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Zero order estimates for functions satisfying generalized functional equations of Mahler type

by

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1. Introduction and results. Zero order estimates for analytic functions are closely related to problems in the theory of transcendental numbers. The basic question, if the value $f(\alpha)$ of a transcendental function f at an algebraic point α is transcendental or—more generally—if the values $f_1(\alpha),\ldots,f_m(\alpha)$ of several algebraically independent functions $f_1,\ldots,f_m$ are algebraically independent for algebraic α, can be changed into the quantitative problem to give lower bounds for $|P(f_1(\alpha),\ldots,f_m(\alpha))|$ in terms of the degree and the height of the polynomial $P\in\mathbb{Z}[y_1,\ldots,y_m]\setminus\{0\}$, and in general zero order estimates are necessary to solve this problem.

In the case of Mahler functions $f:U_1(0)\to\mathbb{C}$, which satisfy (in the simplest case) a functional equation of the form

$$f(z^d)=R(z,f(z))$$

with $d\in\mathbb{N}$, $d\ge 2$, and a rational function $R(z,y)$, the qualitative and the quantitative question are extensively studied. For a historical survey of the qualitative transcendence results see [K], [L], [LP], and transcendence measures can be found in [NT] and in the references given there. The first measures for algebraic independence were proved by Becker [B1] and—using a completely different method—by Nesterenko [Ne3]. Both results are effective in the height, but not in the dependence on the degree of the polynomial P. This is due to the fact that the construction of the auxiliary function, which is needed in the proof, depends on Siegel's lemma. Since this construction is not explicit, a zero order estimate for the auxiliary function is necessary to derive completely effective measures, and at that time no zero order estimate was available.

Using elementary methods, Wass [W] obtained a zero order estimate and gave an effective version of Nesterenko's result. One year earlier Nish-

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ioka derived the following general zero order estimate, which is much better than Wass' result. The proof was published in [Ni1] and is an extension of Nesterenko's elimination-theoretic method in [Ne1]; more exactly, the method of [Ne2] is applied to the polynomial ring $C[z]$ over a field C of characteristic 0, and applications of this theorem were given by Becker [B2], Nishioka [Ni2], and Töpfer [T1], [T2].

THEOREM (Nishioka [Ni1]). *Let $f_1, \ldots, f_m \in C[[z]]$ be formal power series with coefficients in a field C of characteristic* 0 *and satisfy*

$$f_i(z^d) = \frac{A_i(z, f_1(z), \ldots, f_m(z))}{A_0(z, f_1(z), \ldots, f_m(z))} \quad (1 \le i \le m),$$

where $d \in \mathbb{N}$, $d \ge 2$, and $A_i \in C[z, y_1, \ldots, y_m]$ $(0 \le i \le m)$ are polynomials with $\deg_z A_i \le s$ and $\deg_{y_1,\ldots,y_m} A_i \le t$. Suppose that $t^m < d$ and $Q \in C[z, y_1, \ldots, y_m]$ with $\deg_z Q \le M$, $\deg_{y_1,\ldots,y_m} Q \le N$ and $M \ge N \ge 1$. If $Q(z, f_1(z), \ldots, f_m(z)) \ne 0$, then

$$\operatorname{ord}_0 Q(z, f_1(z), \ldots, f_m(z)) \le c_0 M N^{m \log d/(\log d - m \log t)},$$

where $\mu = 1 + s/(d-t)$ and

$$c_0 = \max\left\{ \frac{\operatorname{ord}_0 A_0(z, f_1(z), \ldots, f_m(z))}{d-t}, \right.$$
$$\left. 8m^2 (8dt)^m \mu (12m(8d)^{m-1})^{m \log t/(\log d - m \log t)} \right\}.$$

Recently a more general kind of functional equations was studied by Becker [B3], [B4], [B5]. Suppose that the function f is holomorphic in a neighborhood U of a point $\theta \in \widehat{\mathbb{C}}$, the power series expansion of f at θ has algebraic coefficients, the transformation T is meromorphic in U and algebraic over the function field $\overline{\mathbb{Q}}(z)$ over the algebraic numbers, and f satisfies a functional equation

$$A(z, f(z), f(T(z))) = 0 \tag{1}$$

for $z \in U$ and a polynomial $A(z, y, w)$ with algebraic coefficients. Under certain assumptions on f, T, θ, A, and α Becker [B4] proved that $f(\alpha)$ is transcendental. Quantitative results for functions which satisfy functional equations of the form (1) with polynomial transformations $T(z) \in \overline{\mathbb{Q}}[z]$ and $A(z, y, w) = w - q(y)$, $q \in \overline{\mathbb{Q}}[z]$ with $\deg q = \deg T$, the so-called *Böttcher functions*, can be found in [B5].

Qualitative algebraic independence results for certain rational transformations were given by Becker [B3] for functions $f_1, \ldots, f_m$ satisfying

$$f_i(z) = a_i(z) f_i(T(z)) + b_i(z) \quad (1 \le i \le m) \tag{2}$$

with $a_i, b_i \in \overline{\mathbb{Q}}(z)$ and $T(z) = p(z^{-1})^{-1}$, $p \in \overline{\mathbb{Q}}[z]$ of degree at least 2. In this paper we consider a generalization of (2) and state a zero order estimate which generalizes the above mentioned result of Nishioka. Applications of this result to algebraic independence are given in [T3].

