pi} \exp\left(-i \frac{\pi}{2} \text{sgn}(t) \right) \right)^w.
\]
where \( \mathbb{I}_{(k/2+1-\sigma)} \) means that integration is taken over the vertical line \( u = k/2 + 1 - \sigma \).

We can derive the following lemma by modifying Satz of [3].

**Lemma 2.** Let \( x > 0, \varphi \in \Phi, f \in S_k(\Gamma), \) and \( \chi \) a primitive character mod \( q \). Then the following properties hold.

(a) For \( (k-1)/2 \leq \sigma \leq (k+1)/2 \),
\[
\left( \frac{2\pi}{q} \right)^{-s} \Gamma(s)L_f(s, \chi) = \left( \frac{2\pi}{q} \right)^{-s} \Gamma(s)G_f(s, \varphi, \chi) + i^k \left( \frac{W(\chi)}{|W(\chi)|} \right)^2 \left( \frac{2\pi}{q} \right)^{s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \times G_f(k-s, x^{-1}; \varphi_0, \chi),
\]
where the implied constant is uniform in \( \sigma, \varphi, \) and \( q \) for \( (k-1)/2 \leq \sigma \leq (k+1)/2 \).

(b) Let \( y = qx|t|/(2\pi) \) and \( l \) an integer with \( l > (k+1)/2 \). For \( |t| > t^2 \),
\[
G_f(s, x; \varphi, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} \sum_{j=0}^{l} \varphi(j) \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right)^j \gamma_j(s, |t|^{-1}) + O(\|\varphi^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-1/2}),
\]
where the implied constant is uniform in \( \sigma, \varphi, \) and \( q \) for \( (k-1)/2 \leq \sigma \leq (k+1)/2 \).

Put \( x = 1 \) and \( y = q|x|/(2\pi) \) in Lemma 2. Then we have
\[
L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} \varphi \left( \frac{n}{y} \right) + i^k \left( \frac{W(\chi)}{|W(\chi)|} \right)^2 \left( \frac{2\pi}{q} \right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^k-s} \varphi_0 \left( \frac{n}{y} \right) + R(s),
\]
where
\[
R(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} \sum_{j=1}^{l} \varphi(j) \left( \frac{n}{y} \right) \left( -\frac{n}{y} \right)^j \gamma_j(s, |t|^{-1}) + i^k \left( \frac{W(\chi)}{|W(\chi)|} \right)^2 \left( \frac{2\pi}{q} \right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^k-s} \varphi_0 \left( \frac{n}{y} \right) \gamma_j(k-s, |t|^{-1}) + O(\|\varphi^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-1/2}) + O(\|\varphi_0^{(l+1)}\|_1 y^{(k+1)/2-\sigma}|t|^{-1/2}).
\]
Now we fix a $\varphi$. By (5) and (7), we have
\[
R(s) \ll \sum_{j=1}^{l} |\gamma_j(s, |t|^{-1})| \sum_{n \leq 2y} \frac{|a(n)|}{n^\sigma} \left(\frac{n}{q|t|}\right)^j
\]
\[
+ \left(\frac{2\pi}{q}\right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{j=1}^{l} |\gamma_j(k-s, |t|^{-1})| \sum_{n \leq 2y} \frac{|a(n)|}{n^{k-\sigma}} \left(\frac{n}{q|t|}\right)^j
\]
\[
+ (q|t|)^{(k+1)/2-\sigma}|t|^{-l/2}
\]
\[
\ll (q|t|)^{(k+1)/2-\sigma}|t|^{-1}.
\]
Therefore we have

**Lemma 3.** Let $\varphi \in \Phi$, $f \in S_k(\Gamma)$, $\chi$ a primitive character mod $q$, and $\kappa = 2\pi/q$. Then
\[
L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} \varphi \left(\frac{\kappa n}{|t|}\right)
\]
\[
+ i^k \left(\frac{W(\chi)}{|W(\chi)|}\right)^2 \kappa^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^{k-s}} \varphi_0 \left(\frac{\kappa n}{|t|}\right)
\]
\[
+ O((q|t|)^{(k+1)/2-\sigma}|t|^{-1}),
\]
where the implied constant is uniform in $\sigma$ and $q$ for $(k-1)/2 \leq \sigma \leq (k+1)/2$.

3. **The mean square of** $L_f(s, \chi)$. Throughout this section, we suppose $f$ is in $S_k(\Gamma)$ and $\chi$ is a Dirichlet character mod $q$. The aim of this section is to estimate the mean square
\[
\sum_{\chi}^* \int_{-T}^{T} |L_f(\sigma + it, \chi)|^2 dt
\]
uniformly in $\sigma$ and $q$ for $k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT)$, where $\sum^*$ means a sum running over all primitive characters mod $q$.

