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An explicit version of Birch's Theorem

by

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1. Introduction. Birch [1] has shown that a system of homogeneous polynomials with rational coefficients possesses a non-trivial rational zero provided only that these polynomials are of odd degree, and the system has sufficiently many variables in terms of the number and degrees of these polynomials. While bounds have been obtained for the number of variables which suffice to guarantee the existence of a non-trivial zero, in all but the simplest cases such bounds as are available are too weak to have warranted explicit determination. Rather general versions of the Hardy–Littlewood method have been developed in order to investigate this problem, first by Davenport (see, in particular, [4, 5]), later by Birch [2], and most recently by Schmidt [12], but unfortunately even Schmidt's highly developed version of the Hardy–Littlewood method is disappointingly ineffective in handling systems of higher degree (see [11, 12]).

In this paper we obtain explicit bounds for the number of variables required in Birch's Theorem by using a method involving the Hardy–Littlewood method only indirectly, being based on a refinement of the elementary diagonalisation method of Birch [1] first described in Wooley [15], where we restricted our investigations to systems of cubic and quintic forms. Although the size of our bounds may be aptly described as "not even astronomical" (an eloquent phrase of Birch), it seems that this paper contains the first truly explicit bounds in this problem.

In order to describe our conclusions we require some notation. When k is a field, d and r are natural numbers, and m is a non-negative integer, let $v_{d,r}^{(m)}(k)$ denote the least integer (if any such integer exists) with the property that whenever $s > v_{d,r}^{(m)}(k)$, and $f_i(\mathbf{x}) \in k[x_1, \ldots, x_s]$ $(1 \le i \le r)$ are forms

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of degree d, then the system of equations $f_i(\mathbf{x}) = 0$ $(1 \le i \le r)$ possesses a solution set which contains a k-rational linear space of projective dimension m. If no such integer exists, define $v_{d,r}^{(m)}(k)$ to be $+\infty$. We abbreviate $v_{d,r}^{(0)}(k)$ to $v_{d,r}(k)$, and define $\phi_{d,r}(k)$ in like manner, save that the arbitrary forms of degree d are restricted to be diagonal.

When d is even one has $v_{d,r}^{(m)}(\mathbb{Q}) = \infty$, because definite forms necessarily fail the real solubility condition, and thus we restrict attention to odd d. In previous work [15] we investigated systems of cubic and quintic forms. In particular, we showed that when r is a natural number and m is a non-negative integer, then

$$v_{3,r}^{(m)}(\mathbb{Q}) < (90r)^8(\log(27r))^5(m+1)^5, \tag{1.1}$$

and

$$v_{5,r}^{(m)}(\mathbb{Q}) < \exp(10^{32}((m+1)r\log(3r))^{\kappa}\log(3r(m+1))), \tag{1.2}$$

where

$$\kappa = \frac{\log 3430}{\log 4} = 5.87199\ldots \tag{1.3}$$

We note that earlier work of Lewis and Schulze-Pillot [8, inequality (4)] provides an estimate in many circumstances superior to (1.1), namely

$$v_{3,r}^{(m)}(\mathbb{Q}) \ll r^{11}(m+1) + r^3(m+1)^5. \tag{1.4}$$

Moreover, if one seeks only the existence of a rational point on the intersection of r cubic hypersurfaces, then Schmidt's bound $v_{3,r}^{(0)}(\mathbb{Q}) < (10r)^5$ (see [10, Theorem 1]) is superior to both (1.1) and (1.4).

We now extend our earlier conclusions to arbitrary systems of forms of odd degree, and this will entail recording some further notation. Suppose that A is a subset of $\mathbb{R}$ and Ψ is a function mapping A into A. When α is a real number, write $[\alpha]$ for the largest integer not exceeding α. Then we adopt the notation that whenever x and y are real numbers with $x \ge 1$, then $\Psi_x(y)$ denotes the real number $a_{[x]}$, where $(a_n)_{n=1}^{\infty}$ is the sequence defined by taking $a_1 = \Psi(y)$, and $a_{i+1} = \Psi(a_i)$ $(i \ge 1)$. Finally, when n is a non-negative integer we define the functions $\psi^{(n)}(x)$ by taking $\psi^{(0)}(x) = \exp(x)$, and when $n > 0$ by putting

$$\psi^{(n)}(x) = \psi_{42\log x}^{(n-1)}(x). \tag{1.5}$$

Theorem 1. *Let d be an odd integer exceeding* 5, *and let r and m be non-negative integers with $r \ge 1$. Then*

$$v_{d,r}^{(m)}(\mathbb{Q}) < \psi^{((d-5)/2)}(dr(m+1)).$$

The upper bound contained in Theorem 1 provides the first entirely explicit version of Birch's Theorem for systems of forms of equal degree, corresponding conclusions for mixed degree systems following almost trivially (see Theorem 5.1 below). While systems of forms possessing mild singularities are susceptible to more powerful analyses (see Birch [2], or Tartakovskiĭ [13] for earlier but more restricted work), we stress that our aim in this paper is to provide generally applicable bounds free of any geometric hypotheses. We note that the constant 42 occurring in the definition (1.5) can be reduced with greater effort, especially for large values of the parameters. However, it does not appear feasible to adapt the methods of this paper to replace the level of recursion in (1.5) by any function appreciably smaller than $\log x$. In this context we remark that even Birch's original method [1] could, with sufficient effort, be employed to yield an explicit bound for $v_{d,r}^{(m)}(\mathbb{Q})$, though such a bound would be extraordinarily weak by comparison with that provided by Theorem 1.