THEOREM 1. *Let $f_1, \ldots, f_m \in C[[z]]$ be formal power series with coefficients in a field C of characteristic* 0 *and satisfy*

$$f_i(T(z)) = \frac{A_i(z, \underline{f}(z))}{A_0(z, \underline{f}(z))} \quad (1 \le i \le m),$$

where $\underline{f}(z) = (f_1(z), \ldots, f_m(z))$, $T(z) = T_1(z)/T_2(z)$ is a rational function with $T_1, T_2 \in C[z]$, $d = \max\{\deg T_1, \deg T_2\}$, $\delta = \operatorname{ord}_0 T(z) \ge 2$, and $A_i \in C[z, y_1, \ldots, y_m]$ $(0 \le i \le m)$ are polynomials with $\deg_z A_i \le s$ and $\deg_{y_1,\ldots,y_m} A_i \le t$. Suppose that $t^m < \delta$ and $Q \in C[z, y_1, \ldots, y_m]$ with $\deg_z Q \le M$, $\deg_{y_1,\ldots,y_m} Q \le N$ and $M \ge N \ge 1$. If $Q(z, \underline{f}(z)) \ne 0$, then

$$\operatorname{ord}_0 Q(z, \underline{f}(z)) \le c_1 M N^{m \log d/(\log \delta - m \log t)},$$

where $\mu = 1 + s/(d - t)$ and

$$c_1 = \max\left\{ \frac{\operatorname{ord}_0 A_0(z, \underline{f}(z))}{\delta - t}, \right. \\ \left. \mu d \delta^{-1} m^2 (8\delta t)^m (4m(8\delta)^{m-1})^{\log d/(\log \delta - m \log t) - 1} \right\}.$$

REMARK. In the special case $T(z) = z^d$, we have $\delta = d$, and the assertion of the theorem is just Nishioka's result [Ni1] with a slightly better constant.

COROLLARY 1. *Let $f_1, \ldots, f_m \in C[[z]]$ be formal power series with coefficients in a field C of characteristic* 0 *which satisfy*

$$f_i(z) = a_i(z) f_i(T(z)) + b_i(z) \quad (1 \le i \le m),$$

where $a_i, b_i \in C(z)$ are rational functions, $T(z) = p(z^{-1})^{-1}$ with a polynomial $p \in C[z]$ and $d = \deg p \ge 2$. Suppose that $Q \in C[z, y_1, \ldots, y_m]$ with $\deg_z Q \le M$, $\deg_{y_1,\ldots,y_m} Q \le N$ and $M \ge N \ge 1$. If $Q(z, \underline{f}(z)) \ne 0$, then

$$\operatorname{ord}_0 Q(z, \underline{f}(z)) \le c_1 M N^m$$

with $c_1 = c_1(a_i, b_j, d, m) \in \mathbb{R}_+$ as in Theorem 1.

Proof. Notice that $d = \deg p = \operatorname{ord}_0 T = \delta > t = 1$. ∎

COROLLARY 2. *Let $f_1, \ldots, f_m \in C[[z]]$ be formal power series with coefficients in a field C of characteristic* 0 *which satisfy*

$$f_i(z) = a_i(z) f_i(T(z)) + b_i(z) \quad (1 \le i \le m),$$

where $a_i, b_i \in C(z)$ are rational functions and $T \in C[z]$ is a polynomial with $d = \deg T \ge \delta = \operatorname{ord}_0 T \ge 2$. Suppose that $Q \in C[z, y_1, \ldots, y_m]$ with

$\deg_z Q \le M$, $\deg_{y_1,\dots,y_m} Q \le N$ *and* $M \ge N \ge 1$. *If* $Q(z, \underline{f}(z)) \ne 0$, *then*

$$\operatorname{ord}_0 Q(z,\underline{f}(z)) \le c_1 M N^{m\log d/\log\delta}$$

with $c_1 = c_1(a_i, b_j, d, \delta, m) \in \mathbb{R}_+$ *as in Theorem* 1.

The proof of Theorem 1 depends on the following criterion for algebraic independence over fields of Laurent series. This criterion is based on Nishioka's result [Ni1], hence on the elimination-theoretic method of Nesterenko [Ne1], [Ne2] and Philippon [P1], [P2].

For the statement of the criterion we need some notations. Suppose C is a field of characteristic 0, v the valuation ord_0 of the field $C((z))$ of Laurent series or its unique extension to the algebraic closure $\overline{C((z))}$. For $\underline{\omega} \in \overline{C((z))}^m$ put $v(\underline{\omega}) = \min_{1\le i\le m}\{v(\omega_i)\}$, and for polynomials $Q(z, y_0, y_1, \dots, y_m) \in C[\underline{y}]$ with

$$Q(z,\underline{y}) = \sum_{\mu_0,\dots,\mu_m=0}^{\sigma} q_{\mu_0,\dots,\mu_m}(z) y_0^{\mu_0}\dots y_m^{\mu_m}$$

define

$$v(Q) = \min_{\mu_0,\dots,\mu_m}\{v(q_{\mu_0,\dots,\mu_m})\},\quad N(Q) = \deg_{y_1,\dots,y_m} Q,\quad H(Q) = \deg_z Q.$$

