ACTA ARITHMETICA
LXXXV.2 (1998)

On sums of two cubes: an Ω_+-estimate for the error term

by

M. Kühleitner (Wien), W. G. Nowak (Wien),
J. Schoissengeier (Wien) and T. D. Wooley (Ann Arbor, Mich.)

To Professor Harald Rindler on his 50th birthday

The arithmetic function $r_k(n)$ counts the number of ways to write a natural number n as a sum of two kth powers ($k \geq 2$ fixed). The investigation of the asymptotic behaviour of the Dirichlet summatory function of $r_k(n)$ leads in a natural way to a certain error term $P_{\mathcal{D}_k}(t)$ which is known to be $O(t^{1/4})$ in mean-square. In this article it is proved that $P_{\mathcal{D}_3}(t) = \Omega_+(t^{1/4}(\log\log t)^{1/4})$ as $t \to \infty$. Furthermore, it is shown that a similar result would be true for every fixed $k > 3$ provided that a certain set of algebraic numbers contains a sufficiently large subset which is linearly independent over $\mathbb{Q}$.

1. Introduction. For a bounded convex planar domain $\mathcal{D}$ and a large real parameter t, let $A_{\mathcal{D}}(t)$ denote the number of lattice points (of the standard lattice $\mathbb{Z}^2$) in the "blown up" set $\sqrt{t}\mathcal{D}$, and define as usual the "lattice rest" as $P_{\mathcal{D}}(t) := A_{\mathcal{D}}(t) - \operatorname{area}(\mathcal{D})t$. For the special case where $\mathcal{D}$ is the unit circle $\mathcal{D}_2$ (say), the question of the asymptotic behaviour of $A_{\mathcal{D}_2}(t)$ is the classical circle problem of C. F. Gauß: its history has been described, e.g., in the book of Krätzel [15]. At present, the best results are

$$P_{\mathcal{D}_2}(t) = O(t^{23/73}(\log t)^{315/146}), \tag{1.1}$$

and (1) (for some constant ($c > 0$))

$$P_{\mathcal{D}_2}(t) = \Omega_-(t^{1/4}(\log t)^{1/4}(\log\log t)^{(\log 2)/4}\exp(-c\sqrt{\log\log\log t})). \tag{1.2}$$

1991 *Mathematics Subject Classification*: 11P21, 11N37.

(1) We recall that $F_1(t) = \Omega_*(F_2(t))$ means that $\limsup(*F_1(t)/F_2(t)) > 0$ as $t \to \infty$ where $*$ is either $+$ or $-$ and $F_2(t)$ is positive for t sufficiently large.

They are due to Huxley [11], [12], and Hafner [4], [5], respectively. In fact, (1.2) was a celebrated and comparatively recent refinement of Hardy's classical bound [7], [8]

$$P_{\mathcal{D}_2}(t) = \Omega_-(t^{1/4}(\log t)^{1/4}). \tag{1.3}$$

An Ω_+-estimate (somewhat weaker than (1.2) and (1.3)) has been established by Corrádi and Kátai [2], refining earlier work of Gangadharan [6]:

$$P_{\mathcal{D}_2}(t) = \Omega_+(t^{1/4}\exp(c'(\log\log t)^{1/4}(\log\log\log t)^{-3/4})) \qquad (c' > 0). \tag{1.4}$$

It is usually conjectured that

$$\inf\{\theta \in \mathbb{R} : P_{\mathcal{D}_2}(t) \ll_\theta t^\theta\} = 1/4. \tag{1.5}$$

In favour of this hypothesis, there is the mean-square result

$$\int_0^T (P_{\mathcal{D}_2}(t))^2\,dt = CT^{3/2} + O(T(\log T)^2), \qquad C = \frac{1}{3\pi^2}\sum_{n=1}^{\infty}\frac{(r_2(n))^2}{n^{3/2}}, \tag{1.6}$$

which in this sharp form is due to Kátai [14].

For general $\mathcal{D}$ whose boundary is sufficiently smooth and of nonzero curvature throughout, (1.1) remains true, according to Huxley [11], [12] as well. The second named author [17], [18] showed that (1.3) can be generalized unchanged, as well as a weak form of (1.6), namely

$$\int_0^T (P_{\mathcal{D}}(t))^2\,dt \ll T^{3/2}.$$

Roughly speaking, this means that

$$A_{\mathcal{D}}(t) = \mathrm{area}(\mathcal{D})t + O(t^{1/4}) \quad \text{in mean-square}, \tag{1.7}$$

but that there exists an unbounded sequence of values t for which $\sqrt{t}\mathcal{D}$ contains *less* lattice points than (1.7) would suggest.

Now the open question is: Are there also arbitrarily large t-values for which $\sqrt{t}\mathcal{D}$ contains *more* lattice points than could be expected according to (1.7)—as is known for the circle by (1.4)?

There does not seem to be any chance to attack this problem in general, since the present methods that yield results like (1.4) depend entirely on the algebraic nature of the equation of $\partial\mathcal{D}$. Actually, at least according to the present authors' intuitive feeling, it appears rather more likely that the domain $\sqrt{t}\mathcal{D}$ gets a little extra area quasi "between" lattice points than that the contrary happens. (In this context we remark parenthetically that for the Dirichlet divisor problem, which involves a boundary curve which is convex in the other direction, the Ω_--bound is the more difficult and in fact the weaker one.)

In the present paper we present [apparently as the first example apart from the classical cases] an affirmative answer to the above question for a domain $\mathcal{D}_k$ whose boundary is Lamé's curve $|u|^k + |v|^k = 1$, $k \in \mathbb{N}$. In number-theoretic terms, this involves the arithmetic function $r_k(n)$ which counts the number of ways to write the positive integer n as a sum of the kth powers of two integers taken absolutely:

$$r_k(n) = \#\{(u_1, u_2) \in \mathbb{Z}^2 : |u_1|^k + |u_2|^k = n\}.$$

($k \geq 3$ is a fixed natural number.) To discuss the average order of $r_k(n)$, one is interested in the Dirichlet summatory function

$$A_{\mathcal{D}_k}(t) = \sum_{1 \leq n \leq t^{k/2}} r_k(n),$$

where t is a large real variable. For $k \geq 3$, the asymptotic formula for $A_{\mathcal{D}_k}(t)$ contains a second main term which comes from the points of the boundary curve where the curvature vanishes. It turns out that

$$A_{\mathcal{D}_k}(t) = \operatorname{area}(\mathcal{D}_k)t + B_k \Phi_k(t) t^{1/2-1/(2k)} + P_{\mathcal{D}_k}(t) \tag{1.8}$$

with

$$B_k = 2^{3-1/k}\pi^{-1-1/k}k^{1/k}\Gamma\left(1 + \frac{1}{k}\right),$$

$$\Phi_k(t) = \sum_{n=1}^{\infty} n^{-1-1/k} \sin\left(2\pi n\sqrt{t} - \frac{\pi}{2k}\right).$$

A thorough account on the history (which goes back to van der Corput [22]) and the diverse aspects of this problem can be found in Krätzel's textbook [15].