We need the following lemmas.

**Lemma 4.** Let $0 < \delta < \delta_1$, and let $\varphi(\varrho)$ be a real-valued function in $[0, \infty)$ which is twice continuously differentiable and satisfies $\varphi(\varrho) = 1$ for $0 \leq \varrho \leq \delta$ and $\varphi(\varrho) = 0$ for $\varrho \geq \delta_1$. Let $m$ and $n$ be positive integers, $\kappa$ and $T$ positive real numbers, and $\beta$ a real number which satisfies $-1 < A \leq \beta \leq B < 1$ for some constants $A, B$. Then
\[ \int_0^T \varphi \left( \frac{\kappa t}{T} \right) \varphi \left( \frac{\kappa m}{t} \right) t^{-\beta} \cos \left( t \log \frac{n}{m} \right) dt = \begin{cases} 0 & \text{for } m \geq T \delta_1 / \kappa \text{ or } n \geq T \delta_1 / \kappa, \\ O(T^{1-\beta}) & \text{for } m = n < T \delta / \kappa, \\ O((\kappa n)^{-1-\beta}) & \text{for } m = n \geq T \delta / \kappa, \\ \frac{1}{\log \frac{n}{m}} \sin \left( T \log \frac{n}{m} \right) \varphi \left( \frac{\kappa n}{T} \right) \varphi \left( \frac{\kappa m}{T} \right) T^{-\beta} + O \left( \frac{(\kappa \max(n,m))^{-\beta-1}}{(\log(n/m))^2} \right) & \text{for } m \neq n, \end{cases} \]

where the implied constants are uniform in \( m, n, \kappa, \) and \( \beta. \)

It is easy to prove Lemma 4 by modifying the proof of Lemma 7 of [3].

**Lemma 5.** Let \( f \in S_k(\Gamma) \) and \( \chi \) a Dirichlet character mod \( q. \) Let \( \varepsilon \) be a positive real number and assume \((k - \varepsilon)/2 < \sigma < (k + \varepsilon)/2.\) If \(|t| \leq C\) for some positive constant \( C,\) then

\[ \sum_{\chi} |L_f(s, \chi)|^2 \ll_C \phi(q) q^{k-2\sigma + 2\varepsilon} \left( \int_1^\infty u^{2\sigma-k-1-\varepsilon} du + \int_1^\infty u^{k-2\sigma-1-\varepsilon} du \right) \]

uniformly in \( \sigma \) and \( q, \) where \( \phi \) is the Euler function.

**Proof.** By the automorphic property of \( \sum_{n=1}^\infty \chi(n) a(n) e^{2\pi i n z}, \) which is the twist of \( f \) by the primitive character \( \chi, \)

\[ \left( \frac{2\pi}{q} \right)^{-s} \Gamma(s) L_f(s, \chi) = \int_0^\infty u^{s-1} \sum_{n=1}^\infty \chi(n) a(n) e^{-2\pi n u/q} du \]

\[ = \int_1^\infty u^{s-1} \sum_{n=1}^\infty \chi(n) a(n) e^{-2\pi n u/q} du \]

\[ + i^k \left( \frac{W(\chi)}{|W(\chi)|} \right)^2 \int_1^\infty u^{k-s-1} \sum_{n=1}^\infty \overline{\chi(n)} a(n) e^{-2\pi n u/q} du. \]

Hence

\[ \left( \frac{2\pi}{q} \right)^{-\sigma} \left| \Gamma(s) \right| \cdot \left| L_f(s, \chi) \right| \leq \int_1^\infty u^{\sigma-1} \left| \sum_{n=1}^\infty \chi(n) a(n) e^{-2\pi n u/q} \right| du \]

\[ + \int_1^\infty u^{k-\sigma-1} \left| \sum_{n=1}^\infty \overline{\chi(n)} a(n) e^{-2\pi n u/q} \right| du. \]