THEOREM 2. *Let C be a field of characteristic* 0 *and* $\underline{\omega} \in \overline{C((z))}^m$. *Suppose that there exist increasing functions* $\Psi_1, \Psi_2 : \mathbb{N} \to \mathbb{R}_+$, *positive real numbers* Φ_1, Φ_2, Λ, *a nonnegative integer* k_1 *and for each* $k \in \{0, \dots, k_1\}$ *a set of polynomials* $Q_k^{(1)}, \dots, Q_k^{(n_k)} \in C[z, y_1, \dots, y_m]$ *with the following properties for* $k \in \{0, \dots, k_1\}$, $i \in \{1, \dots, n_k\}$:

(i) $\Phi_2 \ge \Phi_1$, $\Psi_2(k) \ge \max\{\Psi_1(k), -2v(\underline{\omega})\}$, $\Lambda \ge \Psi_2(k+1)/\Psi_1(k)$,

(ii) (a) $N(Q_k^{(i)}) \le \Phi_1$,

(b) $H(Q_k^{(i)}) \le \Phi_2$,

(c) $v(Q_k^{(i)}(\underline{\omega})) \ge \Psi_1(k)$,

(d) $v(\underline{\omega} - \underline{\theta}) \le \Psi_2(k)$ *for all common zeros* $\underline{\theta} \in \overline{C((z))}^m$ *of* $Q_k^{(1)}, \dots, Q_k^{(n_k)}$,

(iii) $\Psi_1(k_1) > 2m(4\Lambda)^{m-1} c_3 \Phi_1^{m-1} \max\{\Phi_1\Psi_2(0), m\Phi_2\}$, *where* $c_3 = 1$ *for* $v(\underline{\omega}) \ge 0$ *and* $c_3 = (2m)^m$ *for* $v(\underline{\omega}) < 0$.

Then we have with $c_4 = m$ *for* $v(\underline{\omega}) \ge 0$ *and* $c_4 = 2^m m^{m+2}$ *for* $v(\underline{\omega}) < 0$,

$$\Psi_1(k_1) \le c_4 (4\Lambda)^m \Phi_1^m \Phi_2.$$

2. Notations and lemmas. For polynomials $Q(z, y_0, y_1, \dots, y_m) \in R[\underline{y}]$ with $R = C[z]$ let $H(Q)$, $N(Q)$, $v(Q)$ be defined as above. If $I \subset R[\underline{y}]$ is a homogeneous ideal, then $h(I)$ denotes the height of I, $\operatorname{rad} I$ is the radical of I, and $Z(I)$ is the zero set of I in $\overline{C((z))}^{m+1} \setminus \{\underline{0}\}$. For the definition of $N(I)$, $H(I)$ (resp. $B(I)$ in [Ni1]) and $v(I(\underline{\beta}))$ for $\underline{\beta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$ the reader is referred to Nishioka's paper [Ni1]. The *projective distance* of $\underline{\beta}, \underline{\theta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$ is defined as

$$V(\underline{\beta}, \underline{\theta}) = -v(\underline{\beta}) - v(\underline{\theta}) + \min_{0 \le i,j \le m} \{v(\beta_i\theta_j - \beta_j\theta_i)\},$$

and for homogeneous ideals I put

$$V(\underline{\beta}, Z(I)) = \sup_{\underline{\theta} \in Z(I)} \{V(\underline{\beta}, \underline{\theta})\}.$$

LEMMA 1. *Suppose that $P \in R[\underline{y}] \setminus \{0\}$ is a homogeneous polynomial, $I = (P)$ is the principal ideal in $R[\underline{y}]$ generated by P, and $\underline{\beta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$. Then*

$$N(I) = N(P), \quad H(I) \le H(P), \quad v(I(\underline{\beta})) \ge v(P(\underline{\beta})) - N(P)v(\underline{\beta}).$$

Proof. See [Ni1], Proposition 1. ∎

LEMMA 2. *Suppose that $\underline{\beta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$, I is an unmixed homogeneous ideal in $R[\underline{y}]$, $h(I) \le m$, and $I = I_1 \cap \ldots \cap I_s \cap I_{s+1} \cap \ldots \cap I_t$ is its irreducible primary decomposition with $I_l \cap R = (0)$ for $l \le s$ and $I_{s+1} \cap \ldots \cap I_t = (b)$, $b \in R \setminus \{0\}$. For $l \le s$ let k_l be the exponent of the ideal I_l and $\mathcal{P}_l = \operatorname{rad} I_l$. Then*

(i) $\sum_{l=1}^{s} k_l N(\mathcal{P}_l) = N(I)$,
(ii) $H(b) + \sum_{l=1}^{s} k_l H(\mathcal{P}_l) = H(I)$,
(iii) $v(b) + \sum_{l=1}^{s} k_l v(\mathcal{P}_l(\underline{\beta})) = v(I(\underline{\beta}))$,
(iv) $0 \le v(b) \le H(b) \le H(I)$.

When $s = t$, the terms $H(b)$ and $v(b)$ are missing.