An alternative to Birch's elementary method for bounding $v_{d,r}^{(m)}(\mathbb{Q})$ is provided by Schmidt's sophisticated version of the Hardy–Littlewood method, described briefly in [12]. We discuss the quality of the bounds which may be wrought from such ideas in an appendix (§6 of this paper). Although we do not carry out sufficient calculations to be confident of the precise bounds stemming from Schmidt's methods, our analysis indicates that they yield bounds qualitatively no stronger than

$$v_{d,r}^{(0)}(\mathbb{Q}) < \phi^{((d-5)/2)}(dr) \quad (d \geq 7), \tag{1.6}$$

where $\phi^{(0)}(x) = \exp(x)$, and $\phi^{(n)}(x) = \phi_{Ax}^{(n-1)}(x)$ $(n \geq 1)$, for a suitable positive constant A. The superiority of Theorem 1 is plain. In this context we note that loose remarks concluding §2 of Schmidt [12] might leave the impression that for a suitable function f, the methods of that section will establish a bound of the shape

$$v_{d,r}^{(0)}(\mathbb{Q}) < \exp_{d-3}(f(d)r) \quad (d \geq 5).$$

As should be clear from §6, however, a bound of the latter strength is wholly beyond reach of such methods when $d \geq 7$.

Our proof of Theorem 1 is based on an inductive strategy which depends for its success on an efficient diagonalisation process described in Wooley [15]. We begin, in §2, by recalling several of the key lemmata of [15] crucial to our subsequent arguments. Here and elsewhere in the proof of Theorem 1, we must extract conclusions simple enough that it remains feasible to keep control of the ensuing induction, yet retain sufficient precision to preserve the quality of the ultimate bounds. It might be said that the construction of a compromise between the latter two objectives represents the major difficulty of our argument. In §3 we consider systems of septic forms, bounds on $v_{7,r}^{(m)}(\mathbb{Q})$ forming the basis for our main induction. The main inductive step itself, in which we bound $v_{d,r}^{(m)}(\mathbb{Q})$ in terms of $v_{d-2,R}^{(M)}(\mathbb{Q})$ for suitable M and R, is established in §4. The proof of Theorem 1 is then completed routinely in §5 by making use of the main conclusions of §§3 and 4. Finally, in §6, we provide an appendix in which we discuss Schmidt's method and its consequences for Birch's Theorem.

Throughout, implicit constants in Vinogradov's notation $\ll$ and $\gg$ depend at most on the quantities occurring as subscripts to the notation.

2. Preliminary lemmata: reduction to diagonal forms. In this section we recall the reduction formulae from Wooley [15] which relate the solubility of arbitrary systems of homogeneous polynomials to diagonal ones. In order to describe these formulae we require some additional notation. Given an r-tuple of polynomials $\mathbf{F} = (F_1, \ldots, F_r)$ with coefficients in a field k, denote by $\nu(\mathbf{F})$ the number of variables appearing explicitly in $\mathbf{F}$. We are interested in the existence of solutions, over k, of systems of homogeneous polynomial equations with coefficients in k. When such a solution set contains a linear subspace of the ambient space, we define its dimension to be that when considered as a projective space. When d is a positive odd integer, denote by $\mathcal{G}_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)$ the set of $(r_d + r_{d-2} + \ldots + r_1)$-tuples of homogeneous polynomials, of which r_i have degree i for $i = 1, 3, \ldots, d$, with coefficients in k, and which possess no non-trivial linear space of solutions of dimension m over k. We define $\mathcal{D}_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)$ to be the corresponding set of diagonal homogeneous polynomials. We then define $w_d^{(m)}(\mathbf{r}) = w_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)$ by

$$w_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k) = \sup_{\mathbf{g} \in \mathcal{G}_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)} \nu(\mathbf{g}),$$

and we define $\phi_d^{(m)}(\mathbf{r}) = \phi_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)$ by

$$\phi_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k) = \sup_{\mathbf{f} \in \mathcal{D}_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k)} \nu(\mathbf{f}).$$

We observe for future reference that both $w_d^{(m)}(\mathbf{r})$ and $\phi_d^{(m)}(\mathbf{r})$ are increasing functions of the arguments m and r_i $(i = 1, 3, \ldots, d)$. For the sake of convenience we abbreviate $w_d^{(m)}(r, 0, \ldots, 0; k)$ to $v_{d,r}^{(m)}(k)$, and note that $w_d^{(0)}(r, 0, \ldots, 0; k) = v_{d,r}(k)$. We also abbreviate $\phi_d^{(m)}(r, 0, \ldots, 0; k)$ to $\phi_{d,r}^{(m)}(k)$, and write $\phi_{d,r}(k)$ for $\phi_{d,r}^{(0)}(k)$.

Next, when $m \geq 2$, we define $\mathcal{H}_d^{(m)}(r; k)$ to be the set of r-tuples, $(F_1, \ldots, F_r)$, of homogeneous polynomials of degree d, with coefficients in k, for which no linearly independent k-rational vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ exist such that $F_i(t_1\mathbf{e}_1 + \ldots + t_m\mathbf{e}_m)$ is a diagonal form in $t_1, \ldots, t_m$ for $1 \leq i \leq r$. We then define $\widetilde{w}_d^{(m)}(r) = \widetilde{w}_d^{(m)}(r;k)$ by

$$\widetilde{w}_d^{(m)}(r;k) = \sup_{\mathbf{h}\in\mathcal{H}_d^{(m)}(r;k)} \nu(\mathbf{h}).$$

Further, we adopt the convention that $\widetilde{w}_d^{(1)}(r;k) = 0$. Note that $\widetilde{w}_d^{(m)}(r;k)$ is an increasing function of the arguments m and r. Moreover, when $s > \widetilde{w}_d^{(m)}(r;k)$ and $F_1, \ldots, F_r$ are homogeneous polynomials of degree d with coefficients in k possessing s variables, then there exist k-rational vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ with the property that $F_i(t_1\mathbf{e}_1 + \ldots + t_m\mathbf{e}_m)$ is a diagonal form in $t_1, \ldots, t_m$ for $1 \le i \le r$.

The efficient diagonalisation process of [15] alluded to in the introduction is embodied in the following lemma, which is nothing other than Lemma 2.1 of [15].