By squaring both sides above and taking \( \sum_{\chi}, \) we have
(8) \[
\frac{1}{2} \left( \frac{2\pi}{q} \right)^{-2\sigma} |\Gamma(s)|^2 \sum\chi |L_f(s, \chi)|^2 \\
\leq \sum\chi^* \left( \int_1^\infty u^{\sigma-1} \left| \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right| du \right)^2 \\
+ \sum\chi^* \left( \int_1^\infty u^{k-\sigma-1} \left| \sum_{n=1}^\infty \overline{\chi(n)a(n)e^{-2\pi nu/q}} \right| du \right)^2.
\]
Let \( \alpha \) be real. By Cauchy’s inequality,
\[
\sum\chi^* \left( \int_1^\infty u^{\alpha-1} \left| \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right| du \right)^2 \\
\leq \sum\chi^* \left( \int_1^\infty u^{2\alpha-1+\epsilon} \left| \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right|^2 du \int_1^\infty u^{-1-\epsilon} du \right) \\
\ll \epsilon \int_1^\infty u^{2\alpha-1+\epsilon} \sum\chi \left( \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right)^2 du.
\]
Here,
\[
\sum\chi \left| \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right|^2 \\
= \phi(q) \sum_{n=1}^\infty \sum_{m=1}^\infty \pi(n)a(m)e^{-2\pi(n+m)u/q} \\
\leq \phi(q) \sum_{n=1}^\infty \sum_{m=1}^\infty (|a(n)|^2 + |a(m)|^2)e^{-2\pi(n+m)u/q} \\
\leq \phi(q) \sum_{n=1}^\infty |a(n)|^2 e^{-2\pi nu/q} \sum_{r=0}^\infty e^{-2\pi r u} \\
\ll \phi(q) \sum_{n=1}^\infty |a(n)|^2 e^{-2\pi nu/q}.
\]
By using partial summation, the right-hand side is
\[
\ll \phi(q) \frac{u^k}{q} \int_1^\infty x^k e^{-2\pi xu/q} dx \\
\ll \epsilon \phi(q) \frac{u^k}{q} \int_1^\infty x^k \left( \frac{xu}{q} \right)^{-k-1-2\epsilon} dx \\
\ll \epsilon \phi(q) \left( \frac{u}{q} \right)^{-k-2\epsilon}.
\]
Hence we have
\[\sum_{\chi}^* \left( \int_1^\infty u^{-1} \left| \sum_{n=1}^\infty \chi(n)a(n)e^{-2\pi nu/q} \right| du \right)^2 \ll \epsilon^2 \phi(q)q^{k+2\epsilon} \int_1^\infty u^{2\alpha-k-1-\epsilon} \, du.\]

Substituting this into (8), we obtain the assertion of Lemma 5.

**Theorem 2.** Let \( f \in S_k(\Gamma) \) and \( \chi \) a Dirichlet character mod \( q \). If \( q \ll T \), then
\[\sum_{\chi}^* \int_{-T}^T |L_f(\sigma+it,\chi)|^2 \, dt \ll \phi(q)T \log(qT), \quad T \to \infty,\]
uniformly in \( \sigma \) and \( q \) for \( k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT) \).

**Proof.** Denote the right-hand side of the formula in the statement of Lemma 3 by \( f_1 + f_2 + f_3 \), say. Let \( C_0 \) be a positive constant for which
\[f_3(\sigma+it) \ll (q|t|)^{(k+1)/2 - \sigma |t|^{-1}}\]
for \( |t| \geq C_0 \). Put
\[A_{\mu\nu}(\sigma,C_0) = \int_{[-T,T]-[-C_0,C_0]} f_{\mu}(\sigma+it)f_{\nu}(\sigma+it) \, dt, \quad \mu, \nu = 1, 2, 3.\]

By Cauchy’s inequality,
\[\left| \sum_{\chi}^* A_{\mu\nu}(\sigma,C_0) \right| \leq \left( \sum_{\chi}^* A_{\mu\mu}(\sigma,C_0) \right)^{1/2} \left( \sum_{\chi}^* A_{\nu\nu}(\sigma,C_0) \right)^{1/2} \leq \frac{1}{2} \sum_{\chi}^* A_{\mu\mu}(\sigma,C_0) + \frac{1}{2} \sum_{\chi}^* A_{\nu\nu}(\sigma,C_0).\]

Hence we have
\[\sum_{\chi}^* \int_{-T}^T |L_f(\sigma+it,\chi)|^2 \, dt \ll \sum_{\mu,\nu=1}^3 \sum_{\chi}^* A_{\mu\nu}(\sigma,C_0) + \sum_{\chi}^* \int_{-C_0}^C_0 |L_f(\sigma+it,\chi)|^2 \, dt \ll \sum_{\nu=1}^3 \sum_{\chi}^* A_{\nu\nu}(\sigma,C_0) + \sum_{\chi}^* \int_{-C_0}^C_0 |L_f(\sigma+it,\chi)|^2 \, dt.\]

We use Lemma 5 with \( \epsilon = 1/2 \) for \( k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT) \) and \( |t| \leq C_0 \) to obtain
\[\sum_{\chi}^* \int_{-C_0}^C_0 |L_f(\sigma+it,\chi)|^2 \, dt \ll C_0 \phi(q).\]
By (9), we have

\[
\sum_{\chi}^{\ast} A_{33}(\sigma, C_0) \ll \phi(q)q^{2/\log(qT)+1} \int_{C_0}^{T} t^{2/\log(qT)-1} \, dt \\
\ll \phi(q)q \log T.
\]