Proof. See [Ni1], Proposition 2. ∎

LEMMA 3. *Suppose that $\underline{\beta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$, $\mathcal{P}$ is a nonzero homogeneous prime ideal of $R[\underline{y}]$ with $\mathcal{P} \cap R = (0)$ and $h(\mathcal{P}) \le m$, $Q \in R[\underline{y}]$ is a homogeneous polynomial with $Q \notin \mathcal{P}$ and*

$$\Lambda(v(Q(\underline{\beta})) - v(\underline{\beta})N(Q)) \ge \min\{X, V(\underline{\beta}, Z(\mathcal{P}))\} > 0,$$

where $v(\mathcal{P}(\underline{\beta})) \ge X$ and $\Lambda \ge 1$. If $r = m+1-h(\mathcal{P}) \ge 2$, then there exists an unmixed homogeneous ideal $I \subset R[\underline{y}]$ with $Z(I) = Z(\mathcal{P}, Q)$, $h(I) = m-r+2$, such that

(i) $N(I) \le N(\mathcal{P})N(Q)$,
(ii) $H(I) \le H(\mathcal{P})N(Q) + N(\mathcal{P})H(Q)$,
(iii) $v(I(\underline{\omega})) \ge X/\Lambda - H(\mathcal{P})N(Q) - N(\mathcal{P})H(Q)$.

If $h(\mathcal{P}) = m$, then the right side of inequality (iii) *is not positive.*

Proof. If $X \le V(\underline{\beta}, Z(\mathcal{P}))$, we know

$$v(Q(\underline{\beta})) - v(\underline{\beta})N(Q) \ge X/\Lambda,$$

and Lemma 3 of [Ni1] yields the assertion. If $V(\underline{\beta}, Z(\mathcal{P})) \le X$, we have

$$v(Q(\underline{\beta})) - v(\underline{\beta})N(Q) \ge V(\underline{\beta}, Z(\mathcal{P}))/\Lambda,$$

and Lemma 4 of [Ni1] implies the assertion. ∎

LEMMA 4. *Suppose* $I \subset R[\underline{y}]$ *is a nonzero unmixed homogeneous ideal*, $I \cap R = (0)$, *and* $r = m + 1 - h(I) \ge 1$. *Then for every* $\underline{\beta} \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$ *we have*

$$N(I)V(\underline{\beta}, Z(I)) \ge v(I(\underline{\beta}))/r - 2H(I).$$

Proof. See Lemma 6 of [Ni1]. ∎

3. Proof of Theorem 2. The proof is analogous to the proof of Theorem 6 in [T1]. As usual in elimination theory, we show by induction that there exist homogeneous prime ideals $\mathcal{P}_l \subset R[\underline{y}]$ with $h(\mathcal{P}_l) = l$ $(l = 1, \ldots, m)$, which satisfy

$$N(\mathcal{P}_l) \le \Phi_1^l, \tag{3}$$

$$H(\mathcal{P}_l) \le l\Phi_1^{l-1}\Phi_2, \tag{4}$$

$$v(\mathcal{P}_l(\underline{\beta})) \ge \frac{\Psi_1(k_1)}{2(4\Lambda)^{l-1}\Phi_1^l} N(\mathcal{P}_l) + \frac{\Psi_1(k_1)}{2(4\Lambda)^{l-1} l \Phi_1^{l-1}\Phi_2} H(\mathcal{P}_l), \tag{5}$$

where $\underline{\beta} = (1, \underline{\omega}) \in \overline{C((z))}^{m+1} \setminus \{\underline{0}\}$ for $\underline{\omega} \in \overline{C((z))}^{m}$ as in Theorem 2. In the last step for $l = m + 1$ Lemma 3 implies the asserted inequality of Theorem 2.

Without loss of generality we may assume that $v(\underline{\omega}) \ge 0$. If $v(\underline{\omega}) < 0$, we suppose that $v(\omega_1), \ldots, v(\omega_\kappa) < 0 \le v(\omega_{\kappa+1}), \ldots, v(\omega_m)$ and apply the transformation

$$\begin{aligned} Q(y_1, \ldots, y_m) \to \overline{Q}&(y_1, \ldots, y_m) \\ &= (y_1 \ldots y_\kappa)^{\deg Q} Q(1/y_1, \ldots, 1/y_\kappa, y_{\kappa+1}, \ldots, y_m) \end{aligned}$$

to all polynomials which occur in the proof. Thus with $\overline{\underline{\omega}} = (1/\omega_1, \ldots, 1/\omega_\kappa, \omega_{\kappa+1}, \ldots, \omega_m)$ we have

$$N(\overline{Q}) \le m \deg Q \le m\Phi_1 = \Phi_1^*, \qquad H(\overline{Q}) = H(Q) \le \Phi_2 \le m\Phi_2 = \Phi_2^*,$$
$$v(\overline{Q}(\overline{\underline{\omega}})) = v((\omega_1 \ldots \omega_\kappa)^{-\deg Q} Q(\underline{\omega})) \ge v(Q(\underline{\omega})) \ge \Psi_1(k) = \Psi_1^*(k).$$

Now we suppose that $\overline{\underline{\theta}} = (\overline{\theta}_1, \ldots, \overline{\theta}_m)$ is a common zero of $\overline{Q}_k^{(1)}, \ldots, \overline{Q}_k^{(n_k)}$. If $\overline{\theta}_i = 0$ for some $i \in \{1, \ldots, \kappa\}$, then $v(\overline{\underline{\omega}} - \overline{\underline{\theta}}) \le v(\overline{\omega}_i) = -v(\omega_i) \le -v(\underline{\omega}) \le$