LEMMA 2.1. *Let d be an odd integer with $d \ge 3$, and let r, n and m be natural numbers. Then*

$$\widetilde{w}_d^{(n+m)}(r;k) \le s + w_{d-2}^{(M)}(\mathbf{R};k),$$

where

$$M = \widetilde{w}_d^{(n)}(r;k), \qquad s = 1 + w_{d-2}^{(N)}(\mathbf{S};k), \qquad N = \widetilde{w}_d^{(m)}(r;k),$$

and for $0 \le u \le (d-1)/2$,

$$R_{2u+1} = r\binom{s+d-2u-2}{d-2u-1} \quad \text{and} \quad S_{2u+1} = r\binom{n+d-2u-2}{d-2u-1}.$$

We must still bound $w_d^{(m)}(\mathbf{r};k)$ in terms of $\widetilde{w}_d^{(M)}(R;k)$, for suitable M and R, and this we choose to do simply in the following two lemmata.

LEMMA 2.2. *Let d be an odd positive number, let r be a natural number, and let m be a non-negative integer. Then*

$$v_{d,r}^{(m)}(k) \le \widetilde{w}_d^{(M)}(r;k),$$

where $M = (m+1)(\phi_{d,r}(k)+1)$.

Proof. This is Lemma 2.2 of [15].

LEMMA 2.3. *Let d be an odd positive number with $d \ge 3$, and let $r_1, r_3, \ldots, r_d$ be non-negative integers with $r_d > 0$. Then for each non-negative integer m one has*

$$w_d^{(m)}(r_d, r_{d-2}, \ldots, r_1; k) \le w_{d-2}^{(M)}(r_{d-2}, \ldots, r_1; k),$$

where $M = v_{d,r_d}^{(m)}(k)$.

Proof. This is Lemma 2.3 of [15].

In order to make use of Lemma 2.2 in our argument, we require an estimate for $\phi_{d,r}(\mathbb{Q})$. For this purpose we use the corollary to Theorem 1 of Brüdern and Cook [3] embodied in the following lemma (this is Lemma 3.2 of [15]; see also Low, Pitman and Wolff [9]). We remark that earlier work of Davenport and Lewis [6], though weaker, would suffice to provide a version of Theorem 1 only slightly inferior to that given.

LEMMA 2.4. *Let d and r be natural numbers with d odd. Then*

$$\phi_{d,r}(\mathbb{Q}) + 1 \le 48rd^3 \log(3rd^2).$$

Before embarking on the analysis of systems of septic forms contained in the next section, we derive some simple estimates for $w_d^{(m)}(\mathbf{r};\mathbb{Q})$ when $d = 3, 5$. Although these estimates are significantly weaker than the best attainable, we emphasise again that our aim is to control the complexity of our subsequent machinations through the use of simple bounds. Since in this and future discussions we will be working only in the rational field $\mathbb{Q}$, we henceforth omit explicit mention of the underlying field from our various notations.

LEMMA 2.5. *Suppose that r_3, r_1 and m are non-negative integers with $r_1 < 3r_3^2$. Then*

$$w_3^{(m)}(r_3, r_1) < (3r_3(m+1))^{25}.$$

Proof. It follows from the argument of Lewis and Schulze-Pillot [8, inequality (4)] (see [15, Lemma 3.3]) that

$$v_{3,r}^{(m)} < (11r)^{11}(m+1) + 50r^3(m+1)^5.$$

Then whenever $r_1 < 3r_3^2$, one finds by elimination of implicit linear equations that

$$w_3^{(m)}(r_3, r_1) = r_1 + v_{3,r_3}^{(m)} < 3r_3^2 + (11r_3)^{11}(m+1) + 50r_3^3(m+1)^5.$$

A modest calculation thus reveals that

$$w_3^{(m)}(r_3, r_1) < (11^{11} + 3 + 50)r_3^{11}(m+1)^5 < (3r_3(m+1))^{25},$$

whence the lemma follows.

LEMMA 2.6. *Suppose that r_5, r_3, r_1 and m are non-negative integers with $r_1 \le 3r_3^2$ and $r_3 < 3r_5^2$. Then*

$$w_5^{(m)}(r_5, r_3, r_1) < \exp((5r_5(m+1))^{49}).$$

Proof. Write $v = v_{5,r_5}^{(m)}$. Then by Theorem 2 of [15] (see (1.2) and (1.3) above), one has

$$v < \exp(10^{32}((m+1)r_5 \log(3r_5))^{\kappa} \log(3r_5(m+1))),$$

where $\kappa = (\log 3430)/(\log 4)$. But Lemma 2.3 shows that

$$w_5^{(m)}(r_5, r_3, r_1) \le w_3^{(v)}(r_3, r_1),$$

whence, in view of the hypotheses of the statement of the lemma, and making use of Lemma 2.5, one finds that

$$\log w_5^{(m)}(r_5, r_3, r_1) < 25 \log(9r_5^2) + 25(10^{32}+1)((m+1)r_5 \log(3r_5))^\kappa \log(3r_5(m+1)).$$

Thus, following a modicum of computation, we obtain the upper bound

$$\begin{aligned}\log w_5^{(m)}(r_5, r_3, r_1) &< 25(\log 9)r_5^2 + 25(10^{32}+1)((\log 3)(m+1)r_5^2)^7 \\ &< (5r_5(m+1))^{49},\end{aligned}$$

and this completes the proof of the lemma.

3. The basis for the induction: systems of septic forms. We establish the bounds for $v_{d,r}^{(m)}$ recorded in Theorem 1 by deriving corresponding bounds for $w_d^{(m)}(\mathbf{r})$, and these we establish by induction on d, bounding $w_d^{(m)}(\mathbf{r})$ in terms of $w_{d-2}^{(M)}(\mathbf{R})$ for suitable M and $\mathbf{R}$. Although it is possible to use the bounds for $w_5^{(m)}(\mathbf{r})$ described in Lemma 2.6 as the basis for this induction, such a strategy entails complications best avoided. Instead, in this section, we establish bounds for $w_7^{(m)}(\mathbf{r})$ which later form the basis of our induction. We begin with the diagonalisation process.

