The continued fraction expansion of α with $\mu(\alpha) = 3$

by

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1. Introduction. For any real α, define $\mu(\alpha)$ by

$$\frac{1}{\mu(\alpha)} = \liminf_{q \to \infty} q\|q\alpha\|,$$

where q is an integer and $\|x\| = \min_{i \in \mathbb{Z}} |x - i|$.

A. Markov [5] made a detailed study of the numbers α such that $\mu(\alpha) < 3$. The set $\{\mu(\alpha) \mid \alpha \in \mathbb{R}\}$ is called the *Lagrange spectrum*.

Theorem A (A. Markov [5]). *The Lagrange spectrum below* 3 *consists of the numbers* $\sqrt{9m^2 - 4}/m$, *where m is a positive integer such that*

$$(1) \qquad m^2 + m_1^2 + m_2^2 = 3mm_1m_2, \qquad m_1 \le m, \ m_2 \le m,$$

for some positive integers m_1 and m_2. Given such a triple m, m_1, m_2, define u to be the least positive residue of $\pm m_1/m_2 \bmod m$ and define v by

$$u^2 + 1 = vm.$$

Define a quadratic form $f_m(x, y)$, called the Markov form, by

$$(2) \qquad f_m(x, y) = mx^2 + (3m - 2u)xy + (v - 3u)y^2,$$

and let α be a root of $f_m(x, 1) = 0$. Then

$$(3) \qquad \mu(\alpha) = \sqrt{9m^2 - 4}/m.$$

Further, given any α such that (3) *holds for some positive integer m, there exist positive integers m_1, m_2 such that* (1) *holds and α is a root of $f(x, 1) = 0$, where $f(x, y)$ is a quadratic form equivalent to* (2), *with u and v as defined above.*

A. Markov ([5], [6]) also got the continued fraction expansion of the root of the Markov form.

THEOREM B (A. Markov [5]). *Any Markov form $f_m(x,y)$ factorizes as follows*:

$$m(x - y(\cfrac{1}{\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cdots}}}))(x + y(\alpha_{-1} + \cfrac{1}{\alpha_{-2} + \cfrac{1}{\alpha_{-3} + \cfrac{1}{\alpha_{-4} + \cdots}}})),$$

where for any integer i, $\alpha_i \in \{1, 2\}$ and

$$\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots = \ldots, 1_{2k(0)}, 2_2, 1_{2k(1)}, 2_2, \ldots, 1_{2k(n)}, 2_2, \ldots,$$

where $u_m = \underbrace{u, \ldots, u}_{m\,times}$ for non-negative integers m and $k(n)$ are non-negative integers which have the following properties:

1. *for any integer i, $k(i) - k(i - 1) \in \{-1, 0, 1\}$,*
2. *if $k(i) - k(i-1) = 1$ for some integer i, then for the first natural number j with $k(i + j) - k(i - (1 + j)) \neq 0$, we have $k(i + j) - k(i - (1 + j)) = -1$,*
3. *if $k(i) - k(i - 1) = -1$ for some integer i, then for the first natural number j with $k(i+j) - k(i - (1+j)) \neq 0$, we have $k(i+j) - k(i-(1+j)) = 1$.*

A. Markov studied the sequences $\{k(n)\}$ with the above properties in [6] and he gave the following theorem.

THEOREM C (A. Markov [6]). *Let $\{k(n)\}$ $(n \in \mathbb{Z})$ be a periodic sequence of integers with the above properties. Then there exist a rational number u and a real number b such that for any integer n,*

$$k(n) = \lfloor nu + b \rfloor - \lfloor (n - 1)u + b \rfloor,$$

where for a real number t, $\lfloor t \rfloor$ is the integral part of t. The converse is also true.

A. Markov called the sequence $\{k(n)\}$ a *Bernoulli sequence*.

Let us denote an ordinary continued fraction expansion with partial quotients $\{a_0, a_1, a_2, \ldots\}$ by

$$[a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Let $W(a, b)$ be the set of finite words, one-sided infinite words and two-sided infinite words in two symbols a and b. If a and b are positive integers, define

$[W] \in \mathbb{R}$ for $W = W_0 W_1 \ldots$ $(W_i \in \{a, b\})$ to be

$$[W] = [0, W_0, W_1, W_2, \ldots] = 0 + \cfrac{1}{W_0 + \cfrac{1}{W_1 + \cfrac{1}{W_2 + \ddots}}}.$$

We denote the word of m w's by w_m, that is,

$$w_m = \underbrace{w \ldots w}_{m \text{ times}},$$

and in the case $m = 0$, w_0 is an empty word.

For $0 \leq x \leq 1$, define a one-sided infinite word $H(x)$ by

$$H(x) = G(x, 1)G(x, 2) \ldots,$$

where the nth coordinate of $H(x)$ is

$$G(x, n) = \lfloor nx \rfloor - \lfloor (n-1)x \rfloor.$$

Define a two-sided infinite word $G(x)$ by

$$G(x) = \ldots G(x, -1)G(x, 0)G(x, 1)G(x, 2) \ldots$$

We remark that the sequences $\{[nx] - [(n-1)x]\}$ have been considered by a number of authors (see [4]). Define a substitution $\phi : W(0, 1) \to W(1, 2)$ by

$$\phi : \begin{cases} 0 & \to & 11, \\ 1 & \to & 22. \end{cases}$$

Using the above notations, we can rewrite Theorems A, B and C as

THEOREM D (A. Markov). *For any $x \in \mathbb{Q} \cap [0, 1]$,*

$$\mu([\phi(H(x))]) < 3.$$

Conversely, if $\mu(\alpha) < 3$ for an irrational number α, then there exists $x \in \mathbb{Q} \cap [0, 1]$ such that α is equivalent to $[\phi(H(x))]$, where real numbers x and y are said to be equivalent if they are related by a unimodular transformation:

$$x = \frac{ay + b}{cy + d},$$

where the integers a, b, c and d are such that $ad - bc = \pm 1$.

By using the sequence $H(x)$, H. Cohn [1] got a result about $\mu = 3$.

THEOREM E (H. Cohn [1]). *For any irrational number $x \in [0, 1]$,*

$$\mu([\phi(H(x))]) = 3.$$

Other examples of α with $\mu(\alpha) = 3$ are found in [10].

EXAMPLE ([10], Chapter 2, §6). *Let $r_1, r_2, \ldots$ be natural numbers with* $\lim_{n \to \infty} r_n = \infty$, *and set*

(4) $$A = 1_{r_1} 2 2 1_{r_2} 2 2 \ldots 1_{r_n} 2 2 \ldots$$

Then $\mu([A]) = 3$.

If $x \in [0,1]$ and $x \neq 0$, then it is easily shown that the maximal length of a string of consecutive 1's in $\phi(H(x))$ is finite. Therefore, the numbers in the example and those in Theorem E are essentially different. It is a natural question to determine for which α we get $\mu(\alpha) = 3$. In this paper, we give a solution to this question. Let us first define some notations.

Let C, D be words in $W(a, b)$. If there exist words E, F (possibly empty) such that $D = ECF$, then we call C a *subword* of D.

Let S be an infinite word in $W(a, b)$. Define $D_S(N)$ and $D'_S(N)$ for any natural number N by

$D_S(N) = \{p \in W(a,b) \mid p$ is a subword of S and $|p| = N\}$,

$D'_S(N) = \{p \in W(a,b) \mid p$ occurs infinitely many times in S and $|p| = N\}$,

where $|p|$ is the number of symbols a or b in p.

From Lemma 3 in Section 2, for $V, W \in W(1,2)$ with $\mu([V]) \leq 3$, if $D'_V(N) = D'_W(N)$ for all N, then $\mu([W]) \leq 3$. And it is not difficult to see that for $W, V_\lambda \in W(1,2)$ with $\mu([V_\lambda]) \leq 3$ ($\lambda \in \Lambda$), if $D'_W(N) = \bigcup_{\lambda \in \Lambda} D'_{V_\lambda}(N)$ for all N, then $\mu([W]) \leq 3$. Therefore, from Theorem D for a subset I' of $[0,1]$ if there exists $W \in W(0,1)$ such that $D'_W(N) = \bigcup_{x \in I'} D'_{H(x)}(N)$ for all N, then $\mu([\phi(W)]) \leq 3$.

Roughly speaking, in this paper we show that if I' is an interval, then a W as above exists and conversely for any one-sided infinite word $S \in W(1,2)$ with $\mu(S) \leq 3$ there exists W with the above condition, $D'_S(N) = D'_{\phi(W)}(N)$ for all N.

To state our theorem, we introduce new sequences which we call *super Bernoulli sequences*.

Let F_N be the Farey sequence for a natural number N. That is,

$$F_N = \{p/q \mid (p, q) = 1, \ p, q \text{ are integers}, \ 0 \leq p/q \leq 1, \ 1 \leq q \leq N\}.$$

For a rational $x = n/m \neq 0$ with $(n, m) = 1$, define a new infinite word $\underline{G(x)} \in W(0,1)$ from $G(x)$ by inserting the finite word $G(u, 1) \ldots G(u, k)$, where $u = \max\{y \in F_m \mid y < x\}$ and k is the denominator of u (if $u = 0$, then we set $k = 1$):

$$\underline{G(x)} = \ldots G(x, -1)G(x, 0)G(u, 1) \ldots G(u, k)G(x, 1)G(x, 2) \ldots$$

For a rational $x = n/m \neq 1$ with $(n, m) = 1$ define

$$\overline{G(x)} = \ldots G(x, -1)G(x, 0)G(u, 1) \ldots G(u, k)G(x, 1)G(x, 2) \ldots,$$

where $u = \min\{y \in F_m \mid x < y\}$ and k is the denominator of u (if $u = 1$, then we set $k = 1$).

For example,

$$\overline{G(0)} = \ldots 00100 \ldots = {}_\infty 010_\infty,$$
$$\underline{G(1/2)} = \ldots 010100101 \ldots = {}_\infty(01)(01)_\infty,$$

where for a word w, $w_\infty = www \ldots$ and ${}_\infty w = \ldots www$.

Let $x, y \in [0, 1]$ and $x \le y$. Let S be a one-sided infinite word $\in W(0, 1)$. If $S \in W(0, 1)$ satisfies one of following conditions (1)–(4) for all natural numbers N, then S is said to be a *super Bernoulli sequence related to* (x, y).

(1) $\displaystyle D'_S(N) = \bigcup_{z \in [x,y]} D_{G(z)}(N),$

(2) $x \in \mathbb{Q}$ and $\displaystyle D'_S(N) = \bigcup_{z \in [x,y]} D_{G(z)}(N) \cup D_{\underline{G(x)}}(N),$

(3) $y \in \mathbb{Q}$ and $\displaystyle D'_S(N) = \bigcup_{z \in [x,y]} D_{G(z)}(N) \cup D_{\overline{G(y)}}(N),$

(4) $x, y \in \mathbb{Q}$ and $\displaystyle D'_S(N) = \bigcup_{z \in [x,y]} D_{G(z)}(N) \cup D_{\underline{G(x)}}(N) \cup D_{\overline{G(y)}}(N).$

If S satisfies one of conditions (i) $(1 \le i \le 4)$, then it is said to be *of type i*. For example, $H(x)$ is a super Bernoulli sequence related to (x, x) of type 1. Our main result is as follows.

THEOREM 3. *Let α be an irrational number with $\mu(\alpha) \le 3$ and with continued fraction expansion $[a_0, a_1, \ldots]$. Then there exists a non-negative integer n such that $a_m \in \{1, 2\}$ for all $m \ge n$, and there exists a one-sided word $S \in W(0, 1)$ which is a super Bernoulli sequence related to (x, y) for some x, y with $0 \le x \le y \le 1$ such that*

$$D'_A(N) = D'_{\phi(S)}(N) \quad \text{for all } N \in \mathbb{N},$$

where $A = a_n a_{n+1} a_{n+2} \ldots$

Conversely, let S be any super Bernoulli sequence related to (x, y) and let $A \in W(1, 2)$ be a one-sided infinite word such that $D'_A(N) = D'_{\phi(S)}(N)$ for all N. Then $\mu([A]) \le 3$, and strict inequality holds if and only if $x = y$ is rational and S is a super Bernoulli sequence of type 1.

In Section 4, we see that if S is a super Bernoulli sequence related to (x, x) of type 1 with $x \in \mathbb{Q}$, then S coincides with $H(x)$ except for a finite number of letters and we can deduce analogously that then A coincides with $\phi(H(x))$ except for a finite number of letters. Therefore, the final line is nothing but the statement of Theorem D.

Let us give an example. For the previous example (4) we have

$$D'_A(N) = D'_{\phi(S)}(N) \quad \text{for all } N,$$

where

$$S = 010010001 \ldots 0_n 1 \ldots,$$

and S is a super Bernoulli sequence related to $(0,0)$ of type 3. We note that if r_n are odd, then A is not represented as $\phi(S)$. For the question whether $A = \phi(S)$ or not in the statement of the theorem we have the following proposition.