$\Psi_2(k)$; otherwise

$$\begin{aligned} v(\overline{\underline{\omega}}-\overline{\underline{\theta}}) &= \min_{\substack{1\le i\le\kappa\\ \kappa+1\le j\le m}} \{-v(\omega_i)-v(\theta_i)+v(\omega_i-\theta_i), v(\omega_j-\theta_j)\} \\ &\le -2v(\underline{\omega})+v(\underline{\omega}-\underline{\theta}) \le 2\Psi_2(k)=\Psi_2^*(k). \end{aligned}$$

Hence (i), (ii) of Theorem 2 are fulfilled with $\Lambda^*=2\Lambda$, $v(\overline{\underline{\omega}})\ge 0$, and (iii) follows from

$$\begin{aligned} \Psi_1^*(k_1) &> 2m(4\Lambda)^{m-1}2^{m-1}(m\Phi_1)^{m-1}\max\{2m\Phi_1\Psi_2(0), m\Phi_2\} \\ &= 2m(4\Lambda^*)^{m-1}\Phi_1^{*\,m-1}\max\{\Phi_1^*\Psi_2^*(0),\Phi_2^*\}. \end{aligned}$$

Therefore we suppose from now on that all assumptions of Theorem 2 are satisfied with $v(\underline{\omega})\ge 0$.

Throughout the proof of Theorem 2 let Q^* denote the homogenization of the polynomial $Q\in R[y_1,\ldots,y_m]$, i.e. $Q^*\in R[y_0,y_1,\ldots,y_m]=R[\underline{y}]$ is homogeneous with $\deg_{\underline{y}} Q^* = \deg_{y_1,\ldots,y_m} Q$ and $Q^*(1,y_1,\ldots,y_m)=Q(y_1,\ldots,y_m)$.

In the first step, $l=1$, we choose one of the polynomials $Q_{k_1}^{(1)},\ldots,Q_{k_1}^{(n_{k_1})}$, say $Q_{k_1}^{(1)}$, and define the unmixed homogeneous ideal $I^{(1)}=(Q_{k_1}^{(1)*})\subset R[\underline{y}]$. Then $h(I^{(1)})=1$ and, by Lemma 1,

$$(6)\quad N(I^{(1)})\le\Phi_1,\quad H(I^{(1)})\le\Phi_2,\quad v(I^{(1)}(\underline{\beta}))\ge v(Q_{k_1}^{(1)}(\underline{\omega}))\ge\Psi_1(k_1).$$

Now suppose that $\mathcal{P}^{(1)},\ldots,\mathcal{P}^{(s)}\subset R[\underline{y}]$ are the associated prime ideals of $I^{(1)}$, which are defined in Lemma 2. Then $N(\mathcal{P}^{(i)})\le\Phi_1$, $H(\mathcal{P}^{(i)})\le\Phi_2$, $h(\mathcal{P}^{(i)})=1$ for $i=1,\ldots,s$. If none of the prime ideals $\mathcal{P}^{(i)}$ satisfies inequality (5), we have

$$v(\mathcal{P}^{(i)}(\underline{\beta}))<\frac{\Psi_1(k_1)}{2\Phi_1}N(\mathcal{P}^{(i)})+\frac{\Psi_1(k_1)}{2\Phi_2}H(\mathcal{P}^{(i)})$$

for $i=1,\ldots,s$, and Lemma 2(iii), (iv) together with Theorem 2(iii) implies

$$v(I^{(1)}(\underline{\beta}))<v(b)+\frac{\Psi_1(k_1)}{2\Phi_1}\sum_{i=1}^{s}k_iN(\mathcal{P}^{(i)})+\frac{\Psi_1(k_1)}{2\Phi_2}\sum_{i=1}^{s}k_iH(\mathcal{P}^{(i)})\le\Psi_1(k_1),$$

but this contradicts the rightmost inequality of (6). Thus at least one prime ideal, say $\mathcal{P}^{(1)}$, satisfies (3)–(5), and we define $\mathcal{P}_1=\mathcal{P}^{(1)}$.

Now we assume that (3)–(5) are fulfilled for $l-1$ with $l\in\{2,\ldots,m\}$. With

$$X=\frac{\Psi_1(k_1)}{2(4\Lambda)^{l-2}\Phi_1^{l-1}}N(\mathcal{P}_{l-1})+\frac{\Psi_1(k_1)}{2(4\Lambda)^{l-2}(l-1)\Phi_1^{l-2}\Phi_2}H(\mathcal{P}_{l-1})$$

the inequalities $v(\mathcal{P}_{l-1}(\underline{\beta}))\ge X>\Psi_2(0)$ hold, the latter by Theorem 2(iii).