Lemma 3.1. *Suppose that m and r are natural numbers. Then*

$$\widetilde{w}_7^{(m)}(r) < \exp_{5\log(7m)}(7rm).$$

Proof. When m and r are natural numbers, write

$$\overline{w}_7^{(m)}(r) = \exp_{5\log(7m)}(7rm). \tag{3.1}$$

We aim to show that for each R and M one has

$$\widetilde{w}_7^{(M)}(R) < \overline{w}_7^{(M)}(R), \tag{3.2}$$

whence the conclusion of the lemma follows. Note that by definition, for each natural number R one has $\widetilde{w}_7^{(1)}(R) = 0$, so that (3.2) certainly holds when $M = 1$. Next suppose that $m > 1$, and that for each R the inequality (3.2) holds whenever $M < m$. We will establish that (3.2) holds for each R when $M = m$, and thus (3.2) will follow for all R and M by induction.

Let m and r be natural numbers with $m \ge 2$. Write $n = [(m+1)/2]$, and note that $n < m$. By Lemma 2.1 one has

$$\widetilde{w}_7^{(m)}(r) \le \widetilde{w}_7^{(2n)}(r) \le s + w_5^{(N)}(\mathbf{R}), \tag{3.3}$$

where

$$N = \widetilde{w}_7^{(n)}(r), \qquad s = 1 + w_5^{(N)}(\mathbf{S}), \tag{3.4}$$

and for $0 \le u \le 3$,

$$R_{2u+1} = r\binom{s+5-2u}{6-2u} \quad \text{and} \quad S_{2u+1} = r\binom{n+5-2u}{6-2u}.$$

We first bound s. Write $\overline{N} = [\overline{w}_7^{(n)}(r)]$, and note that since $n < m$, the inductive hypothesis shows that $N \le \overline{N}$. Note also that for $0 \le u \le 3$ one has $S_{2u+1} \le rn^{6-2u}$. Then the hypotheses required for the application of Lemma 2.6 to bound $w_5^{(\overline{N})}(\mathbf{S})$ are satisfied, and we may conclude from (3.4) that

$$s \le 1 + w_5^{(\overline{N})}(rn^2, rn^4, rn^6) \le w_5^{(\overline{N})}(rn^2, rn^4, 2rn^6),$$

whence

$$s < \exp((5rn^2(\overline{N}+1))^{49}). \tag{3.5}$$

But by (3.1) one has

$$\overline{N} \le \overline{w}_7^{(n)}(r) = \exp_{5\log(7n)}(7rn) \tag{3.6}$$

and

$$\overline{N} = [\exp_{5\log(7n)}(7rn)] \ge 7rn. \tag{3.7}$$

Also, plainly, for each $m \ge 2$ it follows from (3.1) that $\overline{N} \ge \exp_5(7)$. Then by combining (3.5) and (3.7), we obtain

$$s \le \exp((\overline{N}+1)^{147}) < \exp(\overline{N}^{148}). \tag{3.8}$$

Finally, we bound $\widetilde{w}_7^{(m)}(r)$ by substituting (3.8) into (3.3). Note that for $0 \le u \le 3$ one has $R_{2u+1} \le rs^{6-2u}$. Then the hypotheses required for the application of Lemma 2.6 to bound $w_5^{(\overline{N})}(\mathbf{R})$ are satisfied, and we may conclude that

$$\widetilde{w}_7^{(m)}(r) \le s + w_5^{(\overline{N})}(rs^2, rs^4, rs^6) \le w_5^{(\overline{N})}(rs^2, rs^4, 2rs^6),$$

whence

$$\widetilde{w}_7^{(m)}(r) < \exp((5rs^2(\overline{N}+1))^{49}).$$

Write $\overline{s} = \exp(\overline{N}^{148})$. Then by (3.7) and (3.8), one has

$$\begin{aligned}\widetilde{w}_7^{(m)}(r) &< \exp((s(\overline{N}+1))^{98}) \le \exp(\overline{s}^{196}) \\ &= \exp_2(196\overline{N}^{148}) < \exp_2(\overline{N}^{149}).\end{aligned}$$

We therefore deduce from (3.6) that

$$\widetilde{w}_7^{(m)}(r) < \exp_3(149\log \overline{N}) < \exp_3(\exp_{5\log(7n)-1}(7rm)).$$

But on noting that whenever $m \geq 2$ one has

$$5\log\left(7\left[\frac{m+1}{2}\right]\right) \leq 5\log(7m) - 2,$$

we may conclude from (3.1) that

$$\widetilde{w}_7^{(m)}(r) < \exp_{5\log(7m)}(7rm) = \overline{w}_7^{(m)}(r),$$

thereby establishing the inequality (3.2) with $M = m$ and $R = r$. In view of the comments concluding the first paragraph of the proof, this completes the proof of the lemma.

A bound for $v_{7,r}^{(m)}$ may now be obtained by inserting the conclusion of Lemma 3.1 into Lemma 2.2.

LEMMA 3.2. *Suppose that m and r are non-negative integers with $r \geq 1$. Then*

$$v_{7,r}^{(m)} < \exp_{37\log(7r(m+1))}(7r(m+1)).$$

P r o o f. By combining Lemmata 2.2 and 2.4 with Lemma 3.1, one obtains

$$v_{7,r}^{(m)} < \exp_{5\log(7M)}(7rM), \tag{3.9}$$

where

$$\begin{aligned} M &= 16464r(m+1)\log(147r) \leq 16464(\log 147)r^2(m+1) \\ &< 7^6r^2(m+1). \end{aligned} \tag{3.10}$$

But it follows from (3.10) that

$$\log(7M) < \log(7^7r^2(m+1)) \leq 7\log(7r(m+1)),$$

and

$$\log(7rM) < \log(7^7r^3(m+1)) \leq 7\log(7r(m+1)) \leq \exp(7r(m+1)),$$

and hence (3.9) provides the estimate

$$v_{7,r}^{(m)} < \exp_{35\log(7r(m+1))+2}(7r(m+1)).$$

The conclusion of the lemma follows immediately.