Substituting (11) and (12) into (10), gives

\[
\sum_{\chi}^{\ast} \int_{-T}^{T} |L_f(\sigma + it, \chi)|^2 \, dt \ll \sum_{\chi}^{\ast} A_{11}(\sigma) + \sum_{\chi}^{\ast} A_{22}(\sigma) + \phi(q)q \log T,
\]

where

\[
A_{\nu\nu}(\sigma) = \int_{-T}^{T} |f_{\nu}(\sigma + it)|^2 \, dt, \quad \nu = 1, 2.
\]

First, we estimate \(\sum_{\chi}^{\ast} A_{11}(\sigma)\). We have

\[
\sum_{\chi}^{\ast} A_{11}(\sigma) \leq \sum_{\chi} \int_{-T}^{T} |f_{1}(\sigma + it)|^2 \, dt \\
= 2\phi(q) \sum_{n<2T/\kappa} \sum_{m<2T/\kappa} |a(n)|^2 \sum_{(n,q)=1}^{\sigma} (nm)^\sigma \\
\times \int_{0}^{T} \phi\left(\frac{\kappa n}{t}\right) \varphi\left(\frac{\kappa m}{t}\right) \cos\left(t \log \frac{n}{m}\right) \, dt \\
= 2\phi(q) \left\{ \sum_{n<T/(2\kappa)} \frac{|a(n)|^2}{n^{2\sigma}} \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right)^2 \, dt \\
+ \sum_{T/(2\kappa) \leq n < 2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right)^2 \, dt \\
+ \sum_{0} a(n) |a(m)| \int_{0}^{T} \varphi\left(\frac{\kappa n}{t}\right) \varphi\left(\frac{\kappa m}{t}\right) \cos\left(t \log \frac{n}{m}\right) \, dt \right\},
\]

where we set

\[
\sum_{0} = \sum_{n<2T/\kappa} \sum_{m<2T/\kappa} \sum_{(n,q)=1}^{\sigma} (nm)^\sigma \\
\sum_{m \equiv m \, (q)} \sum_{n \not\equiv m \, (q)}.
\]
Applying Lemma 4, we have

\[
\sum_{\chi} \Lambda_{11}(\sigma) \ll \phi(q) \left\{ T \sum_{n<T/(2\kappa)} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{1}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}} \right. \\
+ \sum_{n<2T/\kappa} \frac{|a(n)a(m)|}{(nm)^\sigma |\log \frac{n}{m}|} \\
+ q \sum_{n<2T/\kappa} \frac{|a(n)a(m)|}{(nm)^\sigma \max(n, m)(\log \frac{n}{m})^2} \right\}.
\]

The third sum on the right-hand side is

\[
\leq \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{|\log \frac{n}{m}|} \\
= \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{|\log \frac{n}{m}|} + \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \atop m \not\equiv n \pmod{q}}} \frac{1}{|\log \frac{n}{m}|}.
\]

In the first term we put \( m = n - qr \) to get

\[
\sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{|\log \frac{n}{m}|} < \frac{n}{q} \sum_{1 \leq r<T/(q\kappa)} \frac{1}{r} \ll \frac{n}{q} \log T,
\]

and in the second term we put \( m = n + qr \) to get

\[
\sum_{\substack{m<n \atop m \not\equiv n \pmod{q}}} \frac{1}{|\log \frac{n}{m}|} < \sum_{1 \leq r<T/(q\kappa)} \frac{n+qr}{qr} \ll T + \frac{n}{q} \log T.
\]

Therefore we have

\[
\sum_{n<2T/\kappa} \frac{|a(n)a(m)|}{(nm)^\sigma |\log \frac{n}{m}|} \ll T \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{\log T}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}}.
\]