Furthermore Lemma 4 and Theorem 2(iii) imply

$$V(\underline{\beta}, Z(\mathcal{P}_{l-1})) \geq \frac{X}{(m+1-(l-1))N(\mathcal{P}_{l-1})} - 2\frac{H(\mathcal{P}_{l-1})}{N(\mathcal{P}_{l-1})} > \Psi_2(0).$$

Since

$$X \leq \Psi_1(k_1)\left(\frac{1}{2(4\Lambda)^{l-2}} + \frac{1}{2(4\Lambda)^{l-2}(l-1)}\right) \leq \Psi_1(k_1) \leq \Psi_2(k_1),$$

there exists a number $k_l \in \{0, \ldots, k_1\}$ with

$$\Psi_2(k_l) < \min\{X, V(\underline{\beta}, Z(\mathcal{P}_{l-1}))\} \leq \Psi_2(k_l + 1).$$

We claim that at least one of the polynomials $Q_{k_l}^{(1)*}, \ldots, Q_{k_l}^{(n_{k_l})*}$ does not belong to $\mathcal{P}_{l-1}$. Otherwise $Z(\mathcal{P}_{l-1}) \subset Z(Q_{k_l}^{(1)*}, \ldots, Q_{k_l}^{(n_{k_l})*})$, and then Theorem 2(ii)(d) implies after some calculation

$$\Psi_2(k_l) < V(\underline{\beta}, Z(\mathcal{P}_{l-1})) \leq V(\underline{\beta}, Z(Q_{k_l}^{(1)*}, \ldots, Q_{k_l}^{(n_{k_l})*})) \leq \Psi_2(k_l),$$

but this is a contradiction. Without loss of generality we may assume that $Q_{k_l}^{(1)*} \notin \mathcal{P}_{l-1}$.

Define $\sigma \in \mathbb{R}_+$ by

$$\min\{X, V(\underline{\beta}, Z(\mathcal{P}_{l-1}))\} = \sigma v(Q_{k_l}^{(1)*}(\underline{\beta})) = \sigma v(Q_{k_l}^{(1)}(\underline{\omega})).$$

From Theorem 2(i), (ii)(c) and the choice of k_l we get

$$\sigma \Psi_1(k_l) \leq \sigma v(Q_{k_l}^{(1)}(\underline{\omega})) \leq \Psi_2(k_l + 1) \leq \Lambda \Psi_1(k_l),$$

hence $\sigma \leq \Lambda$ and

$$\Lambda v(Q_{k_l}^{(1)*}(\underline{\beta})) \geq \min\{X, V(\underline{\beta}, Z(\mathcal{P}_{l-1}))\}$$

with $\Lambda \geq 1$ (notice that $v(\underline{\beta}) = v(1) = 0$). By Lemma 3 and Theorem 2(ii), (iii) there exists an unmixed homogeneous ideal $I^{(l)} \subset R[\underline{y}]$ with $h(I^{(l)}) = l$ and

$$N(I^{(l)}) \leq \Phi_1 N(\mathcal{P}_{l-1}) \leq \Phi_1^l, \tag{7}$$

$$H(I^{(l)}) \leq \Phi_1 H(\mathcal{P}_{l-1}) + \Phi_2 N(\mathcal{P}_{l-1}) \leq l\Phi_1^{l-1}\Phi_2, \tag{8}$$

$$v(I^{(l)}(\underline{\beta})) \geq \frac{\Psi_1(k_1)}{(4\Lambda)^{l-1}\Phi_1^{l-1}} N(\mathcal{P}_{l-1}) + \frac{\Psi_1(k_1)}{(4\Lambda)^{l-1}(l-1)\Phi_1^{l-2}\Phi_2} H(\mathcal{P}_{l-1}). \tag{9}$$

Once more we consider the associated prime ideals $\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(s)}$ of the ideal $I^{(l)}$ according to Lemma 2, which satisfy

$$N(\mathcal{P}^{(i)}) \leq \Phi_1^l, \quad H(\mathcal{P}^{(i)}) \leq l\Phi_1^{l-1}\Phi_2.$$

If none of the prime ideals $\mathcal{P}^{(i)}$, $1 \le i \le s$, satisfies (5), from Lemma 2 and (7), (8) we get

$$\begin{aligned} v(I^{(l)}(\underline{\beta})) &\\ &< v(b) + \frac{\Psi_1(k_1)}{2(4\Lambda)^{l-1}\Phi_1^l} \sum_{i=1}^{s} k_i N(\mathcal{P}^{(i)}) + \frac{\Psi_1(k_1)}{2l(4\Lambda)^{l-1}\Phi_1^{l-1}\Phi_2} \sum_{i=1}^{s} k_i H(\mathcal{P}^{(i)}) \\ &\le \frac{\Psi_1(k_1)}{(4\Lambda)^{l-1}\Phi_1^{l-1}} N(\mathcal{P}_{l-1}) + \frac{\Psi_1(k_1)}{(l-1)(4\Lambda)^{l-1}\Phi_1^{l-2}\Phi_2} H(\mathcal{P}_{l-1}), \end{aligned}$$

but this contradicts (9). So at least one prime ideal $\mathcal{P}^{(i_0)}$ satisfies (3)–(5), and we choose $\mathcal{P}_l = \mathcal{P}^{(i_0)}$.