Using Huxley's deep method in its sharpest form, Kuba [16] proved that the new error term $P_{\mathcal{D}_k}(t)$ again satisfies the estimate (1.1). Quite recently, the second named author [19] has been able to show that this analogy partially extends to the order of the mean-square, i.e.,

$$\int_0^T (P_{\mathcal{D}_k}(t))^2 \, dt \ll T^{3/2} \tag{1.9}$$

for a large real parameter T (the $\ll$-constant possibly depending on k). Furthermore [20], it is again true that

$$P_{\mathcal{D}_k}(t) = \Omega_-(t^{1/4}(\log t)^{1/4}). \tag{1.10}$$

Stating (1.9) as before in the form

$$\begin{aligned} A_{\mathcal{D}_k}(t) = {} & \operatorname{area}(\mathcal{D}_k)t + B_k\Phi_k(t)t^{1/2-1/(2k)} \\ & + O(t^{1/4}) \quad \text{in mean-square,} \end{aligned} \tag{1.11}$$

the question arises again if there are arbitrarily large values of t for which $\sqrt{t}\mathcal{D}_k$ contains *more* lattice points than (1.11) suggests.

As we shall see, this is connected with the problem of whether there exists a sufficiently "large" set of coprime pairs (a_1, a_2) such that the numbers [2] $(a_1^{k/(k-1)} + a_2^{k/(k-1)})^{1-1/k}$ together with 1 are linearly independent over the rationals. If so, an affirmative answer to this question can be given.

THEOREM A. *Let*

$$\mathcal{M}^{(0)} := \{\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2 : a_1 > a_2,\ \gcd(a_1, a_2) = 1\}$$

and suppose that, *for a certain fixed value of* $k \geq 3$, *there exists a set* $\mathcal{M} \subset \mathcal{M}^{(0)}$ *such that*:

(I) *For* $X \to \infty$, *the set*

$$\mathcal{R} := \left\{(a_1, a_2) \in \mathbb{N}^2 : \left(\frac{a_1}{\gcd(a_1, a_2)}, \frac{a_2}{\gcd(a_1, a_2)}\right) \in \mathcal{M}^{(0)} - \mathcal{M}\right\}$$

satisfies

$$\#\{(a_1, a_2) \in \mathcal{R} : \max(a_1, a_2) \leq X\} \ll X^{\lambda}$$

for some exponent $\lambda < 2$,

(II) $\{(a_1^{k/(k-1)} + a_2^{k/(k-1)})^{(k-1)/k} : \mathbf{a} \in \mathcal{M}\} \cup \{1\}$ *is linearly independent over* $\mathbb{Q}$.

Then, *for this particular value of* k,

$$\limsup_{t\to\infty} \left(\frac{P_{\mathcal{D}_k}(t)}{t^{1/4}}\right) = +\infty.$$

In fact we shall verify, by some profound algebra and a recent deep estimate of Heath-Brown [9], that a set $\mathcal{M}$ satisfying the hypotheses (I) and (II) exists for $k = 3$: see Theorem B in Section 3. Thus we derive the following unconditional result.

COROLLARY.

$$\limsup_{t\to\infty} \left(\frac{P_{\mathcal{D}_3}(t)}{t^{1/4}}\right) = +\infty. \tag{1.12}$$

Finally, in Section 4 we shall refine this result to the quantitative estimate

$$P_{\mathcal{D}_3}(t) = \Omega_+(t^{1/4}(\log\log t)^{1/4}).$$

2. Proof of Theorem A

Notation. Here and throughout, $C_1, C_2, \ldots$ denote suitable positive constants which depend at most on k. (This applies to all of the constants implied in the O- and $\ll$-, $\gg$- symbols as well.)

---

(2) This expression arises—roughly speaking—when counting the lattice points of $\sqrt{t}\mathcal{D}_k$ by the Poisson summation formula and evaluating the resulting integrals by the method of stationary phase. In terms of convex geometry, $(u^{k/(k-1)} + v^{k/(k-1)})^{1-1/k}$ is the tac-function of $\mathcal{D}_k$.

For any subset $\mathcal{S}$ of $\mathbb{N}^2$ and positive real X, we put

$$\mathcal{S}_X := \{(s_1, s_2) \in \mathcal{S} : \max(s_1, s_2) \le X\}.$$

Further, for any set $\mathcal{T}$ of real or complex numbers and any positive integer k, we write

$$\mathcal{T}^{[k]} := \{t^k : t \in \mathcal{T}\}.$$

LEMMA 1. *For arbitrary $U \in \mathbb{R}^+$ and any $u \in [U-1, U+1]$, we have*

$$P_{\mathcal{D}_k}(u^2) \ge -\frac{8u^{1/2}}{\sqrt{k-1}}\, S^*(u) - C_1 u^{1/2},$$

where

$$\begin{aligned} S^*(u) = \sum_{(m,h)\in\mathcal{D}(U)} & (hm)^{-1+q/2} |(m,h)|_q^{-q+1/2} \\ & \times (\beta_1(h,U)\sin(-2\pi u|(m,h)|_q + \pi/4) \\ & + \beta_2(h,U)\cos(-2\pi u|(m,h)|_q + \pi/4)), \end{aligned}$$

with

$$\beta_1(h,U) = \frac{1}{\pi}\tau\left(\frac{h}{[U]+1}\right), \qquad \beta_2(h,U) = \frac{h(1-h/([U]+1))}{[U]+1},$$

and where q is related to k by $1/k + 1/q = 1$, i.e., $q = k/(k-1)$. Further, $|\cdot|_q$ denotes the q-norm in $\mathbb{R}^2$, i.e.,

$$|(v_1, v_2)|_q = (|v_1|^q + |v_2|^q)^{1/q},$$

and

$$\tau(w) = \pi w(1-w)\cot(\pi w) + w \quad \textit{for } 0 < w < 1.$$

The domain of summation is given by

$$\mathcal{D}(U) = \{(m,h) \in \mathbb{N}^2 : h \le U,\ h < m \le f'_{h,u}(N_J)\},$$

where

$$f_{h,u}(w) = -h(u^k - w^k)^{1/k},$$

and

$$N_J = u(1 - 2^{-J}(1 - 2^{-1/k})) \quad \textit{with } J = \left[\frac{\log((1-2^{-1/k})(U+1))}{\log 2}\right] + 1.$$

We remark that $f'_{h,u}(N_J)$ is independent of u and $f'_{h,u}(N_J) \asymp h2^{J(1-1/k)}$.