PROPOSITION 1. *Let α be an irrational number with $\mu(\alpha) \leq 3$ and with continued fraction expansion $[a_0, a_1, \ldots]$. Suppose that there exists a constant C such that for positive integers k, l, the condition $a_k = a_{k+1} = \ldots = a_{k+l}$ implies $l < C$. Then there exists a non-negative integer n such that $a_m \in \{1, 2\}$ for all $m \geq n$ and there exists a word $S \in W(0,1)$ which is a super Bernoulli sequence related to (x, y) such that*

$$\phi(S) = a_n a_{n+1} a_{n+2} \ldots$$

The paper is organized as follows. In Section 2, we carry out a study of the continued fraction expansion of α with $\mu(\alpha) \leq 3$ analogous to the argument ([2], Chapter 2) in the case of the Markov spectrum. In Section 3, we prove the main result. In Section 4, the existence and some properties of super Bernoulli sequences are proved.

2. Combinatorial calculus of the continued fraction expansion of α

LEMMA 1. *Let $\alpha = [a_0, a_1, \ldots, a_m, \ldots]$ be irrational. Then $\mu(\alpha) = \limsup_{n \to \infty} \mu_n(\alpha)$, where $\mu_n(\alpha) = [0, a_{n-1}, a_{n-2}, \ldots, a_0] + [a_n, a_{n+1}, \ldots]$.*

P r o o f. See [9]. ∎

LEMMA 2. *Let $\alpha = [a_0, a_1, \ldots, a_m, \ldots]$ and $\beta = [b_0, b_1, \ldots, b_m, \ldots]$, where $a_i, b_i \in \{1, 2\}$ for $i = 0, 1, \ldots$ Assume that $a_i = b_i$ for $0 \leq i \leq n$. Then*

$$1/2^{n-1} > |\alpha - \beta|.$$

In addition, assume that $a_{n+1} \neq b_{n+1}$. Then for n odd, $\alpha > \beta$ if and only if $a_{n+1} > b_{n+1}$, while for n even, $\alpha > \beta$ if and only if $a_{n+1} < b_{n+1}$. Furthermore,

$$|\alpha - \beta| > 1/3^{2n+3}.$$

P r o o f. Except for the final inequality, the lemma follows from Lemmas 1 and 2 in Chapter 1 of [2]. Let us prove the final inequality. Define q_m, p_m, q'_m

and p'_m for $m \in \mathbb{N} \cup \{0\}$ as usual by

$$\begin{pmatrix} p_m & * \\ q_m & * \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_m & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} p'_m & * \\ q'_m & * \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_m & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the following formulas are well known:

$$\alpha = \frac{p_m \alpha_{m+1} + p_{m-1}}{q_m \alpha_{m+1} + q_{m-1}}, \qquad \beta = \frac{p'_m \beta_{m+1} + p'_{m-1}}{q'_m \beta_{m+1} + q'_{m-1}},$$

where

$$\alpha_{m+1} = [a_{m+1}, a_{m+2}, \ldots], \qquad \beta_{m+1} = [b_{m+1}, b_{m+2}, \ldots].$$

From the hypothesis, $p_j = p'_j$ and $q_j = q'_j$ for $j = 0, 1, \ldots, n$. Therefore,

$$|\alpha - \beta| = \left| \frac{p_n \alpha_{n+1} + p_{n-1}}{q_n \alpha_{n+1} + q_{n-1}} - \frac{p_n \beta_{n+1} + p_{n-1}}{q_n \beta_{n+1} + q_{n-1}} \right|$$

$$= \frac{|\alpha_{n+1} - \beta_{n+1}|}{|(q_n \beta_{n+1} + q_{n-1})(q_n \alpha_{n+1} + q_{n-1})|}.$$

By induction, $q_j \leq 3^j$ $(j = 0, 1, \ldots, n)$. Therefore,

$$\frac{|\alpha_{n+1} - \beta_{n+1}|}{|(q_n \beta_{n+1} + q_{n-1})(q_n \alpha_{n+1} + q_{n-1})|}$$

$$\geq \frac{|\alpha_{n+1} - \beta_{n+1}|}{(3^{n+1} + 3^{n-1})^2} > \frac{1 + [0, 2, 1, 2, 1, \ldots] - [0, 1, 2, 1, 2, \ldots]}{3^{2n+2}(1 + 1/9)^2} > \frac{1}{3^{2n+3}}. \quad \blacksquare$$

LEMMA 3. *Let $V = v_0 v_1 \ldots$ be a one-sided infinite word with $\mu([V]) \leq 3$, where $v_0, v_1, \ldots \in \{1, 2\}$. Let $W = w_0 w_1 \ldots$ $(w_0, w_1, \ldots \in \{1, 2\})$ be a one-sided infinite word such that $D'_W(N) = D'_V(N)$ for all N. Then $\mu([W]) \leq 3$.*

P r o o f. Using Lemma 1, we show that $\limsup_{n \to \infty} \mu_n([W]) \leq 3$. Let $\varepsilon > 0$ and $1/2^{n-2} < \varepsilon$. It is not difficult to see that there exists $M \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq M$, $w_{m-n} \ldots w_m \ldots w_{m+n}$ occurs infinitely many times in W. From Lemma 2, we have

(5) $\quad |\mu_{m+1}([W]) - ([w_{m-1} \ldots w_{m-n}] + w_m + [w_{m+1} \ldots w_{m+n}])|$

$$\leq |[w_{m-1} \ldots w_0] - [w_{m-1} \ldots w_{m-n}]|$$

$$+ |[w_{m+1} w_{m+2} \ldots] - [w_{m+1} \ldots w_{m+n}]|$$

$$< 1/2^{n-2} < \varepsilon.$$

From the hypothesis, there exists $m' \in \mathbb{N}$ such that for any integer $m'' \geq m'$,

(6) $$\mu_{m''}([V]) \leq 3 + \varepsilon.$$

Since $w_{m-n} \ldots w_m \ldots w_{m+n}$ also occurs infinitely many times in V by hypothesis, there exists $k \in \mathbb{N}$ such that

(7) $\qquad k > m' + n \quad \text{and} \quad v_{k-n+i} = w_{m-n+i} \quad \text{for } i = 0, 1, \ldots, 2n.$

From Lemma 2, we have

(8) $\quad |\mu_{k+1}([V]) - ([v_{k-1} \ldots v_{k-n}] + v_k + [v_{k+1} \ldots v_{k+n}])|$

$$\leq |[v_{k-1} \ldots v_0] - [v_{k-1} \ldots v_{k-n}]|$$

$$+ |[w_{k+1}w_{k+2} \ldots] - [w_{k+1} \ldots w_{k+n}]|$$

$$< 1/2^{n-2} < \varepsilon.$$

Therefore, from (5)–(8) we have $\mu_{m+1}([W]) \leq 3 + 3\varepsilon$, which proves the lemma. ∎

LEMMA 4. *Let $\alpha = [a_0, a_1, \ldots]$ and $\beta = [b_0, b_1, \ldots]$, where $a_i, b_i \in \{1, 2\}$. Set $x = [2, 1, 1, \alpha]$ and $y = [0, 2, \beta]$ and let $0 < \varepsilon < e^{-500}$.*

(1) *If $x + y < 3 + \varepsilon$, then either*
    (A) $\alpha \geq \beta$, *or*
    (B) $\beta > \alpha$ *and $a_m = b_m$ for all $m \in \mathbb{N}$ with $m < -(\log \varepsilon)/8$.*
(2) *If conversely $a_m = b_m$ for every $m < N$, then*
$$|x + y| < 3 + 1/2^N.$$

Proof. (1) By definition,

$$x = 2 + \frac{\alpha + 1}{2\alpha + 1}, \qquad y = \frac{\beta}{2\beta + 1},$$

and

(9) $$x + y - 3 = \frac{\beta - \alpha}{(2\alpha + 1)(2\beta + 1)}.$$

Suppose $\beta > \alpha$; then

(10) $\qquad 0 < \beta - \alpha < (2\alpha + 1)(2\beta + 1)\varepsilon \leq (2[2, 1, 2, 1, \ldots] + 1)^2 \varepsilon < 49\varepsilon.$

Let n be maximal such that $a_m = b_m$ for all $m \leq n$. By Lemma 2,

(11) $$\beta - \alpha > 1/3^{2n+3}.$$

By (10) and (11),

$$n > -\frac{\log \varepsilon}{2 \log 3} - \frac{\log 49}{2 \log 3} - \frac{3}{2}.$$

Since $\varepsilon < e^{-500}$, we have $n > -(\log \varepsilon)/8$.

The last statement of the lemma is immediate from (9) and Lemma 2. ∎

LEMMA 5. *Let $\alpha = [a_0, \ldots, a_n, \ldots]$ be irrational with $\mu(\alpha) \leq 3$. Then there exists $m \in \mathbb{N}$ such that $a_n \in \{1, 2\}$ for $n > m$.*

P r o o f. By Lemma 1, there are only finitely many n such that $a_n \geq 4$. Hence we may assume that $a_n \leq 3$ for all n. Suppose that there are infinitely many n such that $a_n = 3$. For such n, by Lemma 1,

$$\mu_n(\alpha) = [0, a_{n-1}, \ldots, a_1] + 3 + [0, a_{n+1}, a_{n+2}, \ldots].$$

Since $[0, a_{n+1}, a_{n+2}, \ldots] > 1/4$, we have $\mu_n(\alpha) > 3 + 1/4$, which contradicts $\mu(\alpha) \leq 3$. ∎

If the nth letter in $W = w_0 w_1 \ldots$ is v, then we write

$$W = \ldots \overset{n}{v} \ldots$$

For example, in the case of $W = abcabaaab$ we can emphasize the 7th letter a by writing

$$W = abcaba \overset{7}{a} ab \ldots$$

LEMMA 6. *Let* $\alpha = [a_0, \ldots, a_n, \ldots]$ *be irrational with* $\mu(\alpha) \leq 3$ *and* $a_i \in \{1, 2\}$ *for* $i = 0, 1, \ldots$ *Put* $A = a_0 a_1 \ldots \in W(1, 2)$. *Then the words* 121 *and* 212 *do not appear infinitely many times in* A.

P r o o f. Suppose that 121 appears infinitely many times in A, say

$$A = \overset{0}{a_0} \ldots \overset{n-1}{1} \overset{n}{2} \overset{n+1}{1} \ldots$$

By Lemma 1,

$$\mu_n(\alpha) = 2 + [0, \overset{n-1}{1}, \overset{\ldots}{\ldots}, \overset{1}{a_1}] + [0, \overset{n+1}{1}, \overset{\ldots}{\ldots}].$$

By Lemma 2,

$$[0, \overset{n-1}{1}, \overset{\ldots}{\ldots}, \overset{1}{a_1}] \geq [0, 1, 1, 3] = 4/7, \qquad [0, \overset{n+1}{1}, \overset{\ldots}{\ldots}] \geq [0, 1, 1, 3] = 4/7.$$

Therefore,

$$\mu_n(\alpha) \geq 3 + 1/7,$$

contrary to $\mu(\alpha) \leq 3$. The case of 212 is analogous. ∎

LEMMA 7. *Let* $\alpha = [a_0, \ldots, a_n, \ldots]$ *be irrational with* $\mu(\alpha) \leq 3$ *and* $a_i \in \{1, 2\}$ *for* $i = 0, 1, \ldots$ *Put* $A = a_0 a_1 \ldots \in W(1, 2)$. *Then* 111222 *and* 222111 *do not appear infinitely many times in* A.

P r o o f. Suppose that 111222 appears infinitely many times in A. Let $0 < \varepsilon \leq e^{-500}$ and

$$A = \overset{0}{a_0} \ldots \overset{n-3}{1} \overset{n-2}{1} \overset{n-1}{1} \overset{n}{2} \overset{n+1}{2} \overset{n+2}{2} \ldots,$$

where n is so large that $\mu_n(\alpha) < 3 + \varepsilon$. Then

$$[\overset{n}{2}, \overset{n-1}{1}, \overset{n-2}{1}, \overset{n-3}{1}, \ldots, a_1] + [0, \overset{n+1}{2}, \overset{n+2}{2}, \ldots] < 3 + \varepsilon.$$

Since $[\overset{n-3}{1},\ldots,a_1] < [\overset{n+2}{2},\ldots]$, by Lemma 4 we see that for $0 \le m < (-\log\varepsilon)/8$, $a_{n-3-m} = a_{n+2+m}$, contrary to $a_{n-3} \ne a_{n+2}$. The case of 222111 is analogous. ∎

LEMMA 8. *Let* $\alpha = [a_0,\ldots,a_n,\ldots]$ *be irrational with* $\mu(\alpha) \le 3$ *and* $a_i \in \{1,2\}$ *for* $i = 0,1,\ldots$ *Put* $A = a_0 a_1 \ldots \in W(1,2)$. *If* m *is an odd integer, then* $21_m 2$ *and* $12_m 1$ *do not appear infinitely many times in* A.