Note that the conclusion of Lemma 3.2 establishes Theorem 1 when $d = 7$. In order to establish our main inductive step, however, we require a slightly more general result.

LEMMA 3.3. *Suppose that r_{2u+1} $(0 \leq u \leq 3)$ and m are non-negative integers with $r_1 \leq 3r_3^2$, $r_3 \leq 3r_5^2$ and $r_5 < 3r_7^2$. Then*

$$w_7^{(m)}(\mathbf{r}) < \exp_{39\log(7r_7(m+1))}(7r_7(m+1)).$$

Proof. By Lemma 2.3 one has

$$w_7^{(m)}(\mathbf{r}) \le w_5^{(v)}(r_5, r_3, r_1),$$

where $v = v_{7,r_7}^{(m)}$. But in view of the hypotheses concerning r_1, r_3, r_5, we may apply Lemma 2.6 together with Lemma 3.2 to conclude that

$$\begin{aligned}\log_2 w_7^{(m)}(\mathbf{r}) &< 49\log(15r_7^2(1+\exp_{37\log(7r_7(m+1))}(7r_7(m+1))))\\ &< 98\exp_{37\log(7r_7(m+1))-1}(7r_7(m+1))\\ &< \exp_{37\log(7r_7(m+1))}(7r_7(m+1)).\end{aligned}$$

The desired conclusion is almost immediate from the latter inequality.

4. The inductive step: systems of forms of higher degree. Thus far we have established Theorem 1 for $d = 7$. We next establish the inductive step which permits us to prove Theorem 1 for larger exponents, our argument following in spirit the trail laid down in §3. Our argument will be much simplified by making use of the following definition.

Definition. We say that the function $\Psi : [1,\infty) \to [1,\infty)$ satisfies the *exponential growth condition* if it has derivatives of all orders on $[1,\infty)$, and moreover for each non-negative integer n, one has for each $x \in [1,\infty)$ that

$$\frac{d^n\Psi(x)}{dx^n} \ge e^x.$$

When D is an odd integer exceeding 5, we make use of the following hypothesis.

Hypothesis $\mathcal{H}_D(\Psi)$. *For all natural numbers M, and all $\frac{1}{2}(D+1)$-tuples $\mathbf{R} = (R_D, R_{D-2}, \ldots, R_1)$ of non-negative integers satisfying $R_{D-2} < 3R_D^2$ and $R_i \le 3R_{i+2}^2$ $(i = 1, 3, \ldots, D-4)$, one has*

$$w_D^{(M)}(\mathbf{R}) < \Psi(DR_D(M+1)). \tag{4.1}$$

Our initial advance on the inductive step is provided by the diagonalisation process.

Lemma 4.1. *Let d be an odd integer exceeding 7. Suppose that Ψ is a function satisfying the exponential growth condition, and suppose further that the hypothesis $\mathcal{H}_{d-2}(\Psi)$ holds. Then whenever m and r are natural numbers, one has*

$$\widetilde{w}_d^{(m)}(r) < \Psi_{5\log(dm)}(drm).$$

Proof. When m and r are natural numbers, write

$$\overline{w}_d^{(m)}(r) = \Psi_{5\log(dm)}(drm). \tag{4.2}$$

We aim to show that for each R and M one has

$$\widetilde{w}_d^{(M)}(R) < \overline{w}_d^{(M)}(R), \tag{4.3}$$

whence the conclusion of the lemma follows. Since for each natural number R one has $\widetilde{w}_d^{(1)}(R) = 0$, the inequality (4.3) certainly holds for $M = 1$. Suppose next that $m > 1$, and that for each R the bound (4.3) holds whenever $M < m$. We establish that (4.3) holds for each R when $M = m$, and so (4.3) holds for all R and M by induction.

Let m and r be natural numbers with $m \geq 2$. Write $n = [(m+1)/2]$, and note that $n < m$. By Lemma 2.1 one has

$$\widetilde{w}_d^{(m)}(r) \leq \widetilde{w}_d^{(2n)}(r) \leq s + w_{d-2}^{(N)}(\mathbf{R}), \tag{4.4}$$

where

$$N = \widetilde{w}_d^{(n)}(r), \qquad s = 1 + w_{d-2}^{(N)}(\mathbf{S}), \tag{4.5}$$

and for $0 \leq u \leq (d-1)/2$,

$$R_{2u+1} = r\binom{s+d-2u-2}{d-2u-1} \quad \text{and} \quad S_{2u+1} = r\binom{n+d-2u-2}{d-2u-1}.$$

Write $\overline{N} = [\overline{w}_d^{(n)}(r)]$, and note that since $n < m$, the inductive hypothesis of this lemma shows that $N \leq \overline{N}$. Note also that for $0 \leq u \leq (d-1)/2$ one has $S_{2u+1} \leq rn^{d-2u-1}$, so that the hypotheses required for the application of the hypothesis $\mathcal{H}_{d-2}(\Psi)$ to bound $w_{d-2}^{(\overline{N})}(\mathbf{S})$ are satisfied. We may therefore conclude from (4.5) that

$$s \leq 1 + w_{d-2}^{(\overline{N})}(rn^2, rn^4, \ldots, rn^{d-1}) \leq w_{d-2}^{(\overline{N})}(rn^2, rn^4, \ldots, rn^{d-3}, 2rn^{d-1}),$$

whence

$$s < \Psi(drn^2(\overline{N}+1)). \tag{4.6}$$

On recalling that Ψ satisfies the exponential growth condition, it follows from (4.2) that