Next, the fourth sum on the right-hand side of (14) is

\[
\leq \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{\max(n, m)(\log \frac{n}{m})^2} \\
= \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma+1}} \sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{(\log \frac{n}{m})^2} + \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} \sum_{\substack{m<n \atop m \equiv n \pmod{q}}} \frac{1}{m(\log \frac{n}{m})^2}.
\]
In the first term we put $m = n - qr$ to get
\[ \sum_{m < n \atop m \equiv n \pmod{q}} \frac{1}{\left( \log \frac{n}{m} \right)^2} < \frac{n^2}{q^2} \sum_{1 \leq r < n/q} \frac{1}{r^2} \ll \frac{n^2}{q^2}, \]
and in the second term we put $m = n + qr$ to get
\[ \sum_{n < m < \frac{2T}{\kappa} \atop m \equiv n \pmod{q}} \frac{1}{m \left( \log \frac{n}{m} \right)^2} < \sum_{1 \leq r < \frac{2T}{\kappa q \kappa}} \frac{1}{n + qr} \left( \frac{n + qr}{qr} \right)^2 \ll \frac{n^2}{q^2} + \frac{1}{q} \log T. \]
Therefore we have
\[ q \sum_{0} \frac{|a(n)a(m)|}{(nm)^\sigma \max(n, m) \left( \log \frac{n}{m} \right)^2} \ll (\log T) \sum_{n < 2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{1}{q} \sum_{n < 2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}}. \]
Substituting (15) and (16) to (14), we obtain
\[ \sum_{\chi} A_{11}(\sigma) \ll \phi(q) \left( T \sum_{n < 2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma}} + \frac{1}{q} \sum_{n < 2T/\kappa} \frac{|a(n)|^2}{n^{2\sigma-1}} \right). \]
Combining this estimate with (6), we obtain
\[ \sum_{\chi} A_{22}(\sigma) \ll \phi(q) T \log(qT), \quad T \to \infty, \]
uniformly in $\sigma$ and $q$ for $k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT)$.
Second, we estimate $\sum_{\chi} A_{22}(\sigma)$. We have
\[ \sum_{\chi} A_{22}(\sigma) \leq \sum_{\chi} T \int_{-T}^{T} |f_2(\sigma + it)|^2 \, dt \]
\[ = 2\phi(q) \kappa^{2(2\sigma-k)} \sum_{n < 2T/\kappa} \sum_{m < 2T/\kappa} \frac{|a(n)|}{nm^{k-\sigma}} \sum_{n \equiv m \pmod{q}} \frac{\varphi_0(kn/t) \varphi_0(km/t) \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 \cos \left( t \log \frac{n}{m} \right) \, dt.} \]
Note that the interval $[0, T]$ of integration can be replaced by an interval $[(\kappa/2) \max(n, m), T]$, because $\varphi_0(kn/t)\varphi_0(km/t) = 0$ for $0 \leq t \leq (\kappa/2) \max(n, m)$. By Stirling’s formula, we have
\[ \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 = |t|^{2(k-2\sigma)} \left( 1 + O \left( \frac{1}{t^2} \right) \right) \]
for $0 < \sigma < k$ and $|t| \geq C_1$, where $C_1$ is some positive constant. In case $n$ and $m$ satisfy $C_1 \leq (\kappa/2) \max(n,m)$, we have

$$
\int_{(\kappa/2) \max(n,m)}^{T} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 \cos \left( t \log \frac{n}{m} \right) \, dt
$$

$$
= \int_{(\kappa/2) \max(n,m)}^{T} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) t^{2(k-2\sigma)} \cos \left( t \log \frac{n}{m} \right) \, dt + O(1)
$$

uniformly in $\sigma$ and $q$ for $k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT)$. The same result also holds in case $C_1 > (\kappa/2) \max(n,m)$, because in this case

$$
\int_{(\kappa/2) \max(n,m)}^{C_1} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 \cos \left( t \log \frac{n}{m} \right) \, dt = O(1)
$$

and

$$
\int_{(\kappa/2) \max(n,m)}^{C_1} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) t^{2(k-2\sigma)} \cos \left( t \log \frac{n}{m} \right) \, dt = O(1).
$$

Let us denote

$$
\sum_1 = \sum_{n < T/(2\kappa)} \sum_{\substack{n < T/(2\kappa) \\ (n,q) = 1 \\ \sigma \in (\kappa/2) \max(n,m)}} \sum_{\substack{n \equiv m \pmod{q} \\ (m,q) = 1}}
$$

and $\sum_0$ is as before. From the above result, it follows that

$$
\sum_1 \overline{a(n)} a(m) (nm)^{k-\sigma}
$$

$$
\times \int_{(\kappa/2) \max(n,m)}^{T} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) \left| \frac{\Gamma(k-s)}{\Gamma(s)} \right|^2 \cos \left( t \log \frac{n}{m} \right) \, dt
$$

$$
= \sum_1 \overline{\pi(n)} a(m) (nm)^{k-\sigma}
$$

$$
\times \int_{(\kappa/2) \max(n,m)}^{T} \varphi_0 \left( \frac{kn}{t} \right) \varphi_0 \left( \frac{km}{t} \right) t^{2(k-2\sigma)} \cos \left( t \log \frac{n}{m} \right) \, dt
$$

$$
+ O \left( \sum_1 |a(n)| a(m) (nm)^{k-\sigma} \right)
$$

$$
= \sum_{n < T/(2\kappa)} \frac{|a(n)|^2}{n^{2(k-\sigma)}} \int_{0}^{T} \varphi_0 \left( \frac{kn}{t} \right)^2 t^{2(k-2\sigma)} \, dt.
$$
\[\begin{align*}
+ & \sum_{\substack{n<2T/\kappa \leq n<2T/\kappa \equiv 1 \mod q}} \left| a(n) \right|^2 \frac{T}{n^{2(k-\sigma)}} \int_0^T \varphi_0 \left( \frac{\kappa n}{t} \right)^2 t^{2(k-2\sigma)} \, dt \\
& + \sum_{0} \frac{\pi(n)a(m)}{(nm)^{k-\sigma}} \int_0^T \varphi_0 \left( \frac{\kappa n}{t} \right) \varphi_0 \left( \frac{\kappa m}{t} \right) t^{2(k-2\sigma)} \cos \left( t \log \frac{n}{m} \right) \, dt \\
& + O \left( \sum_1 \left| a(n)a(m) \right| \right). \end{align*}\]