P r o o f. We argue by induction on m. For $m = 1$, see Lemma 6. Assume that $m \ge 3$ and the lemma is verified for all positive odd integers smaller than m. Suppose that $21_m 2$ appears infinitely many times in A. Let $\varepsilon > 0$ be a small number and

$$A = \overset{0}{a_0} \ldots \overset{n-m-1}{2} 1_m \overset{n}{2} \ldots,$$

where n is so large that $\mu_n(\alpha) < 3 + \varepsilon$. By Lemmas 6 and 7,

$$A = a_0 \ldots 21_m \overset{n}{2} \overset{n+1}{2} 1_l \ldots,$$

where $l \ge 2$. Then

$$\mu_n(\alpha) = [\overset{n}{2}, \underbrace{1,\ldots,1}_{m\,\text{times}}, \overset{n-m-1}{2}, \ldots, a_1] + [0, \overset{n+1}{2}, \underbrace{\overset{n+2}{1},\ldots,1}_{l\,\text{times}}, \ldots] < 3 + \varepsilon.$$

Suppose $l > m - 2$. Then, by Lemma 2,

$$[\underbrace{1,\ldots,1}_{m-2\,\text{times}}, \overset{n-m-1}{2}, \ldots, a_1] < [\underbrace{\overset{n+2}{1},\ldots,1}_{l\,\text{times}}, \ldots].$$

We may assume that ε is so small that $m < (-\log\varepsilon)/8$ and $\varepsilon < e^{-500}$. By Lemma 4, $a_{n-3-j} = a_{n+2+j}$ for $j = 0,1,\ldots,m$. Therefore, $a_{n+m} = a_{n-m-1} = 2$. But $l > m - 2$ implies $a_{n+m} = 1$, a contradiction. Therefore, $l \le m - 2$ and

$$A = a_0 \ldots 21_m \overset{n}{2} \overset{n+1}{2} 1_l 2 \ldots$$

We see that l is even and $l < m - 2$ by the inductive assumption. Then, by Lemma 2,

$$[\underbrace{1,\ldots,1}_{m-2\,\text{times}}, \overset{n-m-1}{2}, \ldots, a_1] < [\underbrace{\overset{n+2}{1},\ldots,1}_{l\,\text{times}}, 2, \ldots].$$

Therefore Lemma 4 also yields a contradiction. The case of $12_m 1$ is analogous. ∎

LEMMA 9. *Let* $\alpha = [a_0,\ldots,a_n,\ldots]$ *be irrational with* $\mu(\alpha) \le 3$ *and* $a_i \in \{1,2\}$ *for* $i = 0,1,\ldots$ *Let*

$$A = a_0 a_1 \ldots = 1_{p(0)} 2_{p(1)} 1_{p(2)} 2_{p(3)} \ldots \in W(1,2),$$

where $p(0) \in \mathbb{N} \cup \{0\}$ and $p(i) \in \mathbb{N}$ $(i = 1, 2, \ldots)$. Define $A' \in W(1, 2)$ by

$$A' = b_0 b_1 \ldots = 1_{q(0)} 2_{q(1)} 1_{q(2)} 2_{q(3)} \ldots \in W(1, 2),$$

where

$$q(i) = \begin{cases} p(i) + 1 & \text{if } p(i) \text{ is odd,} \\ p(i) & \text{if } p(i) \text{ is even,} \end{cases}$$

for $i = 0, 1, \ldots$ Then $\mu([A']) = \mu([A])$ and $D'_A(N) = D'_{A'}(N)$ for all N.

P r o o f. If $p(i) = \infty$ for some i, then $\mu([A']) = \mu([A])$ is clear. Assume that $p(i) < \infty$ for $i = 0, 1, \ldots$ Let $M \in \mathbb{N}$. By Lemma 8, there exists $k \in \mathbb{N}$ such that $p(i) > 2M + 1$ if $p(i)$ is odd for $i \geq k$. Let $n = \sum_{i=0}^{k} p(i)$. Then for $m > n$,

$$a_m a_{m+1} \ldots a_{m+2M} = b_{m'} b_{m'+1} \ldots b_{m'+2M},$$

where $m' = m + \sharp\{u \mid \sum_{i=0}^{u} p(i) < m \text{ and } p(u) \text{ is odd}\}$. Therefore by Lemma 2,

$$|\mu_{m+M}([A]) - \mu_{m'+M}([A'])| < 1/2^{2M}.$$

Hence $\mu([A']) = \mu([A])$, and by Lemma 8 we have immediately the last statement of the lemma. ∎

LEMMA 10. *Let $\alpha = [a_0, \ldots, a_n, \ldots]$ be irrational with $\mu(\alpha) \leq 3$ and $a_i \in \{1, 2\}$ for $i = 0, 1, \ldots$ Put $A = a_0 a_1 \ldots \in W(1, 2)$. Let $N \geq 4$ be an integer. Then there exists $m \in \mathbb{N}$ such that 1111 and 2222 are not contained in $a_n a_{n+1} \ldots a_{n+N-1}$ at the same time for $n > m$.*

P r o o f. Let $\varepsilon > 0$ be so small that $N + 3 < (-\log \varepsilon)/8$ and $\varepsilon < e^{-500}$. Let $k \in \mathbb{N}$ be such that $\mu_m(\alpha) < 3 + \varepsilon$ for $m \geq k$. By Lemmas 7 and 8 we may assume that $k > 2N$ and $121, 212, 21112, 12221, 111222, 222111$ are not contained in $a_{k-N} a_{k-N+1} \ldots$ Suppose that 1111 and 2222 are contained in $a_n a_{n+1} \ldots a_{n+N-1}$ for some $n \geq k$. Then the following word appears in $a_n a_{n+1} \ldots a_{n+N-1}$:

$$2222(1122)_l 1111 \quad \text{or} \quad 1111(2211)_l 2222 \quad \text{for some } l \in \mathbb{N} \cup \{0\}.$$

Suppose that $2222(1122)_l 1111$ occurs in $a_n a_{n+1} \ldots a_{n+N-1}$. Then

$$A = a_0 \ldots 222 \overset{n'}{2} (1122)_l 1111 \ldots,$$

where $n \leq n' < n + N$. Thus,

$$\mu_{n'}(\alpha) = [\overset{n'}{2}, \underbrace{\overset{n'+1}{1}, 1, 2, 2, \ldots, 1, 1, 2, 2}_{l \text{ times}}, 1, 1, 1, 1, \ldots]$$

$$+ [0, \overset{n'-1}{2}, \overset{n'-2}{2}, \overset{n'-3}{2}, \ldots, a_1]$$

$$< 3 + \varepsilon.$$

Suppose that

$$[\overset{n'+3}{\underbrace{2}},\overset{n'+4}{2},1,1,\ldots,\underbrace{2,2,1,1}},1,1,1,1,\ldots] < [\overset{n'-2}{2},\ldots,a_1].$$
$$\underbrace{\hphantom{2,2,1,1,\ldots,2,2,1,1}}_{l\,\text{times}}$$

Then by Lemma 4,

$$[\overset{n'-2}{2},\ldots,a_1] = [\overset{n'-2}{2},\overset{n'-3}{2},1,1,\ldots,\underbrace{2,2,1,1},1,1,\ldots,a_1].$$
$$\underbrace{\hphantom{2,1,1,\ldots,2,2,1,1}}_{l\,\text{times}}$$

By Lemma 2,

(12) $$[\overset{n'-6}{2},2,1,1,\ldots,\underbrace{2,2,1,1}},1,1,\ldots,a_1]$$
$$\underbrace{\hphantom{2,2,1,1,\ldots,2,2,1,1}}_{l-1\,\text{times}}$$

$$< [\overset{n'-1}{2},\overset{n'}{2},1,1,\ldots,\underbrace{2,2,1,1}},1,1,\ldots].$$
$$\underbrace{\hphantom{2,1,1,\ldots,2,2,1,1}}_{l+1\,\text{times}}$$

On the other hand, we get

$$\mu_{n'-3}(\alpha) = [\overset{n'-3}{2},\underbrace{\overset{n'-4}{1},1,2,2,\ldots,1,1,2,2}},1,1,1,1,\ldots,a_1]$$
$$\underbrace{\hphantom{1,1,2,2,\ldots,1,1,2,2}}_{l-1\,\text{times}}$$

$$+ [0,\overset{n'-2}{2},\overset{n'-1}{2},\overset{n'}{2},\underbrace{1,1,2,2,\ldots,1,1,2,2}},1,1,1,1,\ldots]$$
$$\underbrace{\hphantom{1,1,2,2,\ldots,1,1,2,2}}_{l\,\text{times}}$$

$$< 3 + \varepsilon.$$

Since $4(l-1) < N+1 < (-\log\varepsilon)/8$, (12) contradicts Lemma 4. Therefore,

$$[\overset{n'+3}{2},\overset{n'+4}{2},1,1,\ldots,\underbrace{2,2,1,1}},1,1,1,1,\ldots] \geq [\overset{n'-2}{2},\overset{n'-3}{2},\ldots,a_1].$$
$$\underbrace{\hphantom{2,1,1,\ldots,2,2,1,1}}_{l\,\text{times}}$$

It is easily seen that

$$[\overset{n'-2}{2},\overset{n'-3}{2},\ldots,a_1] = [\overset{n'-2}{2},\overset{n'-3}{2},1,1,\ldots,\underbrace{2,2,1,1}},1,1,\ldots,a_1],$$
$$\underbrace{\hphantom{2,1,1,\ldots,2,2,1,1}}_{p\,\text{times}}$$

where $p \leq l-1$. Thus,

$$\mu_{n'-3}(\alpha) = [\overset{n'-3}{2},\underbrace{\overset{n'-4}{1},1,2,2,\ldots,1,1,2,2}},1,1,1,1,\ldots,a_1]$$
$$\underbrace{\hphantom{1,1,2,2,\ldots,1,1,2,2}}_{p-1\,\text{times}}$$

$$+ [0,\overset{n'-2}{2},\overset{n'-1}{2},\overset{n'}{2},\underbrace{1,1,2,2,\ldots,1,1,2,2}},1,1,1,1,\ldots] < 3 + \varepsilon,$$
$$\underbrace{\hphantom{1,1,2,2,\ldots,1,1,2,2}}_{l\,\text{times}}$$

and we have a contradiction in the same manner. The case of $1111(2211)_2 222$ occurring in $a_n a_{n+1} \ldots a_{N+n-1}$ is analogous. ∎

LEMMA 11. *Let* $\alpha = [a_0, \ldots, a_n, \ldots]$ *be irrational with* $\mu(\alpha) \leq 3$ *and* $a_i \in \{1, 2\}$ *for* $i = 0, 1, \ldots$ *Put* $A = a_0 a_1 \ldots \in W(1, 2)$. *Let* $N \geq 4$ *be an integer, and* p *and* q *be positive even integers less than* N. *Then there exists* $m \in \mathbb{N}$ *such that if either* $221_p 221_q 22$ *or* $112_p 112_q 11$ *is contained in* $a_n a_{n+1} \ldots a_{n+N-1}$ *for some* $n > m$, *then* $|p - q| \leq 2$.

P r o o f. Let $\varepsilon > 0$ be so small that $N + 3 < (-\log \varepsilon)/8$ and $\varepsilon < e^{-500}$. Let $m \in \mathbb{N}$ be such that $\mu_n(\alpha) < 3 + \varepsilon$ for $n \geq m$. Suppose that $221_p 221_q 22$ is contained in $a_n a_{n+1} \ldots a_{n+N-1}$ and $p > q + 2$, and

$$a_n a_{n+1} \ldots a_{n+N-1} = \ldots 221_p \overset{n'}{2} 21_q 22 \ldots$$

By Lemma 1,

$$\mu_{n'}(\alpha) = [\overset{n'}{2}, \underbrace{1, \ldots, 1}_{p \text{ times}}, 2, \ldots, a_1] + [0, \overset{n'+1}{2}, \underbrace{1, \ldots, 1}_{q \text{ times}}, 2, \ldots] < 3 + \varepsilon.$$

Since $p > q + 2$, we have

$$[\overset{n'-2}{\underbrace{1}}, \ldots, 1, 2, \ldots, a_1] < [0, \overset{n'+2}{\underbrace{1}}, \ldots, 1, 2, \ldots],$$
$$\underbrace{}_{p-2 \text{ times}} \qquad \underbrace{}_{q \text{ times}}$$

a contradiction by Lemma 4. Therefore $q \leq p + 2$, and $p \leq q + 2$ in the same manner. For $112_p 112_q 11$, we argue analogously. ∎

LEMMA 12. *Let* $\alpha = [a_0, \ldots, a_n, \ldots]$ *be irrational with* $\mu(\alpha) \leq 3$ *and* $a_i \in \{1, 2\}$ *for* $i = 0, 1, \ldots$ *Put* $A = a_0 a_1 \ldots \in W(1, 2)$. *Let* $N \geq 4$ *be an integer. Then there exists* $m \in \mathbb{N}$ *such that if for positive even integers* p *and* q, *either* $221_p 22$ *and* $221_q 22$, *or* $112_p 11$ *and* $112_q 11$ *are contained in* $a_n a_{n+1} \ldots a_{n+N-1}$ *for some* $n > m$, *then* $|p - q| \leq 2$.