Proof (of Lemma 1). The proof is analogous to that of Nowak [20], formulae (3.2), (3.5), (3.6), by a trivial modification of the method used there. (The key step is again the transition from fractional parts to trigonometric polynomials via a celebrated inequality of Vaaler [21] which now is applied in the shape $a \le b + |a-b|$ in contrast to $a \ge b - |a-b|$ in [20].)

Lemma 2 (Kronecker's approximation theorem; see, e.g., Hlawka–Schoißengeier–Taschner [10], p. 23). *If* $1,\theta_1,\ldots,\theta_s\in\mathbb{R}$ *are linearly independent over* $\mathbb{Q}$, *and* $\alpha_1,\ldots,\alpha_s\in\mathbb{R}$ *are arbitrary*, $U_0,\varepsilon\in\mathbb{R}^+$, *then there exists an integer* $U>U_0$ *such that*

$$\|U\theta_n-\alpha_n\|<\varepsilon \qquad (n=1,\ldots,s),$$

where $\|\cdot\|$ *denotes the distance from the nearest integer.*

Lemma 3. *For a real parameter* $M\ge 1$, *let* F_M *denote the Fejér kernel*

$$F_M(v)=M\left(\frac{\sin(\pi Mv)}{\pi Mv}\right)^2.$$

Then for arbitrary $Q\in\mathbb{R}^+$ *and* $\gamma\in\mathbb{R}$,

$$\int_{-1}^{1}F_M(v)\cos(2\pi Qv+\gamma)\,dv=\max\left(1-\frac{Q}{M},0\right)\cos\gamma+O\left(\frac{1}{Q}\right),$$

where the O-constant is independent of M *and* γ.

Proof. See Lemma 3 in [20].

Proof of the theorem. As in [20], we multiply $S^*(u)$ by the Fejér kernel $F_M(u-U)$, and integrate over u from $U-1$ to $U+1$. (At this stage, M and U are considered as independent large real parameters.) By the definition of $S^*(u)$ in Lemma 1, we obtain

$$\begin{aligned}(2.1)\quad I(U):=\int_{U-1}^{U+1}S^*(u)F_M(u-U)\,du&=\int_{-1}^{1}S^*(U+v)F_M(v)\,dv\\ &=\sum_{\substack{(m,h)\in\mathcal{D}(U)\\ |(m,h)|_q\le M}}(hm)^{-1+q/2}|(m,h)|_q^{-q+1/2}\left(1-\frac{|(m,h)|_q}{M}\right)\\ &\quad\times(\beta_1(h,U)\sin(-2\pi U|(m,h)|_q+\pi/4)\\ &\quad+\beta_2(h,U)\cos(-2\pi U|(m,h)|_q+\pi/4))\\ &\quad+\sum_{(m,h)\in\mathcal{D}(U)}O((hm)^{-1+q/2}|(m,h)|_q^{-q-1/2}).\end{aligned}$$

Let us first estimate the error term sum: $(m,h)\in\mathcal{D}(U)$ implies that $m>h$ and thus $|(m,h)|_q\asymp m$. Consequently,

$$\sum_{(m,h)\in\mathcal{D}(U)}O((hm)^{-1+q/2}|(m,h)|_q^{-q-1/2})\ll\sum_{h\in\mathbb{N}}h^{-1+q/2}\sum_{m>h}m^{-1+q/2-q-1/2}\ll\sum_{h\in\mathbb{N}}h^{-3/2}\ll 1.$$

The next important step is the application of Kronecker's approximation principle. For fixed M sufficiently large, we choose $\delta_M = \varepsilon_0/M$ ($\varepsilon_0 > 0$ a sufficiently small constant), and appeal to Lemma 2 to find a value $U > M^2$ such that

$$\mathcal{B}(M) := \{(m,h) \in \mathbb{N}^2 : m > h,\ |(m,h)|_q \le M\} \subset \mathcal{D}(U) \tag{2.2}$$

and

$$\big\| U|(m,h)|_q - \tfrac{1}{2} \big\| < \delta_M \tag{2.3}$$

for all $(m,h) \in \mathcal{M}$ with $|(m,h)|_q \le M$. We define the sets

$$\mathcal{B}_1(M) = \left\{ (m,h) \in \mathcal{B}(M) : \left(\frac{m}{\gcd(m,h)}, \frac{h}{\gcd(m,h)} \right) \in \mathcal{M} \right\},$$
$$\mathcal{B}_2(M) = \mathcal{B}(M) \cap \mathcal{R},$$

where $\mathcal{M}$ and $\mathcal{R}$ are as in Theorem A. Therefore, for each $(m,h) = (dm', dh')$ $\in \mathcal{B}_1(M)$ with $\gcd(m',h') = 1$, it follows that

$$\begin{matrix}\cos\\ \sin\end{matrix} \left(-2\pi U|(dm',dh')|_q + \frac{\pi}{4} \right) = (-1)^d \frac{1}{\sqrt{2}} + O(d\delta_M).$$

Furthermore, since

$$h < |(m,h)|_q \le M < \sqrt{U},$$

it is clear that, for M large, $\beta_1(h,U)$ is close to $1/\pi$ throughout, and $\beta_2(h,U)$ is small, hence

$$\beta_1(h,U)\sin(-2\pi U|(m,h)|_q + \pi/4) + \beta_2(h,U)\cos(-2\pi U|(m,h)|_q + \pi/4) \\ \le (-1)^{\gcd(m,h)} \frac{1}{\pi\sqrt{2}} + \varepsilon$$

for all $(m,h) \in \mathcal{B}_1(M)$, where $\varepsilon > 0$ is small whenever ε_0 is small. We note that the domain of summation in (2.1) is in fact $\mathcal{B}(M)$ and split up this set into $\mathcal{B}_1(M)$ and $\mathcal{B}_2(M)$. Distinguishing the cases where $\gcd(m,h)$ is odd and even, resp., we arrive at

$$-I(U) \ge \left(\frac{1}{\pi\sqrt{2}} - \varepsilon \right) S(M) - \frac{1}{\sqrt{2}} \left(\frac{1}{\pi\sqrt{2}} + \varepsilon \right) S\left(\frac{M}{2} \right) \\ - C_2 \sum_{(m,h)\in\mathcal{B}_2(M)} (hm)^{-1+q/2} |(m,h)|_q^{-q+1/2} - C_3, \tag{2.4}$$

with

$$S(M) := \sum_{(m,h)\in\mathcal{B}(M)} (hm)^{-1+q/2} |(m,h)|_q^{-q+1/2} \left(1 - \frac{|(m,h)|_q}{M} \right),$$

in view of (2.2) and (2.3).