$$\overline{N} = [\Psi_{5\log(dn)}(drn)] \geq drn. \tag{4.7}$$

Also, plainly, for each $m \geq 2$ it follows from (4.2) that $\overline{N} \geq \exp_5(d)$. Then by combining (4.6) and (4.7) we deduce that

$$s < \Psi((\overline{N}+1)^3) < \Psi(\overline{N}^4). \tag{4.8}$$

Having successfully bounded s, we next estimate $\widetilde{w}_d^{(m)}(r)$ by substituting (4.8) into (4.4). Note first that for $0 \leq u \leq (d-1)/2$ one has $R_{2u+1} \leq rs^{d-2u-1}$, so that the hypotheses required for the application of the hypothesis $\mathcal{H}_{d-2}(\Psi)$ to bound $w_{d-2}^{(\overline{N})}(\mathbf{R})$ are satisfied. We therefore deduce from (4.4) that

$$\widetilde{w}_d^{(m)}(r) \le s + w_{d-2}^{(\overline{N})}(rs^2, rs^4, \ldots, rs^{d-1}) \le w_{d-2}^{(\overline{N})}(rs^2, rs^4, \ldots, rs^{d-3}, 2rs^{d-1}),$$

whence

$$\widetilde{w}_d^{(m)}(r) < \Psi(drs^2(\overline{N}+1)).$$

Write $\overline{s} = \Psi(\overline{N}^4)$. Then on recalling that Ψ satisfies the exponential growth condition, it follows from (4.7) and (4.8) that

$$\widetilde{w}_d^{(m)}(r) < \Psi((s(\overline{N}+1))^2) \le \Psi(\overline{s}^4) = \Psi((\Psi(\overline{N}^4))^4) \le \Psi_2(4\overline{N}^4) < \Psi_2(\overline{N}^5).$$

On making use of the bound

$$\overline{N} \le \overline{w}_d^{(n)}(r) = \Psi_{5\log(dn)}(drn),$$

therefore, we conclude that

$$\widetilde{w}_d^{(m)}(r) < \Psi_3(5\Psi^{-1}(\overline{N})) \le \Psi_3(\Psi_{5\log(dn)-1}(drm)).$$

But on noting that whenever $m \ge 2$ one has

$$5\log\left(d\left[\frac{m+1}{2}\right]\right) \le 5\log(dm) - 2,$$

we find from (4.2) that

$$\widetilde{w}_d^{(m)}(r) < \Psi_{5\log(dm)}(drm) = \overline{w}_d^{(m)}(r),$$

so that (4.3) holds with $M = m$ and $R = r$. On recalling the comments concluding the first paragraph of the proof, the lemma now follows by induction.

Following the lead provided in §3, we next bound $v_{d,r}^{(m)}$ on the hypothesis $\mathcal{H}_{d-2}(\Psi)$ by combining the conclusions of Lemmata 4.1 and 2.2.

LEMMA 4.2. *Let d be an odd integer exceeding 7. Suppose that Ψ is a function satisfying the exponential growth condition, and suppose further that the hypothesis $\mathcal{H}_{d-2}(\Psi)$ holds. Then whenever m and r are non-negative integers with $r \ge 1$, one has*

$$v_{d,r}^{(m)} < \Psi_{41\log(dr(m+1))}(dr(m+1)).$$

Proof. On combining Lemmata 2.2 and 2.4 with Lemma 4.1, one obtains

$$v_{d,r}^{(m)} < \Psi_{5\log(dM)}(drM), \tag{4.9}$$

where

$$M = 48(m+1)rd^3\log(3rd^2) \le 48(\log 3)d^5r^2(m+1) < d^7r^2(m+1). \tag{4.10}$$

It follows from (4.10) that

$$\log(dM) < 8\log(dr(m+1)) \quad \text{and} \quad \log(drM) < \exp(dr(m+1)),$$

and hence (4.9) leads to the upper bound

$$v_{d,r}^{(m)} < \Psi_{40\log(dr(m+1))+2}(dr(m+1)).$$

The conclusion of the lemma follows immediately.

In order to complete the inductive step we must combine the conclusion of the latter lemma with the hypothesis $\mathcal{H}_{d-2}(\Psi)$ in order to bound $w_d^{(m)}(\mathbf{r})$.

LEMMA 4.3. *Let d be an odd integer exceeding 7. Suppose that Ψ is a function satisfying the exponential growth condition, and suppose further that the hypothesis $\mathcal{H}_{d-2}(\Psi)$ holds. Then whenever r_{2u+1} $(0 \le u \le \frac{1}{2}(d-1))$ and m are non-negative integers with $r_i \le 3r_{i+2}^2$ $(i = 1, 3, \ldots, d-4)$ and $r_{d-2} < 3r_d^2$, one has*

$$w_d^{(m)}(\mathbf{r}) < \Psi_{42\log(dr_d(m+1))}(dr_d(m+1)).$$

Proof. By Lemma 2.3 one has

$$w_d^{(m)}(\mathbf{r}) \le w_{d-2}^{(v)}(r_{d-2}, \ldots, r_1),$$

where $v = v_{d,r_d}^{(m)}$. The hypotheses concerning r_i for $i = 1, 3, \ldots, d-2$ permit us the use of the hypothesis $\mathcal{H}_{d-2}(\Psi)$ to bound $w_{d-2}^{(v)}(r_{d-2}, \ldots, r_1)$, and thus on employing Lemma 4.2 to bound v, we deduce that

$$\begin{aligned} w_{d-2}^{(v)}(r_{d-2}, \ldots, r_1) &< \Psi(3(d-2)r_d^2(v+1)) \\ &< \Psi_2(2\Psi_{41\log(dr_d(m+1))-1}(dr_d(m+1))) \\ &\le \Psi_{41\log(dr_d(m+1))+2}(dr_d(m+1)). \end{aligned}$$

The conclusion of the lemma is now immediate.