P r o o f. Let $\varepsilon > 0$ be so small that $N + 3 < (-\log \varepsilon)/8$ and $\varepsilon < e^{-500}$. Let $m \in \mathbb{N}$ be such that $\mu_i(\alpha) < 3 + \varepsilon$ for $i \geq m$. Let $n > m + N$. Suppose that $221_p 22$ and $221_q 22$ are contained in $a_n a_{n+1} \ldots a_{n+N-1}$ and $|p - q| > 2$ for positive even integers p and q. Take the word $221_p 2 T 21_q 22$ ($T \in W(1, 2)$) such that $221_p 22 T 221_q 22$ occurs in $a_n a_{n+1} \ldots a_{n+N-1}$ and $|T|$ is the least possible. By Lemmas 8–11 we see that $|p - q| = 4$ and

$$221_p 2 T 21_q 22 = 221_p 2 (21_{\frac{p+q}{2}} 2)_k 21_q 22,$$

where $k \in \mathbb{N}$. Suppose that $p = q + 4$ without loss of generality, and

$$a_n a_{n+1} \ldots a_{n+N-1} = \ldots \overset{n'}{2} 21_{q+4} 2 (21_{q+2} 2)_k 21_q 2 \ldots$$

We have

$$\mu_{n'+q+6}(\alpha) = [\overset{n'+q+6}{2}, \overbrace{1,\ldots,1}^{q+4\,\text{times}}, 2,\ldots, a_1]$$

$$+ [0, \overbrace{\underbrace{2, 1,\ldots,1}_{q+2\,\text{times}}, 2, \ldots, \underbrace{2, 1,\ldots,1}_{q+2\,\text{times}}, 2}^{k\,\text{times}}, 2, \underbrace{1,\ldots,1}_{q\,\text{times}}, 2, \ldots]$$

$$< 3 + \varepsilon.$$

This is a contradiction, as in the proof of Lemma 10. The case where 221_p22 and 221_q22 are contained in $a_n a_{n+1}\ldots a_{n+N-1}$ is similar. ∎

LEMMA 13. *Let $\alpha = [a_0,\ldots,a_n,\ldots]$ be irrational with $\mu(\alpha) \leq 3$ and $a_i \in \{1,2\}$ for $i = 0, 1,\ldots$ Put $A = a_0 a_1 \ldots \in W(1,2)$. Let $N \geq 4$ be an integer. Then there exists $m \in \mathbb{N}$ such that if p and q are positive even integers and either 221_p22 and 1_q, or 112_p11 and 2_q are contained in $a_n a_{n+1}\ldots a_{n+N-1}$ for some $n > m$, then $|q| \leq p + 2$.*

P r o o f. This can be shown in the same way as Lemma 12. ∎

THEOREM 1. *Let A be a one-sided infinite word in $W(1,2)$:*

$$A = a_0 a_1 \ldots = 1_{p(0)} 2_{p(1)} 1_{p(2)} 2_{p(3)} \ldots \in W(1,2),$$

where $p(i)$ is an even positive integer for $i = 0, 1,\ldots$ Then $\mu([A]) \leq 3$ if and only if the following holds: For any even integer $N > 4$, there exists $m \in \mathbb{N}$ such that for any even $n > m$, $a_n a_{n+1}\ldots a_{n+N-1}$ has one of the following forms:

(i) *If 2222 does not occur in $a_n a_{n+1}\ldots a_{n+N-1}$, then $a_n a_{n+1}\ldots a_{n+N-1}$ coincides with either*

(13) $1_{2r(0)}$ *or* $1_{2r(0)} 2_2 1_{2r(1)} 2_2 \ldots 2_2 1_{2r(k)} 2_2 1_{2r(k+1)}$,

where $r(0), r(k+1) \in \mathbb{N} \cup \{0\}$ and $r(i) \in \mathbb{N}$ $(1 \leq i \leq k)$ satisfy:

(A) *$|r(i) - r(j)| \leq 1$ for all i, j with $1 \leq i, j \leq k$ and $r(0), r(k+1) \leq \max_{1 \leq i \leq k}\{r(i)\}$.*

(B) *If $\delta := r(i+1) - r(i) = \pm 1$ for an integer i with $1 \leq i < k$, then the following holds: if $r(i+1+j) - r(i-j) \neq 0$ for some integer $j > 0$ and $r(i+1+k) - r(i-k) = 0$ for every integer k with $0 < k < j$, then $r(i+1+j) - r(i-j) = -\delta$.*

(ii) *If 1111 does not occur in $a_n a_{n+1}\ldots a_{n+N-1}$, then $a_n a_{n+1}\ldots a_{n+N-1}$ coincides with either*

(14) $2_{2r(0)}$ *or* $2_{2r(0)} 1_2 2_{2r(1)} 1_2 \ldots 1_2 2_{2r(k)} 1_2 2_{2r(k+1)}$,

where $r(0), r(k+1) \in \mathbb{N} \cup \{0\}$ and $r(i) \in \mathbb{N}$ $(1 \leq i \leq k)$ satisfy:

(C) *$|r(i) - r(j)| \leq 1$ for all i, j with $1 \leq i, j \leq k$ and $r(0), r(k+1) \leq \max_{1 \leq i \leq k}\{r(i)\}$.*

(D) *If $\delta = r(i+1) - r(i) = \pm 1$ for an integer i with $1 \le i < k$, then the following holds: if $r(i+1+j) - r(i-j) \ne 0$ for some integer $j > 0$ and $r(i+1+k) - r(i-k) = 0$ for every integer k with $0 < k < j$, then $r(i+1+j) - r(i-j) = -\delta$.*

Proof. *Necessity.* Let $\varepsilon > 0$ be so small that $N + 3 < (-\log \varepsilon)/8$ and $\varepsilon < e^{-500}$. Let $m' \in \mathbb{N}$ be such that $\mu_i(\alpha) < 3 + \varepsilon$ for $i \ge m'$. By Lemmas 10, 12 and 13, for any even integer $N > 4$, there exists $m > m'$ such that for any even $n > m$, if 2222 does not occur in $a_n a_{n+1} \ldots a_{n+N-1}$ and 22 occurs in $a_n a_{n+1} \ldots a_{n+N-1}$, then $a_n a_{n+1} \ldots a_{n+N-1}$ has the form

$$1_{2r(0)} 2_2 1_{2r(1)} 2_2 \ldots 2_2 1_{2r(k)} 2_2 1_{2r(k+1)},$$

where $r(0), r(k+1) \in \mathbb{N} \cup \{0\}$, $r(i) \in \mathbb{N}$ $(1 \le i \le k)$ and

$$|r(i) - r(j)| \le 1 \quad \text{for } 1 \le i, j \le k, \quad r(0), r(k+1) \le \max_{1 \le i \le k} \{r(i)\}.$$

Suppose that for an integer $1 \le i < k$,

$$\delta := r(i+1) - r(i) = \pm 1,$$

and there exists a positive integer $s \le \min(k - (i+1), i-1)$ such that

$$(15) \qquad \begin{aligned} r(i+1+j) - r(i-j) &= 0 \quad \text{for } 1 \le j < s, \\ r(i+1+s) - r(i-s) &= \delta. \end{aligned}$$

Let

$$a_n a_{n+1} \ldots a_{n+N-1} = \ldots 2_2 1_{2r(i)} \overset{n'}{2} 2 1_{2r(i+1)} 2_2 \ldots$$

Suppose that $r(i+1) - r(i) = -1$. Then

$$\mu_{n'}(\alpha) = [\overset{n'}{2}, \underbrace{1, \ldots, 1}_{2r(i)\,\text{times}}, 2, \ldots, a_1] + [0, \overset{n'+1}{2}, \underbrace{1, \ldots, 1}_{2r(i+1)\,\text{times}}, 2, \ldots] < 3 + \varepsilon.$$

Therefore, by Lemma 4,

$$[\underbrace{\overset{n'-3}{1}, \ldots, 1}_{2r(i)-2\,\text{times}}, 2, \ldots, a_1] > [\underbrace{\overset{n'-3}{1}, \ldots, 1}_{2r(i+1)\,\text{times}}, 2, \ldots],$$

or

$$a_{n'-3} \ldots a_{n'-2-i} \ldots a_{n'-2-N} = a_{n'+2} \ldots a_{n'+1+i} \ldots a_{n'+1+N}.$$

But from (15), we have

$$[\underbrace{\overset{n'-3}{1}, \ldots, 1}_{2r(i)-2\,\text{times}}, 2, \ldots, a_1] < [\underbrace{\overset{n'-3}{1}, \ldots, 1}_{2r(i+1)\,\text{times}}, 2, \ldots],$$

and

$$a_{n'-3} \ldots a_{n'-2-i} \ldots a_{n'-2-N} \ne a_{n'+2} \ldots a_{n'+1+i} \ldots a_{n'+1+N}.$$

This is a contradiction. In other cases, we argue analogously.

Sufficiency. There exists $m \in \mathbb{N}$ such that $a_n a_{n+1} \ldots a_{n+2N+3}$ has the form (13) or (14) for any $n > m$. Let $n > m + N + 2$. If $a_{n-1} a_n a_{n+1}$ is neither 122 nor 221, it is easily shown that $\mu_n([A]) \leq 3$. Let $a_{n-1} a_n a_{n+1} = 122$. Assume that $a_{n-N-2} \ldots a_n \ldots a_{n+N+1}$ does not contain 2222. Then $a_{n-N-2} \ldots a_n \ldots a_{n+N+1}$ has the form (13). If $k = 0$, then

$$a_{n-N-2} \ldots a_n \ldots a_{n+N+1} = 11 \ldots 11 \overset{n}{2} 211 \ldots 11,$$

and $\mu_n([A]) \leq 3 + 2^{-(N-3)}$ by Lemma 4. If $k = 1$, then

$$a_{n-N-2} \ldots a_n \ldots a_{n+N+1} = 221 \ldots 1 \overset{n}{2} 211 \ldots 11,$$

and also $\mu_n([A]) \leq 3 + 2^{-(N-3)}$ by Lemma 4. Let $k > 1$. Let

$$a_{n-N-2} \ldots a_n a_{n+1} = 1_{2r(0)} 2_2 \ldots 1_{2r(i-1)} 2_2,$$

$$a_{n+1} \ldots a_{n+N+1} = 1_{2r(i)} 2_2 \ldots 1_{2r(k)} 2_2 1_{2r(k+1)}.$$

If $r(i-1) \leq r(i)$, then clearly $\mu_n([A]) \leq 3$ by Lemma 4. Let $r(i-1) = r(i) + 1$. Then $\mu_n([A]) \leq 3 + 2^{-(N/2-3)}$ by Lemma 4. In other cases, we argue analogously. ∎

3. Super Bernoulli sequences and continued fraction expansions. In this section, we prove our main theorem (Theorem 3). The first step is to introduce B-words which are essentially Bernoulli sequences defined by A. Markov [6]. Lemmas 15 and 17 in this section are mentioned in [6]. We give their new proofs. We apply the theory discussed in [3].

Let $I(0,1)$ be the set of all two-sided infinite words in $W(0,1)$, that is,

$$I(0,1) = \{g \mid g : \mathbb{Z} \to \{0,1\}\}.$$

For $m \in \mathbb{Z}$, we define a transformation σ_m on $I(0,1)$ by setting, for $g \in I(0,1)$,

$$\sigma_m(g)(k) = g(k+m) \quad (k \in \mathbb{Z}).$$

For $g, h \in I(0,1)$, we say that g is equivalent to h, denoted by $g \sim h$, if there exists an integer m such that $\sigma_m(g) = h$.

For a two-sided infinite word $A = \ldots a_{-2} a_{-1} a_0 a_1 \ldots$ ($a_i \in \{0,1\}$) and a substitution γ on $W(0,1)$, we define $\gamma(A)$ by

$$\gamma_i(A) = \ldots s_{-2} s_{-1} s_0 s_1 \ldots \quad (s_i \in \{0,1\}),$$

where $s_0 s_1 \ldots = \gamma_i(a_0) \gamma_i(a_1) \ldots$ and $\ldots s_{-2} s_{-1} = \ldots \gamma_i(a_{-2}) \gamma_i(a_{-1})$. It is easily shown that if $g \sim h$ for $g, h \in I(0,1)$, then $\gamma(g) \sim \gamma(h)$.

In this paper, for two-sided infinite words g, h, if $g \sim h$, then g and h are regarded as the same word.

Let S be a two-sided infinite word in $W(0,1)$. If S has the following properties, then it is said to be a *B-word*:

$$S = \begin{cases} \ldots 0_{r(-1)}10_{r(0)}10_{r(1)}1\ldots 0_{r(k)}1\ldots, & \text{or} \\ \ldots 1_{r(-1)}01_{r(0)}01_{r(1)}0\ldots 1_{r(k)}0\ldots, & \text{or} \\ \ldots 0\ldots 000\ldots 0\ldots, & \text{or} \\ \ldots 1\ldots 111\ldots 1\ldots, & \text{or} \\ \ldots 0\ldots 010\ldots 0\ldots, & \text{or} \\ \ldots 1\ldots 101\ldots 1\ldots, \end{cases}$$

where $r(i)$ $(i \in \mathbb{Z})$ are positive integers with the properties:

(A) $|r(i) - r(j)| \le 1$ for any integers i, j,
(B) if $\delta = r(i+1) - r(i) = \pm 1$, for some integer i, then either
  (1) $r(i+1+j) - r(i-j) = 0$ for all $j \in \mathbb{N}$, or
  (2) there exists $s \in \mathbb{N}$ such that $r(i+1+j) - r(i-j) = 0$ for every j with $1 \le j < s$, and $r(i+1+s) - r(i-s) = -\delta$.