The next lemma provides an asymptotic expansion for $S(M)$ as $M \to \infty$.

LEMMA 4. *For $M \to \infty$,*

$$S(M) \sim \frac{2\pi}{3q} M^{1/2}.$$

Proof. We first evaluate

$$\Sigma(M) := \sum_{(m,h)\in\mathcal{B}(M)} (hm)^{-1+q/2}.$$

To this end we divide $\mathcal{B}(M)$ into subdomains constrained by $Y < hm \le 2Y$, where $Y \in \{M/2, M/4, \ldots\}$, and apply twice the crudest form of the Euler summation formula (see e.g. Krätzel [15], Theorem 1.1). Adding up over Y, by a straightforward calculation we get

$$\Sigma(M) \sim \iint\limits_{\substack{|(v_1,v_2)|_q \le M \\ v_1 \ge v_2 \ge 0}} (v_1 v_2)^{-1+q/2}\, d(v_1, v_2) = \frac{\pi}{2q^2} M^q.$$

Hence, for $M > 1$,

$$\begin{aligned} S(M) &= \int\limits_{1/2}^{M} u^{-q+1/2}\left(1 - \frac{u}{M}\right) d\Sigma(u) \\ &= \int\limits_{1/2}^{M} u^{-q+1/2}\, d\Sigma(u) - \frac{1}{M}\int\limits_{1/2}^{M} u^{-q+3/2}\, d\Sigma(u) \\ &= \left(q - \frac{1}{2}\right)\int\limits_{1/2}^{M} u^{-q-1/2}\Sigma(u)\, du + \left(\frac{3}{2} - q\right)\frac{1}{M}\int\limits_{1/2}^{M} u^{-q+1/2}\Sigma(u)\, du \\ &\sim \frac{\pi}{2q^2}\left(q - \frac{1}{2}\right)\int\limits_{1/2}^{M} u^{-1/2}\, du + \frac{\pi}{2q^2}\left(\frac{3}{2} - q\right)\frac{1}{M}\int\limits_{1/2}^{M} u^{1/2}\, du \sim \frac{2\pi}{3q} M^{1/2}. \end{aligned}$$

This completes the proof of Lemma 4.

It remains to show that the contribution of the remainder term sum in (2.4) is small.

LEMMA 5. *For $M \to \infty$,*

$$\sum_{(m,h)\in\mathcal{B}_2(M)} (hm)^{-1+q/2} |(m,h)|_q^{-q+1/2} \ll M^{1/2-\omega q/2}, \tag{2.5}$$

where $\omega = \frac{1}{2}(2 - \lambda)$, with $\lambda < 2$ from (I) *of Theorem* A.

Proof. We recall that $|(m,h)|_q \asymp m$. The left-hand side of (2.5) is

$$\ll \Bigg\{ \sum_{\substack{m \le M \\ h \le m^{1-\omega}}} + \sum_{\substack{(m,h)\in\mathcal{R}_M \\ m^{1-\omega} < h}} \Bigg\} h^{-1+q/2} m^{-1/2-q/2}.$$

A short calculation shows that the contribution of the first sum is

$$\ll M^{1/2-\omega q/2}.$$

To deal with the second sum, we split up the domain of summation into subdomains constrained by $Y < m \le 2Y$, where $Y \in \{M/2, M/4, \ldots\}$. We obtain (since $1 < q < 2$)

$$\begin{aligned}\sum_{\substack{(m,h)\in\mathcal{R}_{2Y}-\mathcal{R}_Y\\ m^{1-\omega}<h}} h^{-1+q/2}m^{-1/2-q/2} &\ll Y^{-3/2+\omega(1-q/2)} \sum_{(m,h)\in\mathcal{R}_{2Y}} 1\\ &\ll Y^{1/2-\omega(1+q/2)},\end{aligned}$$

using the hypothesis (I) of Theorem A. Adding up over $Y \in \{M/2, M/4, \ldots\}$ we complete the proof of Lemma 5.

Combining Lemmas 4 and 5, and (2.4), we arrive at

$$-I(U) \ge C_4 M^{1/2} - C_5 M^{1/2-\omega q/2} \ge C_6 M^{1/2}.$$

On the other hand, it follows from the definition of $I(U)$ that

$$-I(U) \le \Big(\sup_{U-1\le u\le U+1} (-S^*(u))\Big) \int_{-1}^{1} F_M(v)\,dv \le \sup_{U-1\le u\le U+1} (-S^*(u)).$$

This implies that there exists some $u^* \in [U-1, U+1]$ such that

$$-S^*(u^*) \ge C_7 M^{1/2}.$$

Since, by Kronecker's theorem, this is true for an unbounded sequence of values u^*, it follows from Lemma 1 that

$$\limsup_{t\to\infty} \frac{P_{\mathcal{D}_k}(t)}{t^{1/4}} \ge C_8 M^{1/2}.$$

Since M was arbitrary, this completes the proof of Theorem A.

3. The cubic case: construction of a large linearly independent set

THEOREM B. *Let*

$$\mathcal{M}^{(0)} := \{\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2 : a_1 > a_2,\ \gcd(a_1, a_2) = 1\}.$$

Then there exists a set $\mathcal{M} \subset \mathcal{M}^{(0)}$ *such that*:

(I) *For* $X \to \infty$, *the set*

$$\mathcal{R} := \left\{(a_1, a_2) \in \mathbb{N}^2 : \left(\frac{a_1}{\gcd(a_1,a_2)}, \frac{a_2}{\gcd(a_1,a_2)}\right) \in \mathcal{M}^{(0)} - \mathcal{M}\right\}$$

satisfies $\#(\mathcal{R}_X) \ll X^{\lambda}$ *for some exponent* $\lambda < 2$.