5. The proof of Theorem 1. The machinery of our argument now fully assembled, we crank up the induction which establishes Theorem 1. Note first that, in view of Lemma 3.3, the hypothesis $\mathcal{H}_7(\psi^{(1)})$ holds, where $\psi^{(1)}$ is defined by (1.5). Moreover, $\psi^{(1)}$ plainly satisfies the exponential growth condition. Suppose next that d is an odd integer exceeding 7, and that the hypothesis $\mathcal{H}_{d-2}(\psi^{((d-7)/2)})$ holds. Since, plainly, $\psi^{((d-7)/2)}$ also satisfies the exponential growth condition, it follows from Lemma 4.3 that the hypothesis $\mathcal{H}_d(\psi^{((d-5)/2)})$ holds. We therefore deduce, by induction, that the hypothesis $\mathcal{H}_d(\psi^{((d-5)/2)})$ holds for every odd integer d exceeding 5. Consequently, on applying Lemma 4.2, we conclude that the inequality

$$v_{d,r}^{(m)} < \psi_{41\log(dr(m+1))}^{((d-7)/2)}(dr(m+1)) < \psi^{((d-5)/2)}(dr(m+1))$$

holds for every odd integer exceeding 5. This completes the proof of Theorem 1.

We conclude this section by providing a bound, similar to that recorded in Theorem 1, applicable to systems of forms of mixed degrees.

THEOREM 5.1. *Let d be an odd integer exceeding* 5, *and let $r_1, r_3, \ldots, r_d$ and m be non-negative integers with $r_d \geq 1$. Then*

$$w_d^{(m)}(r_d, \ldots, r_1; \mathbb{Q}) < \psi^{((d-5)/2)}(d(r_1 + r_3 + \ldots + r_d)(m+1)).$$

Proof. Let s be any integer exceeding $\psi^{((d-5)/2)}(d(r_1 + r_3 + \ldots + r_d) \times (m+1))$, and let $\mathcal{F}_{ij}(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_s]$ $(1 \leq j \leq r_i)$ be homogeneous polynomials of degree i for $i = 1, 3, \ldots, d$. We aim to show that the system

$$\mathcal{F}_{ij}(\mathbf{x}) = 0 \quad (1 \leq j \leq r_i;\ i = 1, 3, \ldots, d) \tag{5.1}$$

possesses a solution $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$, whence the theorem follows.

We define a new system of equations by writing

$$\mathcal{G}_{ij}(\mathbf{x}) = (x_1^2 + x_2^2 + \ldots + x_s^2)^{(d-i)/2}\mathcal{F}_{ij}(\mathbf{x}) \quad (1 \leq j \leq r_i;\ i = 1, 3, \ldots, d). \tag{5.2}$$

Plainly, each polynomial $\mathcal{G}_{ij}(\mathbf{x})$ is homogeneous of degree d, and thus the system of equations

$$\mathcal{G}_{ij}(\mathbf{x}) = 0 \quad (1 \leq j \leq r_i;\ i = 1, 3, \ldots, d) \tag{5.3}$$

consists of $r_1 + r_3 + \ldots + r_d$ equations of degree d. In view of the definition of s, it follows from Theorem 1 that the system (5.3) necessarily possesses a linear space of rational solutions of projective dimension m, whence by (5.2) the system (5.1) also possesses such a solution. This completes the proof of the theorem.

6. Appendix: Bounds stemming from Schmidt's method. Our discussion of upper bounds for $v_{d,r}^{(m)}$ would be incomplete without mention of Schmidt's sophisticated version of the circle method, which itself provides an interesting approach to bounding $v_{d,r}^{(0)}$, for odd exponents d. Before outlining the strategy described by Schmidt in §2 of [12], we require some notation. When $F \in \mathbb{Q}[\mathbf{x}]$ is a form of degree $d > 1$, write $h(F)$ for the least number h such that F may be written in the form

$$F = \sum_{i=1}^{h} A_i B_i,$$

with A_i, B_i forms in $\mathbb{Q}[\mathbf{x}]$ of positive degree. When $F_1, \ldots, F_r$ are forms of equal degree $d > 1$, and $\mathbf{F} = (F_1, \ldots, F_r)$, write $h(\mathbf{F}) = \min h(F)$, where the minimum is taken over forms F lying in the rational pencil of $\mathbf{F}$, which is to say the set

$$\{F \in \mathbb{Q}[\mathbf{x}] : F = c_1F_1 + \ldots + c_rF_r \text{ and } (c_1, \ldots, c_r) \in \mathbb{Q}^r \setminus \{\mathbf{0}\}\}.$$

The important achievement of Schmidt in [12] is the development of a version of the Hardy–Littlewood method which, given a system of forms $\mathbf{F} \in \mathbb{Q}[\mathbf{x}]^r$ with $h(\mathbf{F})$ large enough, establishes an asymptotic formula of the expected shape for the number of integral zeros of $\mathbf{F}$ inside a large box. Here, "formula of the expected shape" simply means the product of local densities, embodied in the product of the singular series and singular integral familiar to experts in the circle method. As Schmidt [12, §2] observes, such a result offers the possibility of a reduction process which, given sufficiently many variables, establishes the existence of a non-trivial rational zero to a given system. Roughly speaking, one observes that for forms of odd degree, whenever $h(\mathbf{F})$ is large enough Schmidt's circle method already establishes the existence of a non-trivial rational zero. On the other hand, if $h(\mathbf{F})$ is not large enough, say $h(\mathbf{F}) \le h_0$, then some form F lying in the rational pencil of $\mathbf{F}$ decomposes in the shape

$$F = \sum_{i=1}^{h_0} A_iB_i,$$

with A_i, B_i forms in $\mathbb{Q}[\mathbf{x}]$ of positive degree. Since the forms F_i are of odd degree, one may assume without loss of generality that the A_i are all of odd degree. Consequently, the system $\mathbf{F}$ is soluble non-trivially provided that there is a non-trivial solution to the system

$$\widehat{\mathbf{F}}(\mathbf{x}) = \mathbf{0} \quad \text{and} \quad A_i(\mathbf{x}) = 0 \quad (1 \le i \le h_0),$$

where $\widehat{\mathbf{F}}$ denotes the system $\mathbf{F}$ with a suitable form deleted. Since the A_i all have odd degree smaller than the deleted form, one perceives an obvious reductive strategy which ultimately either shows the system to be non-trivially soluble, or else reduces it to a system of linear equations, which again is non-trivially soluble. We summarise this approach in quantitative form in the following lemma.