Let us define substitutions $\gamma_i : W(0,1) \to W(0,1)$ for $i = 0, 1$ by

$$\gamma_0 \begin{cases} 0 \to 0, \\ 1 \to 01, \end{cases} \qquad \gamma_1 \begin{cases} 0 \to 01, \\ 1 \to 1. \end{cases}$$

LEMMA 14. *Let S be a B-word. Then*:

(1) $\gamma_0(S)$ *and* $\gamma_1(S)$ *are B-words.*
(2) $\gamma_0^{-1}(S)$ *or* $\gamma_1^{-1}(S)$ *exists and it is also a B-word.*

P r o o f. Let

$$S = \ldots 0_{r(-1)}10_{r(0)}10_{r(1)}1\ldots 0_{r(k)}1\ldots$$

Then

$$\gamma_0(S) = \ldots 0_{r(-1)+1}10_{r(0)+1}10_{r(1)+1}1\ldots 0_{r(k)+1}1\ldots$$

Therefore, $\gamma_0(S)$ is a *B*-word. On the other hand, let

$$\gamma_1(S) = \ldots (01)_{r(-1)}1(01)_{r(0)}1(01)_{r(1)}1\ldots (01)_{r(k)}1\ldots$$

Since 00 and 111 do not occur in $\gamma_1(S)$, we see that $\gamma_1(S)$ satisfies the condition (A). As $t(i) \in \{1, 2\}$, we have

$$\gamma_1(S) = \ldots (01)_{r(-1)}1(01)_{r(0)}1(01)_{r(1)}1\ldots (01)_{r(k)}1\ldots$$
$$= \ldots 1_{t(-1)}01_{t(0)}01_{t(1)}0\ldots 1_{t(k)}0\ldots$$

We show that $\gamma_1(S)$ satisfies (B). Assume that $t(i+1) - t(i) = 1$ for an integer i. Then $01_{t(i)}01_{t(i+1)} = 01011$, and $01_{t(i)}01_{t(i+1)}$ is a last part of $(01)_{r(m)}1$ for an integer m, that is, $\ldots 01_{t(i)}01_{t(i+1)} = \ldots (01)_{r(m-1)}1(01)_{r(m)}1$. Let $r(m+1) \ge r(m)$. Then $t(i+1+r(m)-1) - t(i-(r(m)-1)) = -1$ and $t(i+1+u) - t(i-u) = 0$ for $1 \le u \le r(m) - 2$. Therefore in this case $t(i)$ satisfies (B). Let $r(m+1)+1 = r(m)$. If $r(m+1+k) = r(m-k)$ for all $k \in \mathbb{N}$, then $t(i+1+l) = t(i-l)$ for all $l \in \mathbb{N}$. And if $r(m+1+l) = r(m-l)$ for every

l with $1 \le l < u$ and $r(m+1+u) = r(m-u)-1$, then $t(i+1+l) = t(i-l)$ for every l with $1 \le l < \sum_{j=0}^{u} r(m-j) - 1$, and $t(i+1+e) = t(i-e)-1$, where $l = \sum_{j=0}^{u} r(m-j) - 1$. Therefore in this case, $t(i)$ satisfies (B). Therefore, $\gamma_1(S)$ is a B-word. Analogously, $\gamma_i(S)$ for $i = 0, 1$ is a B-word in other cases for S.

Let us show the second statement of the lemma. Let

$$S = \ldots 0_{r(-1)} 1 0_{r(0)} 1 0_{r(1)} 1 \ldots 0_{r(k)} 1 \ldots,$$

where $r(i) \in \mathbb{N}$ ($i \in \mathbb{Z}$) satisfy (A) and (B). Then $\gamma_0^{-1}(S)$ exists and

$$\gamma_0^{-1}(S) = \ldots 0_{r(-1)-1} 1 0_{r(0)-1} 1 0_{r(1)-1} 1 \ldots 0_{r(k)-1} 1 \ldots.$$

Therefore, if $\min\{r(i) \mid i \in \mathbb{Z}\} \ge 2$, then $\gamma_0^{-1}(S)$ is also a B-word. Assume that $\min\{r(i) \mid i \in \mathbb{Z}\} = 1$. Set

$$\begin{aligned}
\gamma_0^{-1}(S) &= \ldots 0_{r(-1)-1} 1 0_{r(0)-1} 1 0_{r(1)-1} 1 \ldots 0_{r(k)-1} 1 \ldots \\
&= \ldots 1_{p(-1)} 0 1_{p(0)} 0 1_{p(1)} 0 \ldots 1_{p(k)} 0 \ldots,
\end{aligned}$$

where $p(i) \ge 1$. First, we show that $|p(i+1) - p(i)| \le 1$ for any $i \in \mathbb{Z}$. Suppose that $p(i+1) - p(i) \ge 2$ for some i. Then

$$S = \ldots 0(01)_{p(i)} 0(01)_{p(i+1)} \ldots$$

Thus, there exists some j such that

$$0(01)_{p(i)} 0(01)_{p(i+1)} = 0_{r(j)} 1 \ldots$$

Then we have

$$\begin{aligned}
&r(j + p(i) + 1) - r(j + p(i)) = -1, \\
&r(j + p(i) + 1 + u) - r(j + p(i) - u) = 0 \quad \text{for } 1 \le u < p(i), \\
&r(j + p(i) + 1 + p(i)) - r(j) = -1.
\end{aligned}$$

But this contradicts the fact that S is a B-word. And we have a contradiction analogously in the case where $p(i) - p(i+1) \ge 2$. Therefore, $|p(i+1) - p(i)| \le 1$ for any $i \in \mathbb{Z}$.

We prove that $|p(i) - p(k)| \le 1$ for any $i, k \in \mathbb{Z}$. Suppose that there exist i, k such that $p(i) - p(k) = 2$, and take such i, k with $|i - k|$ minimal. We may assume that $i > k$. Since $p(j) = p(k) + 1$ for $k < j < i$, we have

$$1_{p(k)} 0 1_{p(k+1)} 0 \ldots 1_{p(i)} 0 = 1_{p(k)} 0 (1_{p(k)+1} 0)_{(i-k-1)} 1_{p(k)+2} 0.$$

Therefore,

$$S = \ldots (01)_{p(k)} \overset{n}{0} ((01)_{p(k)+1} 0)_{(i-k-1)} (01)_{p(k)+2} 0 \ldots$$

Then, for some j,

$$S = \ldots 1 \overset{n}{0} 0_{r(j)-1} 1 0_{r(j+1)} 1 \ldots$$

Since $r(j+1) - r(j) = -1$, we have either $r(j+1+u) - r(j-u) = 0$ for $u = 1, 2, \ldots$, or there exists an integer v such that $r(j+1+v) - r(j-v) = 1$ and $r(j+1+u) - r(j-u) = 0$ for $0 < u \leq v$. In the first case, we have

$$S = \ldots (01)_{p(k)+2} 0((01)_{p(k)+1} 0)_{(i-k-2)} (01)_{p(k)}$$
$$0((01)_{p(k)+1} 0)_{(i-k-1)} (01)_{p(k)+2} 0 \ldots$$

But this contradicts the assumption that $|i - k|$ is minimum. Consider the second case. If $v > (i - k - 1)(p(k) + 1) + p(k) + 1$, then

$$S = \ldots (01)_{p(k)+2} 0((01)_{p(k)+1} 0)_{(i-k-2)} (01)_{p(k)}$$
$$0((01)_{p(k)+1} 0)_{(i-k-1)} (01)_{p(k)+2} 0 \ldots,$$

but this also contradicts $|i - k|$ being minimum. If $v = (i - k - 1)(p(k) + 1) + p(k) + 1$, then

$$S = \ldots (01)_a 0((01)_{p(k)+1} 0)_{(i-k-2)} (01)_{p(k)}$$
$$0((01)_{p(k)+1} 0)_{(i-k-1)} (01)_{p(k)+2} 0 \ldots,$$

where $a \geq p(k) + 3$, contrary to $|p(l+1) - p(l)| \leq 1$ for any integer l. If $v < (i - k - 1)(p(k) + 1) + p(k) + 1$, then $v = b(p(k) + 1) + p(k)$ for some integer $2 \leq b \leq i - k - 1$. Therefore

$$S = \ldots (01)_a 0((01)_{p(k)+1} 0)_{(b-1)} (01)_{p(k)} 0((01)_{p(k)+1} 0)_{(i-k-1)} (01)_{p(k)+2} 0 \ldots,$$

where $a \geq p(k) + 2$. In this case, we have $a = p(k) + 2$, which contradicts the minimality of $|i - k|$. Therefore $|p(i) - p(k)| \leq 1$ for any $i, k \in \mathbb{Z}$.

Now we prove that $\gamma_0^{-1}(S)$ satisfies the condition (B). Suppose that there exist $i \in \mathbb{Z}$ and $u \in \mathbb{N}$ such that $p(i+1) - p(i) = p(i+1+u) - p(i-u) \neq 0$ and $p(i+1+j) - p(i-j) = 0$ for $1 \leq j < u$. Suppose that $p(i+1) - p(i) = 1$. Then

$$S = \ldots (01)_{p(i-u)} \ldots 0(01)_{p(i)} \overset{n}{0} (01)_{p(i+1)} \ldots 0(01)_{p(i+1+u)} \ldots$$

Thus, for some integer j,

$$S = \ldots 1 \overset{n}{0} 0_{r(j)-1} 1 0_{r(j+1)} 1 \ldots$$

Therefore $r(j+1) - r(j) = -1$, $r(j+1+v) - r(j-v) = -1$ and $r(j+1+k) - r(j-k) = 0$ for $1 \leq k < v$, where $v = \sum_{m=i-u}^{i} r(m)$. But this contradicts the assumption that S is a B-word. Therefore $\gamma_0^{-1}(S)$ satisfies (B). The case $p(i+1) - p(i) = -1$ is similar.

We reason analogously in other cases for S. ∎

Now we introduce the following transformation T on $[0, 1]$:

$$T(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \in I_0 = [0, 1/2), \\[2mm] \dfrac{2x-1}{x} & \text{if } x \in I_1 = [1/2, 1], \end{cases}$$

and define
$$\phi_0(x) = \frac{x}{1+x}, \qquad \phi_1(x) = \frac{1}{2-x}.$$
Then $T \circ \phi_i = \mathrm{id}$ on $[0, 1]$ and $\phi_0 \circ T = \mathrm{id}$ on I_i for $i = 0, 1$. Define a function $G : [0, 1] \rightarrow W(0, 1)$ by
$$G : x \rightarrow G(x).$$

We need the following theorem. Originally, it was stated for $H(x)$ instead of $G(x)$, but it is not difficult to show that it holds for $G(x)$.

THEOREM F (S. Ito, S. Yasutomi [3]). *The following diagrams commute for $i = 0, 1$:*

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{\;G\;} & W(0, 1) \\
{\scriptstyle \phi_i} \downarrow & & \downarrow {\scriptstyle \gamma_i} \\
I_i & \xrightarrow{\;G\;} & W(0, 1)^i
\end{array}
$$

and

$$
\begin{array}{ccc}
[0, 1] & \xrightarrow{\;G\;} & W(0, 1) \\
{\scriptstyle T} \uparrow & & \uparrow {\scriptstyle \gamma_i^{-1}} \\
I_i & \xrightarrow{\;G\;} & W(0, 1)^i
\end{array}
$$

where $W(0, 1)^i$ is the image of I_i under G.

LEMMA 15 (A. Markov [6]). *Let S be a B-word. Then for any finite subword M in S there exists $x \in [0, 1]$ such that M is a subword of $G(x)$.*

Proof. By Lemma 14 and its proof, there exist words $S_0, S_1, \ldots$ such that $S_0 = S$ and $f_i(S_i) = S_{i-1}$ for $i = 1, 2, \ldots$, where $f_i \in \{\gamma_0, \gamma_1\}$. Define a sequence $\{i_n\}_{n=1}^{\infty}$ as follows:
$$i_n = \begin{cases} 0 & \text{if } f_i = \gamma_0, \\ 1 & \text{if } f_i = \gamma_1. \end{cases}$$
Consider two cases:

CASE 1: there exists an integer m such that $i_m = i_n$ for any $n \geq m$,
CASE 2: otherwise.

CASE 1. It is easily seen that if $i_m = 0$ then
$$S_m = \begin{cases} \ldots 0 \ldots 000 \ldots 0 \ldots, & \text{or} \\ \ldots 0 \ldots 010 \ldots 0 \ldots, \end{cases}$$
and if $i_m = 1$ then
$$S_m = \begin{cases} \ldots 1 \ldots 111 \ldots 1 \ldots, & \text{or} \\ \ldots 1 \ldots 101 \ldots 1 \ldots \end{cases}$$

Let $i_m = 0$. If $S_m = \ldots 0 \ldots 000 \ldots 0 \ldots$, then $S_m = G(0)$ and by Theorem F, we have

$$S = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(G(0)) = G(\phi_{i_1} \circ \ldots \circ \phi_{i_m}(0)).$$

Therefore, in this case the assertion holds. Let $S_m = \ldots 0 \ldots 010 \ldots 0 \ldots$ Then

$$S = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(\ldots 0 \ldots 010 \ldots 0 \ldots).$$

Therefore, for a large integer k,

$$\gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(0_k 10_k) \supset M.$$

On the other hand, we see easily that

$$G\left(\frac{1}{k+1}\right) = \ldots 0_k 10_k 1 \ldots$$

Therefore, by Theorem F we have

$$G\left(\phi_{i_1} \circ \ldots \circ \phi_{i_m}\left(\frac{1}{k+1}\right)\right) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}\left(G\left(\frac{1}{k+1}\right)\right) \supset M,$$

and so, in this case the assertion also holds. If $i_m = 1$, the lemma is obtained analogously.