Since \(-4/\log(qT) \leq -2(k-2\sigma) \leq 4/\log(qT)\), by using Lemma 4, we see that the right-hand side of the above is

\[\ll T \sum_{n<T/(2\kappa)} \frac{|a(n)|^2}{n^{2(k-\sigma)}} + \frac{1}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2(k-\sigma)-1}} + \sum_0 \frac{|a(n)a(m)|}{(nm)^{k-\sigma} |\log \frac{n}{m}|} \]

\[+ q \sum_0 \frac{|a(n)a(m)|}{(nm)^{k-\sigma} \max(n,m) \left( \log \frac{n}{m} \right)^2} + \sum_1 \frac{|a(n)a(m)|}{(nm)^{k-\sigma}} \]

uniformly in \(\sigma\) and \(q\) for \(k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT)\). By (15), (16), and the estimate

\[\sum_1 \frac{|a(n)a(m)|}{(nm)^{k-\sigma}} \leq \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2(k-\sigma)}} \sum_{m<2T/\kappa} \frac{1}{n^{2(k-\sigma)}} \ll T \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2(k-\sigma)}}, \]

we have

\[\sum_\chi A_{22}(\sigma) \ll \phi(q) \left( T \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2(k-\sigma)}} + \frac{\log T}{q} \sum_{n<2T/\kappa} \frac{|a(n)|^2}{n^{2(k-\sigma)-1}} \right), \]

hence, by (6), we obtain

\[(18) \sum_\chi A_{22}(\sigma) \ll \phi(q) T \log(qT), \quad T \to \infty, \]

uniformly in \(\sigma\) and \(q\) for \(k/2 - 1/\log(qT) \leq \sigma \leq k/2 + 1/\log(qT)\).

Combining (13), (17), (18), and the assumption \(q \ll T\), we obtain the assertion of Theorem 2.

**Corollary 1.** Under the same notation as in Theorem 2, we have

\[\sum_\chi \int_{-T}^T \left| L_f(k/2 + it, \chi) \right|^2 \, dt \ll \phi(q) T (\log(qT))^3, \quad T \to \infty, \]

uniformly in \(q\).
Proof. Put \( r = (\log(qT))^{-1} \). Since

\[
|L'_f(k/2 + it, \chi)|^2 \ll r^{-3} \int_{|z-k/2-it|=r} |L_f(z, \chi)|^2|dz|,
\]

we have

\[
\sum^* \chi \int_{-T}^{T} |L'_f(k/2 + it, \chi)|^2|dt \\ll r^{-3} \sum^* \chi \int_{|z-k/2-it|=r} |L_f(z, \chi)|^2|dz| |dt|.
\]

From Theorem 2, it follows that

\[
\sum^* \chi \int_{-T}^{T} \int_{|z-k/2-it|=r} |L_f(z, \chi)|^2|dz| |dt|
\leq 2 \left\{ \int_{k/2-r}^{k/2+r} \left( \sum^* \chi \int_{-T}^{T} |L_f(\sigma + it, \chi)|^2|dt| \right)^{3/2} d\sigma \right\}^{1/2}
\times 2 \left\{ \int_{k/2-r}^{k/2+r} \left( 1 - \left( \frac{\sigma - k/2}{r} \right)^2 \right)^{-1/4} d\sigma \right\}^{2/3}
\ll r \phi(q) T \log(qT) \ll \phi(q) T.
\]

This proves the corollary.

Corollary 2. Let \( \chi \) be a Dirichlet character mod \( q \), and \( \chi^* \) the primitive character inducing \( \chi \). Let \( \delta \) be a positive real number such that \( \delta \ll T \), and \( T\chi^* \) a finite subset of \([-T, T]\) with \(|t-t'| \geq \delta\) for any distinct \( t \) and \( t' \) in \( T\chi^* \). If \( q \ll T \), then

\[
\sum \chi \sum_{t \in T\chi^*} |L_f(k/2 + it, \chi^*)|^2 \ll \left( \frac{1}{\delta} + \log(qT) \right) qT \log(qT), \quad T \to \infty,
\]

uniformly in \( q \).

Corollary 2 can be derived from Theorem 2 and Corollary 1 by the same argument as the proof of Corollary 10.4 of [6].