(II) $\{(a_1^{3/2}+a_2^{3/2})^{2/3} : \mathbf{a}\in\mathcal{M}\}\cup\{1\}$ *is linearly independent over* $\mathbb{Q}$.

Two pivotal tools from the literature

PROPOSITION 1. *For any fixed* $\varepsilon>0$ *and* $Z\to\infty$,

$$\#\{(x_1,\ldots,x_4)\in\mathbb{N}^4 : x_i\le Z, x_1^3+x_2^3=x_3^3+x_4^3,\\ (x_1,x_2)\notin\{(x_3,x_4),(x_4,x_3)\}\}\ll Z^{4/3+\varepsilon}.$$

Proof. This is a special case of a recent deep result of Heath-Brown [9].

PROPOSITION 2. *Let* $k>1$ *be a positive integer and* $\mathbb{F}$ *an algebraic number field over* $\mathbb{Q}$ *which contains all kth roots of unity. Define* $\mathbb{F}^{1/k}:=\mathbb{F}(\{x\in\mathbb{C}:x^k\in\mathbb{F}\})$ *and consider* $\mathbb{F}_*:=\mathbb{F}-\{0\}$ *as a group with respect to multiplication. Let* $\{t_{\mathcal{X}}\}$ *be the image of a one-one map* $\mathcal{X}\to t_{\mathcal{X}}$ *from the factor group* $\mathbb{F}_*/\mathbb{F}_*^{[k]}$ *into* $\mathbb{C}$ *such that* $t_{\mathcal{X}}^k\mathbb{F}_*^{[k]}=\mathcal{X}$ *for each coset* $\mathcal{X}\in\mathbb{F}_*/\mathbb{F}_*^{[k]}$. *Then the set* $\{t_{\mathcal{X}}\}$ *is a basis of* $\mathbb{F}^{1/k}$ *over* $\mathbb{F}$.

Proof. This is a result of the classical Kummer theory. For a textbook reference, see e.g. Bourbaki [1], Chap. 5, §11, No. 8, Theorem 4(c).

Preparations for the proof of Theorem B. First of all, let

$$\mathcal{M}^{(1)}:=\{(a_1,a_2)\in\mathcal{M}^{(0)}:a_2<a_1-a_1^{0.99}\};$$

then it is clear that

$$\#(\mathcal{M}^{(0)}-\mathcal{M}^{(1)})_X\ll X^{1.99}. \tag{3.1}$$

Let further

$$\mathcal{U}^{(n)}:=\{\mathbf{a}\in\mathcal{M}^{(1)}:a_1^3-a_2^3=n\}\qquad(n\in\mathbb{N}),$$

$$\mathcal{Y}:=\{n\in\mathbb{N}:\ \text{there exist } \mathbf{a},\mathbf{b}\in\mathcal{M}^{(1)},\mathbf{a}\ne\mathbf{b},\ \text{with } a_1^3-a_2^3=b_1^3-b_2^3=n\}$$

and put

$$\mathcal{U}:=\bigcup_{n\in\mathcal{Y}}\mathcal{U}^{(n)},\qquad \mathcal{M}^{(2)}:=\mathcal{M}^{(1)}-\mathcal{U}.$$

We claim that there exists some $\lambda_1<2$ such that

$$\#(\mathcal{U}_X)\ll X^{\lambda_1}. \tag{3.2}$$

To prove this, consider any $\mathbf{a}\in\mathcal{U}_X$: It follows that $a_2<a_1\le X$ and that there exists some $\mathbf{b}\in\mathcal{M}^{(1)},\mathbf{b}\ne\mathbf{a}$, such that $a_1^3-a_2^3=b_1^3-b_2^3$. By definition of $\mathcal{U}_X$ and $\mathcal{M}^{(1)}$,

$$b_1^{2.99}\le(b_1-b_2)(b_1^2+b_1b_2+b_2^2)=a_1^3-a_2^3\le X^3,$$

hence $b_2<b_1\le X^{3/2.99}$. According to Proposition 1, the number of possible quadruples (a_1,a_2,b_1,b_2) (which of course is not less than the number of possible pairs (a_1,a_2)) is

$$\ll X^{\frac{3}{2.99}(\frac{4}{3}+\varepsilon)}=X^{\lambda_1},\qquad \lambda_1<2.$$

Further, let

$$\mathcal{W} := \{(a_1, a_2) \in \mathcal{M}^{(2)} : a_1, a_2 \in \mathbb{N}^{[2]}\}.$$

Then it is trivial that

$$\#(\mathcal{W}_X) \ll X. \tag{3.3}$$

On the set $\mathcal{M}^{(3)} := \mathcal{M}^{(2)} - \mathcal{W}$ we define an equivalence relation as follows:

$$\mathbf{a} \sim \mathbf{b} \Leftrightarrow \frac{a_1^3 - a_2^3}{b_1^3 - b_2^3} \in \mathbb{Q}^{[3]}.$$

To construct finally the set $\mathcal{M}$ announced in Theorem B, we simply select from each of the equivalence classes arising from this relation that element $\mathbf{b}$ with minimal $b_1^3 - b_2^3$. We proceed to verify that this selection comprehends "almost all" elements of $\mathcal{M}^{(3)}$:

LEMMA 6. *The set $\mathcal{N} := \mathcal{M}^{(3)} - \mathcal{M}$ satisfies $\#(\mathcal{N}_X) \ll X^{\lambda_2}$ with some $\lambda_2 < 2$.*

Before proving this result, we notice that, along with (3.1)–(3.3), it implies that

$$\#(\mathcal{M}^{(0)} - \mathcal{M})_X \ll X^{\lambda} \quad (\lambda < 2). \tag{3.4}$$

Moreover, by construction,

$$\mathbf{a} \sim \mathbf{b} \Leftrightarrow \mathbf{a} = \mathbf{b} \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathcal{M}. \tag{3.5}$$

Proof (of Lemma 6). We can write $\mathcal{N}$ as

$$\mathcal{N} = \{\mathbf{a} \in \mathcal{M}^{(3)} : \text{ there exists } \mathbf{b} \in \mathcal{M}^{(3)} \text{ with } \mathbf{a} \sim \mathbf{b}, b_1^3 - b_2^3 < a_1^3 - a_2^3\}.$$

Thus we have to estimate the number of all $\mathbf{a} \in \mathcal{M}_X^{(3)}$ for which there exists some $\mathbf{b} \in \mathcal{M}^{(3)}$ such that

$$\frac{a_1^3 - a_2^3}{b_1^3 - b_2^3} = \frac{p^3}{q^3} \tag{$*$}$$

with $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, $p > q$.