CASE 2. For $n = 1, 2, \ldots$ let

$$A_n = \gamma_{i_1} \circ \ldots \circ \gamma_{i_n}(0), \qquad B_n = \gamma_{i_1} \circ \ldots \circ \gamma_{i_n}(1).$$

Clearly,

$$\lim_{n \to \infty} |A_n| = \infty, \qquad \lim_{n \to \infty} |B_n| = \infty.$$

Therefore, there exists an integer k such that

$$\min(|A_k|, |B_k|) > |M|.$$

Since $S = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(S_k)$, M is contained in either (i) $A_k A_k$, (ii) $B_k B_k$, or (iii) $A_k B_k$ or $B_k A_k$.

In cases (i)–(iii), we have respectively either

$$G(\phi_{i_1} \circ \ldots \circ \phi_{i_m}(0)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(G(0)) \supset M,$$
$$G(\phi_{i_1} \circ \ldots \circ \phi_{i_m}(1)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(G(1)) \supset M,$$
$$G(\phi_{i_1} \circ \ldots \circ \phi_{i_m}(1/2)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m}(G(1/2)) \supset M,$$

which proves the assertion. ∎

LEMMA 16. *Let $x \in [0, 1)$ be rational. Then there exists $n \in \mathbb{N} \cup \{0\}$ such that $T^n(x) = 0$.*

Proof. See Proposition 1.1 of [4]. ∎

For $x \in [0, 1]$, define an infinite sequence $\{i_1, i_2, \ldots\}$ by the condition

$$i_n = i \quad \text{if} \quad T^{n-1}(x) \in I_i.$$

We call this sequence the *name* of x.

LEMMA 17 (A. Markov [6]). *For any $x \in [0,1]$, the sequence $G(x)$ is a B-word.*

P r o o f. The sequences $G(0) = \ldots 000 \ldots$ and $G(1) = \ldots 111 \ldots$ are B-words. First, let $0 < x < 1$ be rational, and the sequence $i_1, i_2, \ldots$ be the name of x. Then by Lemma 16, there exists $n \in \mathbb{N} \cup \{0\}$ such that $T^n(x) = 0$. Therefore, by Theorem F we have

$$G(x) = G(\phi_{i_1} \circ \ldots \circ \phi_{i_n}(0)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_n}(G(0)).$$

Since $G(0)$ is a B-word, so is $G(x)$ by Lemma 15.

Let x be irrational. Then there exist $x_n \in [0,1] \cap \mathbb{Q}$ for all $n \in \mathbb{N}$ such that $G(x, i) = G(x_n, i)$ for $-n \leq i \leq n$. Hence $G(x)$ is a B-word. ■

THEOREM 2. *Let A be a one-sided infinite word in $W(1,2)$. Let*

$$A = a_0 a_1 \ldots = 1_{p(0)} 2_{p(1)} 1_{p(2)} 2_{p(3)} \ldots \in W(1,2),$$

where $p(i)$ is an even positive integer for $i = 0, 1, \ldots$, and put $S = \phi^{-1}(A) = s_0 s_1 \ldots$, where $s_i \in \{0, 1\}$. For $\mu([A]) \leq 3$, the following condition is necessary and sufficient: For every $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that if $n \geq m$, then $s_n s_{n+1} \ldots s_{n+N-1}$ is a subword of $G(x)$ for some $x \in [0,1]$.

P r o o f. *Necessity.* There exists $m \in \mathbb{N}$ such that for any integer $n \geq m$, $s_n s_{n+1} \ldots s_{n+N-1}$ occurs infinitely many times in S. Set $M = s_n s_{n+1} \ldots \ldots s_{n+N-1}$. Then, at least one of the words $0M0$, $0M1$, $1M0$, $1M1$ occurs infinitely many times in S. By induction, there exist sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots \in \{0, 1\}$ such that $a_n \ldots a_1 M b_1 \ldots b_n$ occurs infinitely many times in S for $n = 1, 2, \ldots$ Define a two-sided word M' by

$$M' = \ldots a_2 a_1 M b_1 b_2 \ldots$$

Theorem 1 implies that M' is a B-word. Therefore, by Lemma 15 there exists $x \in [0,1]$ such that M is a subword of $G(x)$, and then $s_n s_{n+1} \ldots s_{n+N-1}$ is a subword of $G(x)$ for some $x \in [0,1]$.

Sufficiency. By assumption, for $n \geq m$, $s_n s_{n+1} \ldots s_{n+N-1}$ is a subword of $G(x)$ for some $x \in [0,1]$. Since $G(x)$ is a B-word by Lemma 17, $\phi(M)$ satisfies the conditions of Theorem 1. Therefore, $\mu([A]) \leq 3$. ■

Define $G(x, y, n)$ for real numbers x, y and an integer n by

$$G(x, y, n) = \lfloor nx + y \rfloor - \lfloor (n-1)x + y \rfloor.$$

LEMMA 18. *Let $\alpha \in [0,1]$ and $N \in \mathbb{N}$. Set $\{-k\alpha \bmod 1 \mid k = 1, \ldots, N\} \cup \{0, 1\} = \{a_0^{(\alpha, N)}, \ldots, a_{P_{(\alpha, N)}}^{(\alpha, N)}\}$ with $a_0^{(\alpha, N)} = 0 < a_1^{(\alpha, N)} < \ldots < a_{P_N}^{(\alpha, N)} = 1$. Define a function f on $[0,1]$ by*

$$f(x) = \begin{cases} k & \text{if } x \in [a_k^{(\alpha, N)}, a_{k+1}^{(\alpha, N)}), \\ 0 & \text{if } x = 1. \end{cases}$$

Then, for $x, y \in [0,1]$,

(1) $G(\alpha, x, 1) \ldots G(\alpha, x, N) = G(\alpha, y, 1) \ldots G(\alpha, y, N)$ *if* $f(x) = f(y)$,
(2) $G(\alpha, x, 1) \ldots G(\alpha, x, N) \neq G(\alpha, y, 1) \ldots G(\alpha, y, N)$ *if* $f(x) \neq f(y)$.

P r o o f. (1) Suppose that $x, y \in [0, 1]$, $x > y$, $f(x) = f(y)$ and $G(\alpha, x, i)$ $\neq G(\alpha, y, i)$ for some $1 \leq i \leq N$. Since $[i\alpha + x] > [i\alpha + y]$, there exists z such that $y < z \leq x$ and $i\alpha + z \equiv 0 \bmod 1$. Therefore, $z = a_k$ for some $0 \leq k \leq P_{(\alpha, N)}$, contrary to $f(x) = f(y)$.

(2) Let

$$0 < \varepsilon < \min\{|a_i^{(\alpha, N)} - a_{i-1}^{(\alpha, N)}| \mid i = 1, \ldots, P_{(\alpha, N)}\}.$$

Then, for $k = 1, \ldots, P_{(\alpha, N)}$,

$$G(\alpha, a_k, 1) \ldots G(\alpha, a_k, N) \neq G(\alpha, a_k - \varepsilon, 1) \ldots G(\alpha, a_k - \varepsilon, N).$$

It is easily shown that for $x \in [0, 1)$, $\{u \in [0, 1) \mid G(\alpha, x, 1) \ldots G(\alpha, x, N) = G(\alpha, u, 1) \ldots G(\alpha, u, N)\}$ is a connected set. This yields (2). $\blacksquare$

LEMMA 19. *Let S be an arbitrary finite word in $W(0, 1)$ and set*

$$P_S = \{x \in [0, 1] \mid S \text{ is a subword of } G(x)\}.$$

Then P_S is a connected set in $[0, 1]$.

P r o o f. Let P_S be not empty. Let $u, v \in P_S$ and $u \leq v$. We show that for any $z \in [u, v]$, $z \in P_S$. By hypothesis, there exist integers n, m such that

(16) $\qquad S = G(u, n)G(u, n+1) \ldots G(u, n + |S| - 1)$

$$= G(v, m)G(v, m+1) \ldots G(v, m + |S| - 1).$$

Set $u_1 = \{(n-1)u\}$, $v_1 = \{(m-1)v\}$, $x = (v_1 - u_1)/(u - v)$ and $y = (vu_1 - uv_1)/(v - u)$. Then (x, y) is a solution of the equation

$$\begin{cases} Y = uX + u_1, \\ Y = vX + v_1. \end{cases}$$

Set $z_1 = y - zx$. Then (x, y) is on the line $\{(X, Y) \mid Y = zX + z_1\}$. For $p, q \in \mathbb{R}$, let

$$\langle p, q \rangle = \begin{cases} [p, q] & \text{if } p \leq q, \\ [q, p] & \text{otherwise.} \end{cases}$$

It is not difficult to see that $kz + z_1 \in \langle ku + u_1, kv + v_1 \rangle$ for $k = 0, 1, \ldots, |S| - 1$. From (16), we have $\lfloor ku + u_1 \rfloor = \lfloor kv + v_1 \rfloor$ and so $\lfloor kz + z_1 \rfloor = \lfloor ku + u_1 \rfloor = \lfloor kv + v_1 \rfloor$ for such k. Hence,

$$S = G(z, z_1, 1) \ldots G(z, z_1, |S|).$$

If z is not rational, then the fact that $\{\{nz_1\} \mid n = 0, 1, \ldots\}$ is dense in $[0, 1]$ implies that there exists $l \in \mathbb{N}$ such that $\lfloor kz + z_1 \rfloor = \lfloor kz + \{lz\} \rfloor$ for $k = 0, 1, \ldots, |S| - 1$. Therefore,

$$S = G(z, l+1) \ldots G(z, l + |S|),$$

and $z \in P_S$.

Assume that z is rational and set $z = p/q$, where $p, q \in \mathbb{N}$ and $(p, q) = 1$. Set $r = \lfloor qz_1 \rfloor$. Then Lemma 18 shows that

$$G(z, z_1, 1) \ldots G(z, z_1, |S|) = G(z, r/q, 1) \ldots G(z, r/q, |S|).$$

From the hypothesis on p and q, there exists $t \in \mathbb{N} \cup \{0\}$ such that $tp \equiv r \bmod q$ and $t < q$. Therefore,

$$G(z, t + 1) \ldots G(z, t + |S|) = G(z, r/q, 1) \ldots G(z, r/q, |S|).$$

Hence, $z \in P_S$. ∎

Let $S^1 = \mathbb{R}/\mathbb{Z}$ and $I = [0, 1]$. Define a function $\tau_n : I \to I \times S^1$ for $n = 0, 1, \ldots$ by

$$\tau_n(x) = (x, -nx),$$

and define

$$\Delta_N = I \times S^1 - \bigcup_{n=0}^{N} \tau_n(I).$$

LEMMA 20. *Let* $P = (x_1, y_1)$, $Q = (x_2, y_2) \in \Delta_N$. *Then* P *and* Q *are in the same connected component of* Δ_N *if and only if*

$$G(x_1, y_1, 1) \ldots G(x_1, y_1, N) = G(x_2, y_2, 1) \ldots G(x_2, y_2, N).$$

P r o o f. The necessity is immediate by Lemma 18.

For a component C of Δ_N we denote by $g(C)$ the word

$$g(C) = G(x_1, y_1, 1) \ldots G(x_1, y_1, N) \quad \text{for some } (x_1, y_1) \in C,$$

which depends only on C. Let π be the projection $I \times S^1 \to I$. Let C be a connected component of Δ_N. Put $p(C) = \{D \mid D$ is a connected component of Δ_N such that $g(D) = g(C)\}$. Suppose that $C_1, C_2 \in p(C)$ and $\pi(C_1) \cap \pi(C_2) \neq \emptyset$. Then there exist $(\alpha, v_1) \in C_1$ and $(\alpha, v_2) \in C_2$. By the definition of Δ_N, we have

$$(\{\alpha\} \times S^1) \cap \Delta_N = \{\alpha\} \times (a_0^{(\alpha, N)}, a_1^{(\alpha, N)}) \cup \ldots \cup \{\alpha\} \times (a_{P_{(\alpha, N)}-1}^{(\alpha, N)}, a_{P_{(\alpha, N)}}^{(\alpha, N)}).$$

If $f(v_1) \neq f(v_2)$, where $f(\cdot)$ is defined in Lemma 18, then

$$G(u, v_1, 1) \ldots G(u, v_1, N) \neq G(u, v_2, 1) \ldots G(u, v_2, N).$$

This is a contradiction. Thus $f(v_1) = f(v_2)$, and therefore $C_1 = C_2$. On the other hand, it is easily seen that

$$P_{g(C)} = \pi\Big(\bigcup_{D \in p(C)} D \Big).$$

Since $P_{p(C)}$ is a connected set by Lemma 19, we know that $p(C) = \{C\}$. Hence, the assertion follows. ∎

LEMMA 21. *Let* S *be a finite word in* $W(0, 1)$. *Then*

$$|P_S| \leq 2/|S|.$$

P r o o f. We may assume that $P_S \neq \emptyset$. By Lemma 20 there exists a component C of $\Delta_{|S|}$ such that $P_S = \pi(C)$. From the definition of $\Delta_{|S|}$,

$$|\pi(C)| \leq 2/|S|. \quad \blacksquare$$

LEMMA 22. *Let* $x, y \in [0, 1]$. *Then*

$$D_{G(x)}(N) = D_{G(x,y)}(N) \quad \text{for all } N,$$

where

$$G(x, y) = \ldots G(x, y, -1)G(x, y, 0)G(x, y, 1) \ldots,$$

and $G(x, y, n) = \lfloor nx + y \rfloor - \lfloor (n-1)x + y \rfloor$ *for* $n \in \mathbb{Z}$.