In the last step for $l = m+1$ the prime ideal $\mathcal{P}_m \subset R[\underline{y}]$ satisfies (3)–(5), and Theorem 2(iii) implies once more

$$\Psi_2(0) < \min\{X, V(\underline{\beta}, Z(\mathcal{P}_m))\} \le \Psi_2(k_1),$$

so that we can find $k_{m+1} \in \{0, \ldots, k_1\}$ with

$$\Psi_2(k_{m+1}) < \min\{X, V(\underline{\beta}, Z(\mathcal{P}_m))\} \le \Psi_2(k_{m+1} + 1)$$

and some $\nu \in \{1, \ldots, n_{k_{m+1}}\}$ such that $Q_{k_{m+1}}^{(\nu)*} \notin \mathcal{P}_m$. Thus Lemma 3 with $r = 1$ implies

$$\begin{aligned} 0 &\ge X/\Lambda - \Phi_1 H(\mathcal{P}_m) - \Phi_2 N(\mathcal{P}_m) \\ &\ge \left(\frac{\Psi_1(k_1)}{2(4\Lambda)^{m-1}\Lambda\Phi_1^m} - \Phi_2 \right) N(\mathcal{P}_m) \\ &\quad + \left(\frac{\Psi_1(k_1)}{2(4\Lambda)^{m-1}m\Lambda\Phi_1^{m-1}\Phi_2} - \Phi_1 \right) H(\mathcal{P}_m), \end{aligned}$$

and this completes the proof of Theorem 2. ∎

4. Proof of Theorem 1. To apply Theorem 2, we begin with the polynomial $Q \in R[y_1, \ldots, y_m]$ and define a sequence $(Q_k)_{k \in \mathbb{N}_0}$ of polynomials in $R[y_1, \ldots, y_m]$ with certain functions $\Phi_1, \Phi_2, \Psi_1, \Psi_2 : \mathbb{N} \to \mathbb{R}_+$ such that

$$N(Q_k) \le \Phi_1(k), \quad H(Q_k) \le \Phi_2(k), \quad \Psi_1(k) \le v(Q_k(\underline{\omega})) \le \Psi_2(k)$$

for $k \in \mathbb{N}_0$ and $\underline{\omega} = (f_1(z), \ldots, f_m(z))$. Then we choose the parameter k_1 with respect to $H(Q)$ and $N(Q)$, such that (iii) is satisfied with $\Phi_1 = \Phi_1(k_1)$ and $\Phi_2 = \Phi_2(k_1)$. To fulfill (ii)(d), we notice that $v(\underline{\omega}) \ge 0$, and for each zero $\underline{\theta} \in \overline{C((z))}^m$ of the polynomial Q_k the inequalities

$$\begin{aligned} \Psi_2(k) &\ge v(Q_k(\underline{\omega})) = v(Q_k(\underline{\omega}) - Q_k(\underline{\theta})) \\ &\ge v(Q_k) + v(\underline{\omega} - \underline{\theta}) \ge v(\underline{\omega} - \underline{\theta}) \end{aligned}$$

hold. Then Theorem 2 yields a bound for $\Psi_1(k_1)$ and thereby a bound for $v(Q(\underline{\omega})) = \operatorname{ord}_0 Q(z, \underline{f}(z))$.

Without loss of generality we suppose that $T(z) = T_1(z)/T_2(z)$ with $T_2(0) \neq 0$, and inductively we define for $k \in \mathbb{N}_0$,

$$Q_0(z, y_1, \ldots, y_m) = Q(z, y_1, \ldots, y_m),$$

$$\begin{aligned} Q_k(z, y_1, \ldots, y_m) &\\ = T_2(z)^{H(Q_{k-1})} & A_0(z, y_1, \ldots, y_m)^{N(Q_{k-1})} \\ &\times Q_{k-1}\left(T(z), \frac{A_1(z, y_1, \ldots, y_m)}{A_0(z, y_1, \ldots, y_m)}, \ldots, \frac{A_m(z, y_1, \ldots, y_m)}{A_0(z, y_1, \ldots, y_m)}\right). \end{aligned}$$

Then for all $k \in \mathbb{N}_0$ we have

$$Q_k \in C[z, y_1, \ldots, y_m], \quad N(Q_k) \leq tN(Q_{k-1}) \leq t^k N,$$
$$H(Q_k) \leq dH(Q_{k-1}) + sN(Q_{k-1}) \leq d^k M + sN\frac{d^k - t^k}{d - t} \leq \mu M d^k$$

with $\mu = 1 + s/(d-t)$. Since $T_2(0) \neq 0$ and $v(T(z)) = \delta$, we get for the zero order of

$$Q_k(z, \underline{f}(z)) = T_2(z)^{H(Q_{k-1})} A_0(z, \underline{f}(z))^{N(Q_{k-1})} Q_{k-1}(T(z), \underline{f}(T(z)))$$

the bound

$$\begin{aligned} \delta \operatorname{ord}_0 Q_{k-1}(z, \underline{f}(z)) &\leq \operatorname{ord}_0 Q_k(z, \underline{f}(z)) \\ &\leq \delta \operatorname{ord}_0 Q_{k-1}(z, \underline{f}(z)) + N(Q_{k-1}) \operatorname{ord}_0 A_0(z, \underline{f}(z)), \end{aligned}$$

and this implies with $\nu = v(Q(\underline{\omega})) = \operatorname{ord}_0 Q(z, \underline{f}(z))$,