CASE 1: $\mathbf{a} \in \mathcal{M}_X^{(3)}$ *such that there exists* $\mathbf{b} \in \mathcal{M}^{(3)}$ *for which* $(*)$ *holds with* $p \le X^{2/5}$. Since $(*)$ is equivalent to

$$(a_1 q)^3 + (b_2 p)^3 = (b_1 p)^3 + (a_2 q)^3, \tag{$**$}$$

we may appeal once again to Heath-Brown's Proposition 1: By the argument used to establish (3.2), $b_2 < b_1 \le X^{3/2.99}$, and of course $a_2 < a_1 \le X$, thus the number of "non-diagonal" solutions [3] of $(**)$ is

$$\ll X^{(\frac{3}{2.99} + \frac{2}{5})(\frac{4}{3} + \varepsilon)} \ll X^{\lambda_3} \quad \text{with } \lambda_3 < 2.$$

[3] It is obvious to see that the "diagonal" solutions excluded in the statement of Proposition 1 lead to the impossible cases $a_1 = a_2$, $b_1 = b_2$, and $\mathbf{a} = \mathbf{b}$, respectively.

(Properly speaking, we hereby count all quadruples (a_1q, b_2p, b_1p, a_2q) which solve $(**)$, but for a_1q fixed there are only $\ll X^{\varepsilon}$ possibilities for a_1; then a_2 is uniquely determined by q and a_2q.)

Case 2: $\mathbf{a} \in \mathcal{M}_X^{(3)}$ *such that for all* $\mathbf{b} \in \mathcal{M}^{(3)}$ *satisfying* $(*)$ *necessarily* $p > X^{2/5}$. We can rewrite $(*)$ as

$$(a_1 - a_2)\underbrace{(a_1^2 + a_1a_2 + a_2^2)}_{Q(\mathbf{a})}q^3 = (b_1^3 - b_2^3)p^3,$$

hence p^3 divides $(a_1 - a_2)Q(\mathbf{a})$. It is well known that $\gcd(a_1 - a_2, Q(\mathbf{a}))$ is either 1 or 3 for a_1, a_2 coprime. Thus we can write $p = e_0p_1p_2$ with

$$e_0 \in \{1, 3\}, \quad \gcd(p_1, p_2) = 1, \quad p_1^3 \,|\, (a_1 - a_2), \quad p_2^3 \,|\, Q(\mathbf{a}).$$

Consequently, $p_1^3 \le a_1 - a_2 \le X$, hence $p_1 \le X^{1/3}$, and thus

$$p_2 > \tfrac{1}{3}X^{2/5-1/3} = \tfrac{1}{3}X^{1/15}.$$

Let $Q(\mathbf{a}) = lp_2^3$ with $l \in \mathbb{N}$. There correspond at most $\ll (lp_2^3)^{\varepsilon} \ll X^{2\varepsilon}$ pairs $\mathbf{a}$ to each fixed choice of (l, p_2). Therefore, the total number of possible values of $\mathbf{a}$ is

$$\ll X^{2\varepsilon} \sum_{p_2 > X^{1/15}/3} \sum_{l \ll p_2^{-3}X^2} 1 \ll X^{2\varepsilon} \sum_{p_2 > X^{1/15}/3} X^2 p_2^{-3} \ll X^{28/15+2\varepsilon}$$

which completes the proof of Lemma 6.

Proof of Theorem B: *Verification of (I)*. For $(a_1, a_2) \in \mathbb{N}^2$, let

$$(a_1', a_2') = \left(\frac{a_1}{\gcd(a_1, a_2)}, \frac{a_2}{\gcd(a_1, a_2)}\right).$$

With this notation,

$$\mathcal{R} = \{(a_1, a_2) \in \mathbb{N}^2 : (a_1', a_2') \in \mathcal{M}^{(0)} - \mathcal{M}\}.$$

We put further, for $Y \in \mathbb{R}^+$,

$$\mathcal{R}^{(Y)} := \{(a_1, a_2) \in \mathcal{R} : Y/2 < a_1' \le Y\}.$$

Consequently,

$$\begin{aligned}\#(\mathcal{R}_X) &\ll \sum_{Y = X, X/2, \ldots} \#(\mathcal{R}_X^{(Y)}) \ll \sum_{Y = X, X/2, \ldots} (X/Y)\#(\mathcal{M}^{(0)} - \mathcal{M})_Y \\ &\ll X \sum_{Y = X, X/2, \ldots} Y^{\lambda - 1} \ll X^{\lambda} \quad (\lambda < 2),\end{aligned}$$

by an appeal to (3.4).

Verification of (II). For $\mathbf{a} = (a_1, a_2) \in \mathbb{N}^2$, put

$$\alpha(\mathbf{a}) = (|(a_1, a_2)|_{3/2})^3 = a_1^3 + a_2^3 + 2(a_1a_2)^{3/2}. \tag{3.6}$$

Then we have to show that $\{1, \alpha(\mathbf{a}^{(1)})^{1/3}, \ldots, \alpha(\mathbf{a}^{(J)})^{1/3}\}$ is linearly independent over $\mathbb{Q}$ for arbitrary $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(J)} \in \mathcal{M}$. We apply Proposition 2 with $k = 3$ and $\mathbb{F} = \mathbb{Q}(\alpha(\mathbf{a}^{(1)}), \ldots, \alpha(\mathbf{a}^{(J)}), \xi)$, where $\xi = \frac{1}{2}(-1 + i\sqrt{3})$ is a third root of unity. It readily follows that $\{1, \alpha(\mathbf{a}^{(1)})^{1/3}, \ldots, \alpha(\mathbf{a}^{(J)})^{1/3}\}$ is linearly independent even over $\mathbb{F}$, provided we can show that the cosets $\alpha(\mathbf{a}^{(j)})\mathbb{F}_*^{[3]}$ are pairwise distinct for $j = 0, 1, \ldots, J$ (with $\mathbf{a}^{(0)} := (1, 0)$ for convenience of notation). Let us assume the contrary, i.e., that for some $r, s, 0 \le r < s \le J$,

$$\frac{\alpha(\mathbf{a}^{(r)})}{\alpha(\mathbf{a}^{(s)})} = z^3, \qquad z \in \mathbb{F}.$$