The proof is easy. $\blacksquare$

LEMMA 23. *Let* $\alpha \in [0, 1]$ *and* $N \in \mathbb{N}$. *Then there exist* $\varepsilon > 0$ *and* $N_0 \in \mathbb{N}$ *such that for any* $\beta \in [0, 1]$ *with* $|\beta - \alpha| < \varepsilon$, *each subword of* $G(\alpha)$ *of length* N *is contained in every subword of* $G(\beta)$ *of length larger than* N_0.

P r o o f. Assume that α is irrational. For $x \in [0, 1]$ and $n \in \mathbb{N}$, define:

$$M_{(x,n)} = \max\{|a_k^{(x,n)} - a_{k+1}^{(x,n)}| \mid k = 0, 1, \ldots, P_{(x,n)} - 1\},$$
$$m_{(x,n)} = \min\{|a_k^{(x,n)} - a_{k+1}^{(x,n)}| \mid k = 0, 1, \ldots, P_{(x,n)} - 1\},$$

where $a_k^{(x,n)}$ $(k = 0, 1, \ldots, P_{(x,n)})$ and $P_{(x,n)}$ are defined in Lemma 18. Since α is irrational, there exists $K \in \mathbb{N}$ such that

$$M_{(\alpha,K)} < \tfrac{1}{3} m_{(\alpha,N)}.$$

Then it is not difficult to show that there exists $\varepsilon > 0$ such that if $|\beta - \alpha| < \varepsilon$ then $M_{(\beta,K)} < \tfrac{1}{3} m_{(\beta,N)}$ and $G(\beta)$ contains every subword of $G(\alpha)$ of length N. Let $\beta \in [0, 1]$ and $|\beta - \alpha| < \varepsilon$. Let S be a subword of $G(\alpha)$ of length N. Let S' be an arbitrary subword of $G(\beta)$ of length $2K$. By Lemma 18 there exists an integer $0 \leq k \leq P_{(\beta,n)} - 1$ such that if $v \in [a_k^{(\beta,n)}, a_{k+1}^{(\beta,n)})$ then $S = G(\beta, v, 1) \ldots G(\beta, v, N)$. Moreover, there exists $u \in [0, 1]$ such that

$$S' = G(\beta, u, 1) \ldots G(\beta, u, 2K).$$

Since $M_{(\beta,K)} < \tfrac{1}{3} m_{(\beta,N)}$, there exists a natural number $j \leq K$ such that $j\beta + u \in [a_k^{(\beta,n)}, a_{k+1}^{(\beta,n)})$. Therefore,

$$S = G(\beta, u, j + 1) \ldots G(\beta, u, j + N).$$

Hence, S is a subword of S'.

Suppose that α is rational. First, let $\alpha = 0$. Then a subword of $G(0)$ of length N is equal to 0_N. It is not difficult to show that if $0 \leq \beta \leq 1/(N+1)$ then $G(\beta) = \ldots 10_{R(0)} 10_{R(1)} \ldots$ and $R(i) \geq N$ for every $i \in \mathbb{Z}$. Therefore, any subword of $G(\beta)$ of length $2N$ has 0_N as a subword.

Let $\alpha = 1$. Then a subword of $G(1)$ of length N is 1_N. It is not difficult to show that if $N/(N+1) \leq \beta \leq 1$ then $G(\beta) = \ldots 01_{R(0)} 01_{R(1)} \ldots$ and $R(i) \geq N$ for $i \in \mathbb{Z}$. Therefore, any subword of $G(\beta)$ of length $2N$ has 1_N as a subword. Thus the lemma holds for $\alpha = 0, 1$.

Suppose that $\alpha \neq 0, 1$. By Lemma 16, there exists $n \in \mathbb{N} \cup \{0\}$ such that $T^n(\alpha) = 0$. Therefore, there exists $m \in \mathbb{N} \cup \{0\}$ such that $T^m(\alpha) = 1/2$. Then, by Theorem F, we have

$$\phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_0(1) = \alpha, \qquad \phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_1(0) = \alpha,$$

where $\{i_1, i_2, \ldots\}$ is the name of α. Define words A_1, B_1, A_2 and B_2 by

$$A_1 = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m} \circ \gamma_0(0), \qquad A_2 = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m} \circ \gamma_1(0),$$
$$B_1 = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m} \circ \gamma_0(1), \qquad B_2 = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m} \circ \gamma_1(1).$$

Then $\gamma_0(1) = \gamma_1(0) = 01$ implies $B_1 = A_2$. By Theorem F, we see that

$$G(\alpha) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_m} \circ \gamma_0(G(1)) = \ldots B_1 B_1 \ldots$$

It is easily seen that any subword of $G(\alpha)$ of length N is contained in $(B_1)_N$. Define real numbers u and v by

$$u = \phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_0\left(\frac{N}{N+1}\right), \qquad v = \phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_1\left(\frac{1}{N+1}\right).$$

Then

$$[u, \alpha] = \phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_0\left(\left[\frac{N}{N+1}, 1\right]\right),$$
$$[\alpha, v] = \phi_{i_1} \circ \ldots \circ \phi_{i_m} \circ \phi_1\left(\left[0, \frac{1}{N+1}\right]\right).$$

Therefore, $\beta \in [u, \alpha]$ implies

$$G(\beta) = \ldots (B_1)_{r(0)} A_1 (B_1)_{r(1)} A_1 \ldots,$$

where $r(i)$ are integers and $r(i) \geq N$ for $i \in \mathbb{Z}$, and if $\beta \in [\alpha, v]$ then

$$G(\beta) = \ldots (A_2)_{r(0)} B_2 (A_2)_{r(1)} B_2 \ldots,$$

where $r(i)$ have the same property. Since $|A_1|, |B_2| < |B_1|$, every subword of $G(\beta)$ of length $2N|B_1|$ contains $(B_1)_N$ as a subword. Therefore the lemma holds in this case. ∎

LEMMA 24. *Let $x \in [0, 1] \cap \mathbb{Q}$. Then for $i = 1, 2$,*

$$\overline{G(\phi_i(x))} = \gamma_i(\overline{G(x)}) \qquad \text{if } x \neq 1,$$
$$\underline{G(\phi_i(x))} = \gamma_i(\underline{G(x)}) \qquad \text{if } x \neq 0.$$

P r o o f. Assume that $x \neq 1$. Put $x = n/m$, let $i_1, i_2, \ldots$ be the name of x, and set $\bar{x} = \min\{y \in F_m \mid y > x\}$. It is not difficult to show that

$\overline{T(x)} = T(\overline{x})$. Since $\overline{0} = 1$ and $\phi_{i_1} \circ \ldots \circ \phi_{i_k}(0) = x$ where k is defined by $T^k(x) = 0$ and $T^{k-1}(x) \neq 0$, we have

$$\phi_{i_1} \circ \ldots \circ \phi_{i_k}(1) = \overline{x}.$$

Therefore, by Theorem F, we have

$$G(x) = G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(0)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(G(0)),$$
$$G(\overline{x}) = G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(1)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(G(1)).$$

Then it is not difficult to show that

$$\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(0) = G(x,1) \ldots G(x,m),$$
$$\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(1) = G(\overline{x},1) \ldots G(\overline{x},h),$$

where h is the denominator of $\overline{x}$. Therefore,

$$\overline{G(x)} = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\ldots 0 \ldots 010 \ldots 0 \ldots).$$

Since the name of $\phi_i(x)$ is $i, i_1, \ldots, i_k, \ldots$, we have

$$\overline{G(\phi_i(x))} = \gamma_i \circ \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\ldots 0 \ldots 010 \ldots 0 \ldots) = \gamma_i(\overline{G(x)}).$$

Other cases can be proved analogously. ∎

LEMMA 25. (1) *Let* $(x_0, y_0), (x_1, y_1) \in [0,1]^2$ *be distinct with* $x_i \leq y_i$ *for* $i = 0, 1$. *Let* S_i *be any super Bernoulli sequence related to* (x_i, y_i) *for* $i = 0, 1$. *Then* S_0 *and* S_1 *are different.*

(2) *Let* $(x, y) \in [0,1]^2$ *and* $x \leq y$. *Let* S *and* S' *be super Bernoulli sequences related to* (x, y) *of type* $i, j \in \{1, 2, 3, 4\}$, *respectively. Then* S *and* S' *are different if* $i \neq j$.

Proof. (1) Since $(x_0, y_0) \neq (x_1, y_1)$, there exists an irrational number $w \in [x_0, y_0] \triangle [x_1, y_1]$. We assume without loss of generality that $w \in [x_0, y_0]$ and $w \notin [x_1, y_1]$. Then, by Lemma 21, there exists $M \in \mathbb{N}$ such that for $N > M$,

$$D_{G(w)}(N) \cap \bigcup_{z \in [x_1, y_1]} D_{G(z)}(N) = \emptyset.$$

Therefore if S_1 is a super Bernoulli sequence related to (x_1, y_1) of type 1, then $S_0 \neq S_1$.

Assume that S_1 is a super Bernoulli sequence related to (x_1, y_1) of type 3. Then, from the proof of Lemma 24, we have

$$\overline{G(y_1)} = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\ldots 0 \ldots 010 \ldots 0 \ldots) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\overline{G(0)}),$$

where $\{i_1, i_2, \ldots\}$ is the name of y_1 and $T^k(y_1) = 0$ and $T^{k-1}(y_1) \neq 0$. For any $u > 0$, we see easily that for all N,

$$D_{\overline{G(0)}}(N) \subset \bigcup_{z \in [0, u]} D_{G(z)}(N).$$

Therefore, for each $M \in \mathbb{N}$,

$$D_{\gamma_{i_1} \circ \ldots \circ \gamma_{i_k} \overline{G(0)}}(M) \subset \bigcup_{z \in [0,u]} D_{\gamma_{i_1} \circ \ldots \circ \gamma_{i_k} G(z)}(M).$$

By Lemma 24, we have

$$D_{\overline{G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(0))}}(M) \subset \bigcup_{z \in [0,u]} D_{G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(z))}(M).$$

Since $\phi_{i_1} \circ \ldots \circ \phi_{i_k}$ is increasing and $\phi_{i_1} \circ \ldots \circ \phi_{i_k}(0) = y_1$, we have for all M,

$$D_{\overline{G(y_1)}}(M) \subset \bigcup_{w \in [y_1,v]} D_{G(w)}(M),$$

where $v = \phi_{i_1} \circ \ldots \circ \phi_{i_k}(u)$. Therefore, for any $u \in [0,1]$ with $u > y_1$, we have

$$D_{\overline{G(y_1)}}(M) \subset \bigcup_{w \in [y_1,u]} D_{G(w)}(M).$$

By Lemma 21 there exists $M' \in \mathbb{N}$ and $u \in [0,1]$ with $u > y_1$ such that for any $N > M'$,

$$D_{G(w)}(N) \cap \bigcup_{z \in [y_1,u]} D_{G(z)}(N) = \emptyset.$$

Therefore, for any $N > M'$ we have

$$D_{G(w)}(N) \cap \left(D_{\overline{G(y_1)}}(M) \cup \bigcup_{z \in [x_1,y_1]} D_{G(z)}(N) \right) = \emptyset.$$

Hence, $S_0 \neq S_1$. In other cases we argue in the same way.

(2) We assume that S_0 is a super Bernoulli sequence related to (x, y) of type 1, and S_1 is of type 3. Let $i_1, i_2, \ldots$ be the name of w. Then, as in the proof of Lemma 24, we have

$$\overline{G(y)} = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\ldots 0 \ldots 010 \ldots 0 \ldots),$$

where k is least such that $T^k(y) = 0$. We show that $\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01)$ is not a subword of $G(y)$. Suppose otherwise. As in the proof of Lemma 24, we have

$$G(y) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(G(0)).$$

Since $G(T(y)) = \gamma_{i_2} \circ \ldots \circ \gamma_{i_k}(G(0))$, $\gamma_{i_2} \circ \ldots \circ \gamma_{i_k}(01)$ is a subword of $G(T(y))$. Therefore, by induction on k, 01 is a subword of $G(T^k(y)) = G(0)$, a contradiction. Therefore, $\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01)$ is not a subword of $G(y)$. By Theorem F we have

$$G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(1)) = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k} G(1),$$

and $\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01)$ is a subword of $G(\phi_{i_1} \circ \ldots \circ \phi_{i_k}(1/2))$. Since $\phi_{i_1} \circ \ldots \circ \phi_{i_k}$ is an increasing function, we know that $\phi_{i_1} \circ \ldots \circ \phi_{i_k}(1/2) > y$. Therefore,

the fact that $P_{\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(1)}$ is a connected set, by Lemma 19, implies

$$P_{\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01)} \cap [x, y] = \emptyset.$$

Therefore,

$$\gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01) \in D_{\overline{G(y)}}(K) \quad \text{and} \quad \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(01) \notin \bigcup_{z \in [x,y]} D_{G(z)}(K).$$

Thus, S does not coincide with S'. Other cases are analogous. ∎

Proof of Theorem 3. Let α be irrational with $\mu(\alpha) \leq 3$, and with continued fraction expansion $[a_0, a_1, \ldots]$. By Lemma 5 there exists $m \in \mathbb{N} \cup \{0\}$ such that $a_n \in \{1, 2\}$ for all $n \geq m$. Define a word A by

$$A = a_m a_{m+1} \ldots$$

By Lemma 9, we only have to study the word A' defined in Lemma 9, instead of A. The word A' has the form

$$A' = 1_{q(0)} 2_{q(1)} 1_{q(2)} 2_{q(3)} \ldots,$$

where $q(i)$ are even non-negative integers for $i = 0, 1, \ldots$ Set $S = s_0 s_1 \ldots = \phi^{-1}(A')$, and define a subset Ω of $[0, 1]$ as follows: $x \in \Omega$ if and only if for any $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $\beta \in [0, 1]$ such that $|\beta - x| < \varepsilon$ and there exists a subword of $G(\beta)$ of length N which occurs infinitely many times in S.