Lemma 6.1. *Let d be an odd integer exceeding* 1, *let $r_1, r_3, \ldots, r_d$ be non-negative integers with $r_d \ge 1$, and define*

$$\widetilde{h}_d(\mathbf{r}; k) = 2^{4k}k!r_kdw_d^{(0)}(\mathbf{r}; \mathbb{Q}_p) \qquad (3 \le k \le d).$$

Then

$$\text{(6.1)} \qquad w_d^{(0)}(\mathbf{r}; \mathbb{Q}) \le \max_{k=3,5,\ldots,d} w_d^{(0)}(r_d, r_{d-2}, \ldots, r_k - 1, r_{k-2} + \widetilde{h}_d(\mathbf{r}; k), r_{k-4}, \ldots, r_1; \mathbb{Q}).$$

Proof. The conclusion (6.1) is immediate from Theorem II of [12] and its supplement, on applying the argument described by Schmidt in §2 of [12].

We are now in a position to illustrate the use of Schmidt's strategy. We concentrate on systems of septic forms, such systems displaying the salient features and problems of the method. Although we will be somewhat rough in our estimates, it should be clear that Lemma 6.1 is incapable of doing substantially better.

Consider bounds for $w_7^{(0)}(r_7, r_5, r_3, 0)$. Write $R = R(\mathbf{r}) = r_7 + r_5 + r_3$, and suppose throughout that R is large. We recall an immediate consequence of Corollary 1.1 of Wooley [14] (improving on earlier work of Leep and Schmidt [7]), which provides the estimate

$$w_d^{(0)}(r_d, r_{d-2}, \ldots, r_1; \mathbb{Q}_p) \le (r_d + r_{d-2} + \ldots + r_1)^{2^{d-1}} d^{2^d}.$$

It follows that there is an integral constant $c_1 > 0$ such that whenever $r_5 > 0$, the best bounds stemming from Lemma 6.1 cannot be substantially stronger than

$$w_7^{(0)}(r_7, r_5, r_3, 0) \le w_7^{(0)}(r_7, r_5 - 1, r_3 + R^{c_1}, 0). \tag{6.2}$$

Meanwhile, when $r_5 = 0$ and $r_7 > 0$, there is an integral constant $c_2 > 0$ such that the corresponding best available bound from Lemma 6.1 cannot be substantially stronger than

$$w_7^{(0)}(r_7, 0, r_3, 0) \le w_7^{(0)}(r_7 - 1, R^{c_2}, r_3, 0). \tag{6.3}$$

We next bound $v_{7,r}^{(0)}$ by use of (6.2) and (6.3), as suggested by Schmidt's strategy. When r is large, one obtains

$$\begin{aligned} v_{7,r}^{(0)} = w_7^{(0)}(r, 0, 0, 0) &\le w_7^{(0)}(r - 1, r^{c_2}, 0, 0) \\ &\le w_7^{(0)}(r - 1, r^{c_2} - 1, c_3 r^{c_1 c_2}, 0), \end{aligned}$$

for an integral constant $c_3 > 0$. Next, by repeated application of (6.2) one obtains for $n = 1, \ldots, r^{c_2}$ the estimate

$$w_7^{(0)}(r, 0, 0, 0) \le w_7^{(0)}(r - 1, r^{c_2} - n, c_{n+2} r^{c_1^n c_2}, 0),$$

with $c_n > 0$ uniformly bounded above by a fixed real number. Consequently, when r is large enough one has

$$v_{7,r}^{(0)} \le w_7^{(0)}(r - 1, 0, [\exp_2(r^{a_1})], 0),$$

for a suitable fixed real number $a_1 > 0$. We next apply (6.3), and then again repeatedly apply (6.2). We now obtain

$$v_{7,r}^{(0)} \le w_7^{(0)}(r - 2, 0, [\exp_4(r^{a_2})], 0),$$

for a suitable fixed real number $a_2 > 0$. Repeating this iteration pattern, we ultimately obtain

$$v_{7,r}^{(0)} \le w_7^{(0)}(0,0,[\exp_{2r}(r^a)],0),$$

for a suitable fixed real number $a > 0$. It therefore follows from Schmidt [10, Theorem 1] that

$$v_{7,r}^{(0)} \ll \exp_{2r+1}(r). \tag{6.4}$$

Notice that in deriving the upper bound (6.4), it is conceivable that in following Schmidt's strategy, we are forced by the structure of our implicit forms to apply the bounds (6.2) and (6.3) in the indicated fashion. Thus we conclude that Schmidt's method is incapable of providing estimates substantially stronger than (6.4). By contrast, Theorem 1 of the present paper establishes the bound

$$v_{7,r}^{(0)} \le \exp_{42\log(7r)}(7r),$$

which is plainly stronger for large r.

For larger odd exponents d, the worst bounds that arise from Schmidt's strategy are obtained as follows. One repeatedly applies Lemma 6.1 so as to reduce the number of implicit equations of lowest degree until none of that degree remain. Here one ignores implicit linear equations. One then takes the next lowest degree for which there are implicit equations, applies Lemma 6.1 so as to reduce the number of such equations, this in turn spawning many equations of lower degree. An analysis only slightly more careful than that above will reveal that the bounds attainable by such an approach take the shape (1.6) indicated in the introduction.

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