Taking the norm on both sides, we conclude that

$$\frac{N_{\mathbb{F}/\mathbb{Q}}(\alpha(\mathbf{a}^{(r)}))}{N_{\mathbb{F}/\mathbb{Q}}(\alpha(\mathbf{a}^{(s)}))} \in \mathbb{Q}^{[3]}. \tag{3.7}$$

Since $\alpha(\mathbf{a}^{(1)}), \ldots, \alpha(\mathbf{a}^{(J)}), \xi$ are quadratic irrationals, the degree of $\mathbb{F}$ over $\mathbb{Q}$ is a power of 2, say 2^l. As an easy consequence, for $\mathbf{a} \in \mathcal{M}$,

$$N_{\mathbb{F}/\mathbb{Q}}(\alpha(\mathbf{a})) = (a_1^3 - a_2^3)^{2^l}.$$

For $r = 0$, (3.7) means that

$$((a_1^{(s)})^3 - (a_2^{(s)})^3)^{2^l} \in \mathbb{Q}^{[3]}, \quad \text{hence} \quad (a_1^{(s)})^3 - (a_2^{(s)})^3 \in \mathbb{N}^{[3]},$$

which contradicts Fermat's Last Theorem with the classical exponent 3. For $r > 0$, (3.7) becomes

$$\left(\frac{(a_1^{(r)})^3 - (a_2^{(r)})^3}{(a_1^{(s)})^3 - (a_2^{(s)})^3}\right)^{2^l} \in \mathbb{Q}^{[3]}, \quad \text{hence} \quad \frac{(a_1^{(r)})^3 - (a_2^{(r)})^3}{(a_1^{(s)})^3 - (a_2^{(s)})^3} \in \mathbb{Q}^{[3]},$$

which means that $\mathbf{a}^{(r)} \sim \mathbf{a}^{(s)}$, thus, in view of (3.5), $\mathbf{a}^{(r)} = \mathbf{a}^{(s)}$, the desired contradiction. This completes the proof of Theorem B.

4. A quantitative refinement. A comparison with the classical results on the circle and divisor problems shows that our (1.12) corresponds to the achievements of Ingham [13]. It is natural to ask for stronger estimates comparable to those of Gangadharan [6] and Corrádi–Kátai [2]. However, it is immediate that the very sharp bounds in [2] depend on the special multiplicative structure of the function $r_2(n)$ and its reappearance after the exponential sum transformation. Further, even the attempt to generalize Gangadharan's elaborate argument leads to overwhelming technical difficulties.

In this paper we shall use a different approach based on tools from the theory of uniform distribution (in particular on the Erdős–Turán–Koksma inequality) to establish a result which may be compared to those of Gangadharan [6].

THEOREM C. *As* $t \to \infty$,

$$P_{\mathcal{D}_3}(t) = \Omega_+(t^{1/4}(\log\log t)^{1/4}),$$

i.e.,

$$\limsup_{t\to\infty}\left(\frac{P_{\mathcal{D}_3}(t)}{t^{1/4}(\log\log t)^{1/4}}\right) > 0.$$

Proof. The idea is to use a quantitative version of Kronecker's approximation principle. Recalling (2.3) and the analysis nearby, this has to be applied to the set of numbers

$$\{\alpha(\mathbf{a})^{1/3} : \mathbf{a} \in \mathcal{M},\ \alpha(\mathbf{a}) \le M^3\} =: \{\theta_j : j = 1,\dots,s\}$$

(in arbitrary order), with $\alpha(\mathbf{a})$ as in (3.6). Let s denote the cardinality of this set; then obviously $s \asymp M^2$. Our first task is to establish a lower bound for $\|\sum_{j=1}^{s} h_j\theta_j\|$ where $\mathbf{h} = (h_1,\dots,h_s) \in \mathbb{Z}^s - \{\mathbf{0}\}$ and $\|\cdot\|$ denotes the distance from the nearest integer.

LEMMA 7. *There exists a positive constant* c_1 *such that*

$$\|h_1\theta_1 + \dots + h_s\theta_s\| \ge \frac{1}{\varrho(\max(|h_1|,\dots,|h_s|))}$$

for all $\mathbf{h} \in \mathbb{Z}^s - \{\mathbf{0}\}$, *where* $\varrho(t) := (3sMt)^{c_1^s}$.

Proof. The numbers θ_j are all algebraic integers of degree 6. For $\theta_j = \alpha(\mathbf{a})^{1/3} = |(a_1,a_2)|_{3/2}$, we put $\theta_j^- := (a_1^{3/2} - a_2^{3/2})^{2/3}$. It is simple to show that the conjugates of θ_j are $\xi^r\theta_j$, $\xi^r\theta_j^-$, $r = 0,1,2$, $\xi = \frac{1}{2}(-1+i\sqrt{3})$ a third root of unity. We consider the field extension $\mathbb{F} = \mathbb{Q}(\theta_1,\theta_1^-,\dots,\theta_s,\theta_s^-,\xi)$. The corresponding Galois group $G = \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ contains at most c_1^s elements χ where c_1 is a suitable positive constant. For fixed $\mathbf{h}$, let $-h_0$ be the nearest integer to $h_1\theta_1 + \dots + h_s\theta_s$. It is clear that

$$\Big|\prod_{\chi\in G}\chi(h_0 + h_1\theta_1 + \dots + h_s\theta_s)\Big| \ge 1,$$

since the left-hand side is the modulus of the norm of a nonzero algebraic integer. (Note that $1,\theta_1,\dots,\theta_s$ are linearly independent over $\mathbb{Q}$.) Furthermore, for every $\chi \in G$,

$$|\chi(h_0 + h_1\theta_1 + \dots + h_s\theta_s)| \le 3s|\mathbf{h}|_\infty \max_{1\le j\le s}|\theta_j| \le 3s|\mathbf{h}|_\infty M.$$

Consequently,

$$|h_0 + h_1\theta_1 + \dots + h_s\theta_s| \ge (3s|\mathbf{h}|_\infty M)^{-c_1^s},$$

which establishes Lemma 7.