4. Proof of Theorem 1. Our argument is a modification of the proof of the zero density estimates of Dirichlet \( L \)-functions in [6], so we give only a sketch.
Let $L_f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ for $\sigma > (k+1)/2$. We define $\mu_f(n)$ by

$$\frac{1}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s}$$

for $\sigma > (k+1)/2$. By the Euler product expansion of $L_f(s)$ and the estimate $|a(n)| \leq n^{(k-1)/2}d(n)$ (see [2]), where $d(n)$ is the divisor function, it is easy to see that the following properties hold:

$$|\mu_f(n)| \leq n^{(k-1)/2}d(n),$$

$$\sum_{d|n, d>0} \mu_f(d)a\left(\frac{n}{d}\right) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $L_f(s, \chi)$ and $L_f(s, \chi^*)$ have the same zeros for $\sigma \geq k/2$, it is enough to consider $N_f(\sigma_0, T, \chi^*)$ instead of $N_f(\sigma_0, T, \chi)$. Let $A_1$ be a positive real number, and let $X$ and $Y$ be parameters satisfying $2 \leq X \leq Y \leq (qT)^{A_1}$. We define

$$M(s, \chi^*) = \sum_{n\leq X} \frac{\mu_f(n)\chi^*(n)}{n^s}.$$

Then it follows that, for $\sigma > (k+1)/2$,

$$L_f(s, \chi^*)M(s, \chi^*) = \sum_{n=1}^{\infty} \frac{h(n)\chi^*(n)}{n^s},$$

where $h(n) = \sum_{d|n, 0<d\leq X} \mu_f(d)a(n/d)$ has the following properties: $h(1) = 1$, $h(n) = 0$ for $2 \leq n \leq X$, and $|h(n)| \leq n^{(k-1)/2}d(n)^3$ for $n > X$. By using the Mellin integral formula, we have

$$e^{-1/Y} + \sum_{n>X} h(n)\chi^*(n)n^{-s}e^{-n/Y}$$

$$= \frac{1}{2\pi i} \int_{(k+1)/2+1-i\infty}^{(k+1)/2+1+i\infty} L_f(s+w, \chi^*)M(s+w, \chi^*)Y^w \Gamma(w) \, dw$$

for $\sigma > -1$. Let $\varrho = \beta + i\gamma$ be a zero of $L_f(s, \chi^*)$ such that $\sigma_0 \leq \beta \leq (k+1)/2$ and $-T \leq \gamma \leq T$, and take $s = \varrho$ in the equation above. Since $L_f(\varrho + w, \chi^*)M(\varrho + w, \chi^*)Y^w \Gamma(w)$ is holomorphic for $-1/2 \leq \Re w$, the path of integration in the above can be moved to the line $\Re w = k/2 - \beta$. Therefore, if $Y$ is large, every $\varrho$ counted by $N_f(\sigma_0, T, \chi^*)$ has at least one of the following properties:

(a) $\left| \sum_{X<n \leq Y^2} h(n)\chi^*(n)n^{-\varrho}e^{-n/Y} \right| \geq \frac{1}{5},$
where \( z = A_2 \log(qT) \) for a large absolute constant \( A_2 \). Let \( \mathcal{R}(\chi^*) \) be a set of \( \varrho \)'s which are well-spaced, that is, \( 3z \leq |\gamma - \gamma'| \) for any distinct \( \varrho = \beta + i\gamma \) and \( \varrho' = \beta' + i\gamma' \). We denote by \( R(\chi^*) \) the number of elements of \( \mathcal{R}(\chi^*) \).

From (2) and the definition of \( R(\chi^*) \), it follows that

\[
N_f(\sigma_0, T, \chi^*) \ll R(\chi^*)(\log(qT))^2,
\]

hence

\[
\sum_{\chi} N_f(\sigma_0, T, \chi) = \sum_{\chi} N_f(\sigma_0, T, \chi^*) \ll R(\log(qT))^2,
\]

where \( R = \sum_{\chi} R(\chi^*) \). The sets \( \mathcal{R}_1(\chi^*) \) and \( \mathcal{R}_2(\chi^*) \) are defined to be the subsets of \( \mathcal{R}(\chi^*) \) such that every element of \( \mathcal{R}_1(\chi^*) \) satisfies the condition (a), and every element of \( \mathcal{R}_2(\chi^*) \) satisfies the condition (b). Denote by \( R_j(\chi^*) \) the number of elements of \( \mathcal{R}_j(\chi^*) \), \( j = 1, 2 \). Put

\[
\mathcal{R}_j = \bigcup_{\chi} \mathcal{R}_j(\chi^*) \quad \text{and} \quad R_j = \sum_{\chi} R_j(\chi^*), \quad j = 1, 2,
\]

and we shall estimate \( R_1 \) and \( R_2 \).