$$\Psi_1(k) = \delta^k \nu \leq \operatorname{ord}_0 Q_k(z, \underline{f}(z)) \leq \delta^k \nu + \frac{\delta^k - t^k}{\delta - t} N v(A_0(\underline{\omega})) \leq 2\delta^k \nu = \Psi_2(k),$$

if we assume without loss of generality that $\nu \geq N v(A_0(\underline{\omega}))/(\delta - t)$. With

$$\Phi_1 = N t^{k_1}, \quad \Phi_2 = \mu M d^{k_1}, \quad \Lambda = 2\delta, \quad \Psi_1(k) = \nu \delta^k, \quad \Psi_2(k) = 2\nu\delta^k$$

we can apply Theorem 2. Therefore we choose

$$k_1 = \left[\frac{(m-1)\log(8\delta) + \log(4m) + m \log N}{\log \delta - m \log t}\right] + 1,$$

and this implies

$$\nu \delta^{k_1} \geq 4m(8\delta)^{m-1} \nu N^m t^{m k_1}.$$

Now we must distinguish between two cases. If $\Psi_1(k_1)$ does not satisfy (iii) of Theorem 2, then

$$\Psi_1(k_1) \leq 2m^2(8\delta)^{m-1} \Phi_1^{m-1} \Phi_2 \leq m^2 (8\delta)^m \Phi_1^m \Phi_2.$$

Otherwise we get the same upper bound from Theorem 2 and deduce

$$\begin{aligned}\nu &\le \mu m^2(8\delta)^m(dt^m\delta^{-1})^{k_1}MN^m\\ &\le \mu d\delta^{-1}m^2(8\delta t)^m(4m(8\delta)^{m-1})^{\log d/(\log\delta - m\log t)-1}\\ &\quad\times MN^{m\log d/(\log\delta - m\log t)}.\end{aligned}$$

This completes the proof of Theorem 1. ∎

References

[B1] P.-G. Becker-Landeck, *Maße für algebraische Unabhängigkeit nach einer Methode von Mahler*, Acta Arith. 50 (1988), 279–293.

[B2] P.-G. Becker, *Effective measures for algebraic independence of the values of Mahler type functions*, ibid. 58 (1991), 239–250.

[B3] —, *Algebraic independence of the values of certain series by Mahler's method*, Monatsh. Math. 114 (1992), 183–198.

[B4] —, *Transcendence of the values of functions satisfying generalized Mahler type functional equations*, J. Reine Angew. Math. 440 (1993), 111–128.

[B5] —, *Transcendence measures for the values of generalized Mahler functions in arbitrary characteristic*, Publ. Math. Debrecen 45 (1994), 269–282.

[K] K. K. Kubota, *Linear functional equations and algebraic independence*, in: Transcendence Theory: Advances and Applications, A. Baker and D. W. Masser (eds.), Academic Press, New York, 1977, 227–229.

[L] J. H. Loxton, *Automata and transcendence*, in: New Advances in Transcendence Theory, A. Baker (ed.), Cambridge Univ. Press, Cambridge, 1988, 215–228.

[LP] J. H. Loxton and A. J. van der Poorten, *Transcendence and algebraic independence by a method of Mahler*, in: Transcendence Theory: Advances and Applications, A. Baker and D. W. Masser (eds.), Academic Press, New York, 1977, 211–226.

[Ne1] Yu. V. Nesterenko, *Estimates for the orders of zeros of functions of a certain class and applications in the theory of transcendental numbers*, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), 253–284 (in Russian); English transl.: Math. USSR-Izv. 11 (1977), 239–270.

[Ne2] —, *On algebraic independence of algebraic powers of algebraic numbers*, Mat. Sb. 123 (165) (1984), 435–459 (in Russian); English transl.: Math. USSR-Sb. 51 (1985), 429–454.

[Ne3] —, *On a measure of the algebraic independence of the values of certain functions*, Mat. Sb. 128 (170) (1985), 545–568 (in Russian); English transl.: Math. USSR-Sb. 56 (1987), 545–567.

[Ni1] K. Nishioka, *On an estimate for the orders of zeros of Mahler type functions*, Acta Arith. 56 (1990), 249–256.

[Ni2] —, *Algebraic independence measures of the values of Mahler functions*, J. Reine Angew. Math. 420 (1991), 203–214.

[NT] K. Nishioka and T. Töpfer, *Transcendence measures and nonlinear functional equations of Mahler type*, Arch. Math. (Basel) 57 (1991), 370–378.

[P1] P. Philippon, *Critères pour l'indépendance algébrique*, Inst. Hautes Etudes Sci. Publ. Math. 64 (1986), 5–52.

[P2] P. Philippon, *Critères pour l'indépendance algébrique dans les anneaux diophantiens*, C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), 511–515.

[T1] T. Töpfer, *An axiomatization of Nesterenko's method and applications on Mahler functions*, J. Number Theory 49 (1994), 1–26.

[T2] —, *An axiomatization of Nesterenko's method and applications on Mahler functions II*, Compositio Math. 95 (1995), 323–342.

[T3] —, *Algebraic independence of the values of generalized Mahler functions*, Acta Arith. 70 (1995), 161–181.

[W] N. C. Wass, *Algebraic independence of the values at algebraic points of a class of functions considered by Mahler*, Dissertationes Math. 303 (1990).

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