Let next ω_N denote the s-dimensional sequence $(n\theta_1, \ldots, n\theta_s)_{n=N+1}^{2N}$, where N is any positive integer. In terms of the theory of uniform distribution (4) modulo 1, it follows that the discrepancy of this sequence satisfies

$$D(\omega_N) \le c_2^s s! \frac{\log N \log \gamma(N)}{\gamma(N)}, \tag{4.1}$$

with γ denoting the inverse function of ϱ, provided that N is so large compared to s that $\gamma(N) \ge e$ (say). In fact, (4.1) is essentially the estimate in Theorem 1.80 of Drmota and Tichy [3], p. 70, with the dependence on the dimension s worked out explicitly. Now let $\square$ denote the s-dimensional cube $\left[\frac{1}{2} - \varepsilon_0 M^{-1}, \frac{1}{2} + \varepsilon_0 M^{-1}\right]^s$ (cf. (2.3)). Then it follows that (with $\langle\cdot\rangle$ denoting the fractional part)

$$\begin{aligned}\#\{n \in \mathbb{N} : N < n \le 2N, (\langle n\theta_1\rangle, \ldots, \langle n\theta_s\rangle) \in \square\} &\ge N\mathrm{vol}(\square) - ND(\omega_N)\\ &\ge N((2\varepsilon_0)^s M^{-s} - c_3^s s! N^{-c_1^{-s}} (\log N)^2)\\ &\ge N((c_4 M)^{-c_5 M^2} - (c_6 M)^{c_7 M^2} N^{-c_8^{-M^2}} (\log N)^2),\end{aligned}$$

using Stirling's formula and the fact that $s \asymp M^2$. A short computation shows that this last expression is > 0 if we choose

$$N = N^*(M) := \left[c_9^{c_{10}^{M^2}}\right] \tag{4.2}$$

with sufficiently large constants c_9, c_{10}. Consequently, for arbitrary M there exists at least one integer U in $]N^*(M), 2N^*(M)]$ such that $(\langle U\theta_1\rangle, \ldots$ $\ldots, \langle U\theta_s\rangle) \in \square$ which, in the notation of Section 2, means that (2.3) is true for all $(m, h) \in \mathcal{M}$ with $|(m, h)|_{3/2} \le M$. The rest of the analysis in Section 2 applies as before, yielding the existence of some $u_* \in [U-1, U+1]$ satisfying

$$-S^*(u_*) \ge C_7 M^{1/2}$$

and, in view of Lemma 1, also

$$P_{\mathcal{D}_3}(u_*^2) \ge C_8 u_*^{1/2} M^{1/2}.$$

The condition $U \in\,]N^*(M), 2N^*(M)]$ (along with (4.2)) on the one hand ensures that u_* tends to $+\infty$ if so does M, and on the other hand implies that

$$M \gg (\log\log u_*)^{1/2}.$$

Thus

$$P_{\mathcal{D}_3}(u_*^2) \ge C_9 u_*^{1/2} (\log\log u_*)^{1/4}$$

for an unbounded sequence of values u_*. This completes the proof of Theorem C.

---

(4) For an enlightening introduction to this area, the reader may consult, e.g., the recent textbook of Drmota and Tichy [3].

Acknowledgements. The authors would like to express their sincere gratitude to Professors Franz Halter-Koch, Robert Tichy, and Kurt Girstmair, and to Drs. Christoph Baxa and Gerald Kuba, for helpful comments on different parts of this work.

References

[1] N. Bourbaki, *Algebra II*, Springer, Berlin, 1990.

[2] K. Corrádi and I. Kátai, *A comment on K. S. Gangadharan's paper "Two classical lattice point problems"*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 17 (1967), 89–97 (in Hungarian).

[3] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Math. 1651, Springer, Berlin, 1997.

[4] J. L. Hafner, *New omega theorems for two classical lattice point problems*, Invent. Math. 63 (1981), 181–186.

[5] —, *On the average order of a class of arithmetical functions*, J. Number Theory 15 (1982), 36–76.

[6] K. S. Gangadharan, *Two classical lattice point problems*, Proc. Cambridge Philos. Soc. 57 (1961), 699–721.

[7] G. H. Hardy, *On the expression of a number as the sum of two squares*, Quart. J. Math. 46 (1915), 263–283.

[8] —, *On Dirichlet's divisor problem*, Proc. London Math. Soc. (2) 15 (1916), 1–25.

[9] D. R. Heath-Brown, *The density of rational points on cubic surfaces*, Acta Arith. 79 (1997), 17–30.

[10] E. Hlawka, J. Schoißengeier and R. Taschner, *Geometric and Analytic Number Theory*, Springer, Berlin, 1991.

[11] M. N. Huxley, *Exponential sums and lattice points II*, Proc. London Math. Soc. 66 (1993), 279–301.

[12] —, *Area, lattice points, and exponential sums*, London. Math. Soc. Monographs (N.S.) 13, Oxford, 1996.

[13] A. E. Ingham, *On two classical lattice point problems*, Proc. Cambridge Philos. Soc. 36 (1940), 131–138.

[14] I. Kátai, *The number of lattice points in a circle*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 8 (1965), 39–60.

[15] E. Krätzel, *Lattice Points*, Deutsch. Verlag Wiss., Berlin, 1988.

[16] G. Kuba, *On sums of two k-th powers of numbers in residue classes II*, Abh. Math. Sem. Univ. Hamburg 63 (1993), 87–95.

[17] W. G. Nowak, *An Ω-estimate for the lattice rest of a convex planar domain*, Proc. Roy. Soc. Edinburgh Sect. A 100 (1985), 295–299.

[18] —, *On the average order of the lattice rest of a convex planar domain*, Proc. Cambridge Philos. Soc. 98 (1985), 1–4.

[19] —, *On sums of two k-th powers: a mean-square bound for the error term*, Analysis 16 (1996), 297–304.

[20] —, *Sums of two k-th powers: an Omega estimate for the error term*, Arch. Math. (Basel) 68 (1997), 27–35.

[21] J. D. Vaaler, *Some extremal problems in Fourier analysis*, Bull. Amer. Math. Soc. (2) 12 (1985), 183–216.

[22] J. G. van der Corput, *Over roosterpunkten in het plate vlak*, Thesis, Groningen, 1919.

Manfred Kühleitner and Werner Georg Nowak
Institut für Mathematik
und angewandte Statistik
Universität für Bodenkultur
A-1180 Wien, Austria
E-mail: kleitner@mail.boku.ac.at
nowak@mail.boku.ac.at
Web: http://www.boku.ac.at/math/nth.html

Johannes Schoißengeier
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
A-1090 Wien, Austria
E-mail: schoi@pap.univie.ac.at
Web: http://radon.mat.univie.ac.at
/Groups/Schoissengeier.html

Trevor D. Wooley
Department of Mathematics
University of Michigan
East Hall, 525 East University Avenue
Ann Arbor, Michigan 48109-1109
U.S.A.
E-mail: wooley@math.lsa.umich.edu
Web: http://www.math.lsa.umich.edu/~wooley/

Received on 10.10.1997 (3276)