First, we estimate \( R_1 \). For every \( \varrho \) in \( \mathcal{R}_1 \),

\[
\max_{1 \leq l \leq l_0 + 1} \left\{ \left| \sum_{2^{l-1}X < n \leq 2^lX} h(n)\chi^*(n)n^{-\varrho}e^{-n/Y} \right| \right\} \geq \frac{1}{15\log Y}
\]

for large \( Y \), where \( l_0 = \left\lfloor (\log 2)^{-1}\log(X^{-1}Y^2) \right\rfloor \). Hence, there exists \( U \) such that \( X < U \leq Y^2 \) and the inequality

\[
\left| \sum_{U < n \leq 2U} \sum_{n \leq Y^2} h(n)\chi^*(n)n^{-\varrho}e^{-n/Y} \right| \geq \frac{1}{15\log Y}
\]

holds for more than \( R_1/(4\log Y) \) zeros of \( \mathcal{R}_1 \). Therefore, by Theorem 7.6 of [6],

\[
R_1 \ll (\log Y)^3 \sum_{\chi} \sum_{\varrho \in \mathcal{R}_1(\chi^*)} \left| \sum_{U < \varrho \leq 2U} \sum_{n \leq Y^2} h(n)\chi^*(n)n^{-\varrho}e^{-n/Y} \right|^2
\]

\[
\ll (qTX^{k-2\sigma_o} + Y^{k+1-2\sigma_o})(\log(qT))^{67}.
\]
Second, we estimate $R_2$. For every $\varrho$ in $\mathcal{R}_2$,
\[
\int_{-z}^{z} |L_f(k/2 + i(\gamma + \varrho), \chi^*)M(k/2 + i(\gamma + \varrho), \chi^*)| \times Y^{k/2 - \beta} |\Gamma(k/2 - \beta + iv)| \, dv \geq \frac{2\pi}{5}.
\]

Let $t_\varrho = \gamma + \varrho$ be a value for which $|L_f(k/2 + i(\gamma + \varrho), \chi^*)M(k/2 + i(\gamma + \varrho), \chi^*)|$ is maximal. Since $z/CK - z|\Gamma(k/2 - \beta + iv)|dv \ll 1/CK - 1/\beta - k/2dv \ll \log(qT)$, we have
\[
|L_f(k/2 + it_\varrho, \chi^*)M(k/2 + it_\varrho, \chi^*)| \gg Y^{\sigma_0 - k/2(\log(qT))^{-1}}.
\]

Hence,
\[
Y^{\sigma_0 - k/2(\log(qT))^{-1}} R_2 \ll \sum_{\chi} \sum_{\varrho \in \mathcal{R}_2(\chi^*)} |L_f(k/2 + it_\varrho, \chi^*)M(k/2 + it_\varrho, \chi^*)|
\leq \left( \sum_{\chi} \sum_{\varrho \in \mathcal{R}_2(\chi^*)} |L_f(k/2 + it_\varrho, \chi^*)|^2 \right)^{1/2}
\times \left( \sum_{\chi} \sum_{\varrho \in \mathcal{R}_2(\chi^*)} |M(k/2 + it_\varrho, \chi^*)|^2 \right)^{1/2}.
\]

Since $|t_\varrho - t_\varrho^\prime| \geq z$, we can use Corollary 2 under the assumption $q \ll T$:
\[
\sum_{\chi} \sum_{\varrho \in \mathcal{R}_2(\chi^*)} |L_f(k/2 + it_\varrho, \chi^*)|^2 \ll qT(\log(qT))^2.
\]

From Theorem 7.6 of [6], if $X \leq qT$, then
\[
\sum_{\chi} \sum_{\varrho \in \mathcal{R}_2(\chi^*)} |M(k/2 + it_\varrho, \chi^*)|^2 \ll qT(\log(qT))^6.
\]

Therefore, if $q \ll T$ and $X \leq qT$, we obtain
\[
(21) \quad R_2 \ll Y^{k/2 - \sigma_0} qT(\log(qT))^5.
\]

Substituting (20) and (21) into (19) gives
\[
\sum_{\chi} N_f(\sigma_0, T, \chi) \ll (qT X^{k-2\sigma_0} + Y^{k+1-2\sigma_0} + qTY^{k/2-\sigma_0})(\log(qT))^69,
\]
and putting $X = qT$, $Y = (qT)^{(k/2+1-\sigma_0)}$, we now obtain (3) uniformly in $\sigma_0$ and $q$ for $k/2 + 1/\log(qT) \leq \sigma_0 \leq (k + 1)/2$. 
Finally, the estimate (4) can be derived by a different treatment of $R_1$ and $R_2$. This is almost identical to the proof of Theorem 12.1 of [6], so we omit the details.

References


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