From the definition, it is not difficult to see that Ω is closed. Let us show that it is connected. Let $x_1, x_2 \in \Omega$ and assume that $x_1 < x_2$. Take z satisfying $x_1 < z < x_2$. We show that $z \in \Omega$. Let $\varepsilon > 0$ be small and $N \in \mathbb{N}$. We may assume that $1/(N-1) < \varepsilon$. By Theorem 2, there exists $m \in \mathbb{N}$ such that for any subword p of $S_{(m)}$ of length N there exists $\beta \in [0, 1]$ such that p is also a subword of $G(\beta)$, where $S_{(m)} = s_m s_{m+1} \ldots$ We may assume that any subword of $S_{(m)}$ of length N occurs infinitely many times in $S_{(m)}$. Since $x_i \in \Omega$ for $i = 1, 2$, there exists $y_i \in [0, 1]$ such that $|x_i - y_i| < \varepsilon$ and there exists a subword p_i of $G(y_i)$ of length N which occurs infinitely many times in $S_{(m)}$. Let $s_{c_i} s_{c_i+1} \ldots s_{c_i+N-1}$ be an occurrence of p_i for $i = 1, 2$. Assume that $c_1 < c_2$. By Theorem 2, there exist $t_k \in [0, 1]$ for $k = 0, 1, \ldots, c_2 - c_1$ such that $s_{c_1+k} s_{c_1+k+1} \ldots s_{c_1+k+N-1}$ is a subword of $G(t_k)$. By Lemma 21, we have $|t_k - t_{k+1}| < 2/(N-1) < 2\varepsilon$ for $k = 0, 1, \ldots, c_2 - c_1 - 1$. Therefore, there exists t_l such that $|z - t_l| < \varepsilon$. Since $s_{c_1+k} s_{c_1+k+1} \ldots s_{c_1+k+N-1}$ occurs infinitely many times in S, we conclude that $z \in \Omega$.

Let us show that Ω is not empty. We can choose words q_N for $N \in \mathbb{N}$ such that q_N occurs infinitely many times in S and $|q_N| = N$. Then, by Theorem 2, for each N there exists $\tau_N \in [0, 1]$ such that q_N is contained in $G(\tau_N)$. Since $[0, 1]$ is a compact set, there exists a subsequence τ_{i_k} ($k = 1, 2, \ldots$) which converges to a point $\gamma \in [0, 1]$. It is not difficult to show that $\gamma \in \Omega$. Therefore, Ω is not empty.

Since Ω is a connected and closed set, there exist $x, y \in [0, 1]$ with $x \le y$ such that $\Omega = [x, y]$. Let us show that

$$(17) \qquad D'_S(N) \supset \bigcup_{z \in [x,y]} D_{G(z)}(N) \quad \text{for all } N \in \mathbb{N}.$$

Suppose that $z \in [x, y]$ and p is an arbitrary subword of $G(z)$ of length N. Then, by Lemma 23, there exist $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ such that for $\beta \in [0, 1]$ with $|\beta - z| < \varepsilon$, each subword of $G(z)$ of length N is contained in every subword of $G(\beta)$ of length greater than N_0. Since $z \in \Omega$, there exists $\gamma \in [0, 1]$ such that $|\gamma - z| < \varepsilon$, and there exists a subword q of $G(\gamma)$ of length N_0 such that q occurs infinitely many times in S. Since p is contained in q, p occurs infinitely many times in S. Therefore, $p \in D'_S(N)$, that is, (17) holds.

Let us study the set $D'_S(N) \backslash \bigcup_{z \in [x,y]} D_{G(z)}(N)$. Suppose that it is not empty and let $L \in D'_S(N) \backslash \bigcup_{z \in [x,y]} D_{G(z)}(N)$. Then we can take a sequence of words $L_0, L_1, \dots$ which satisfies the following conditions:

- $L_0 = L$,
- $L_{i+1} = e_i L_i f_i$ for some $e_i, f_i \in \{0, 1\}$ for $i = 0, 1, \dots$,
- L_i occurs infinitely many times in S for $i = 0, 1, \dots$

Then, by Theorem 2, there exists $\tau_i \in [0, 1]$ for $i = 0, 1, \dots$ such that L_i is a subword of $G(\tau_i)$. By Lemma 21, τ_i converges in $[0, 1]$ to some $v = \lim_{i \to \infty} \tau_i$. It is easy to see that $v \in [x, y] \; (= \Omega)$.

Let us show that $v = x$ or $v = y$. Suppose that $x < v < y$. Then there exists $l \in \mathbb{N}$ such that $\tau_l \in [x, y]$. Since L is a subword of $G(\tau_l)$, $L \in \bigcup_{z \in [x,y]} D_{G(z)}(N)$, contrary to the definition of L. Therefore, $v = x$ or $v = y$.

CASE 1: $v = x$. Suppose that x is irrational. Since the boundary points of $\pi(C)$ are rational where C is any component of Δ_N, there exists $\varepsilon > 0$ such that if $y \in [0, 1]$ and $|x - y| < \varepsilon$, then $D_{G(x)}(N) = D_{G(y)}(N)$. As $\lim_{i \to \infty} \tau_i = v$ there exists $j \in \mathbb{N}$ such that $|\tau_j - x| < \varepsilon$. Therefore, since L is a subword of $D_{G(\tau_j)}(N)$, L is a subword of $D_{G(x)}(N)$, contrary to the definition of L.

Suppose that $v = x$ and x is rational. Define a two-sided infinite word Γ by

$$\Gamma = \lim_{i \to \infty} L_i = \dots e_1 e_0 L f_0 f_1 \dots$$

By Theorem 1, Γ is a B-word. From Lemma 14 and its proof, if Γ has 00 as a subword, then $\gamma_0^{-1}(\Gamma)$ exists and $\gamma_1^{-1}(\Gamma)$ does not; if Γ has 11 as a subword, then $\gamma_1^{-1}(\Gamma)$ exists and $\gamma_0^{-1}(\Gamma)$ does not; and if Γ has neither 00 nor 11 as a subword, that is, $\Gamma = G(1/2)$, then $\gamma_1^{-1}(\Gamma)$ and $\gamma_0^{-1}(\Gamma)$ exist. Define a sequence $i_1, i_2, \dots \in \{0, 1\}$ inductively as follows:

If Γ has 00 as a subword then $i_1 = 0$, if Γ has 11 as a subword then $i_0 = 1$, and if $\Gamma = G(1/2)$ then $i_0 = 1$.

Assume that $i_1, \ldots, i_k$ are defined; set

$$\Gamma_k = \gamma_{i_k}^{-1} \circ \ldots \circ \gamma_{i_1}^{-1}(\Gamma).$$

Then $i_{k+1} = 0$ if Γ_k has 00 as a subword, $i_{k+1} = 1$ if Γ_k has 11 as a subword, and $i_{k+1} = 1$ if $\Gamma_k = G(1/2)$.

Let $u \in [0, 1]$ be the number whose name is $\{i_1, i_2, \ldots\}$. Suppose that u is irrational. In the same way as in the proof of Lemma 15, L_i is a subword of $G(u)$ for $i = 0, 1, \ldots$ Therefore, by Lemma 21, $u = x$. But this contradicts the fact that x is rational. Hence, u is rational and there exists $j \in \mathbb{N}$ such that if $k \geq j$, then $i_k = i_j$. It is not difficult to see that

$$\Gamma_k \in \{G(0), \overline{G(0)}, G(1), \underline{G(1)}\}.$$

Since $\Gamma = \gamma_{i_1} \circ \ldots \circ \gamma_{i_k}(\Gamma_k)$, by Lemma 24 we have

$$\Gamma \in \{G(\phi'(0)), \overline{G(\phi'(0))}, G(\phi'(1)), \underline{G(\phi'(1))}\},$$

where $\phi' = \phi_{i_1} \circ \ldots \circ \phi_{i_k}$.

Let us show that $\Gamma = \underline{G(\phi'(1))}$. Suppose that $\Gamma = G(\phi'(0))$ or $G(\phi'(1))$, that is, there exists $t \in \overline{[0, 1]}$ such that $\Gamma = G(t)$. Then, $t \in \underline{[x, y]}$ by Lemma 21, contrary to $L \notin \bigcup_{z \in [x,y]} D_{G(z)}(N)$. Suppose that $\Gamma = \overline{G(\phi'(0))}$. Let us show that $\phi'(0) = x$. By Lemma 24,

(18) $$\overline{G(\phi'(0))} = \gamma'(\overline{G(0)}) = \ldots \gamma'(0) \ldots \gamma'(0)\gamma'(1)\gamma'(0) \ldots,$$

where $\phi' = \gamma_{i_0} \circ \ldots \circ \gamma_{i_k}$. By Theorem F,

$$G(\phi'(0)) = \gamma'(G(0)) = \ldots \gamma'(0) \ldots \gamma'(0) \ldots$$

Since L_i is a subword Γ for $i = 0, 1, \ldots$, for any n there exists k such that $\gamma'(0)_n$ is a subword of L_k. Therefore, Lemma 21 implies $\phi'(0) = x$.

Let us show that for all M,

(19) $$D_{\overline{G(x)}}(M) \subset \bigcup_{z \in [x,y]} D_{G(z)}(M).$$

By (18), any element of $D_{\overline{G(x)}}(M)$ is a subword of $\gamma'(0)_M \gamma'(1)\gamma'(0)_M$, and since $\phi'(0) = x$, there exists $M' > M$ with $\phi'(1/M') \in [x, y]$. By Theorem F we have

$$G(\phi'(1/M')) = \gamma'(\ldots 0_{M'-1}10_{M'-1}1 \ldots)$$
$$= \ldots \gamma'(0)_{M'-1}\gamma'(1)\gamma'(0)_{M'-1}\gamma'(1) \ldots$$

Therefore, any element of $D_{\overline{G(x)}}(M)$ is a subword of $G(\phi'(1/M'))$. Hence, (19) holds and $\Gamma \neq \overline{G(\phi'(0))}$. Therefore, $\Gamma = \underline{G(\phi'(1))}$. Then we have $x = \phi'(1)$ in the same way. Thus $L \in D_{\underline{G(x)}}(N)$.

CASE 2: $v = y$. Then analogously y is rational and $L \in D_{\overline{G(y)}}(N)$. Therefore we have the first statement of the theorem.

We now prove the last statement of the theorem. Let $0 \leq x \leq y \leq 1$, let $S = s_0 s_1 s_2 \ldots \in W(0,1)$ be a super Bernoulli sequence related to (x,y) and let $A \in W(1,2)$ be a one-sided infinite word such that $D'_A(N) = D'_{\phi(S)}(N)$ for all N.

Theorem 2 implies immediately $\mu([A]) \leq 3$.

Let $x = y$ be rational. By Proposition 3 in Section 4, since S is a super Bernoulli sequence related to (x,y) of type 1, there exists m such that

$$s_m s_{m+1} \ldots = G(x,1)G(x,2) \ldots,$$

and there exists k such that

$$a_k a_{k+1} \ldots = \phi(G(x,1)G(x,2) \ldots).$$

Theorem D yields $\mu([\phi(S)]) < 3$.

Conversely, suppose that $\mu([A]) < 3$. Then, by Theorem D, there exist a rational number $\alpha \in [0,1]$ and a natural number m such that

$$a_m a_{m+1} \ldots = \phi(G(\alpha,1)G(\alpha,2) \ldots).$$

Therefore S is also a super Bernoulli sequence related to (α, α) of type 1. Then Lemma 25 yields that $x = y = \alpha$ and S is not a super Bernoulli sequence related to (α, α) of type i for $i = 2, 3, 4$. ∎

Proof of Proposition 1. By Lemma 5 there exists $m \in \mathbb{N} \cup \{0\}$ such that $a_n \in \{1, 2\}$ for $n \geq m$. Define a word A by

$$A = a_m a_{m+1} \ldots,$$

and denote it by

$$A = 1_{p(0)} 2_{p(1)} 1_{p(2)} 2_{p(3)} \ldots,$$

where $p(i) \in \mathbb{N} \cup \{0\}$ for $i = 0, 1, \ldots$ From Lemma 8 and the assumption on A, there exists $k \in \mathbb{N}$ such that if $i \geq k$ then $p(i)$ is even. Therefore, from the proof of Theorem 3 we obtain the assertion. ∎

4. On super Bernoulli sequences. In this section we prove the existence and some properties of super Bernoulli sequences.

PROPOSITION 2. *For each $x, y \in [0,1]$ with $x \leq y$, there exists a super Bernoulli sequence related to (x,y) of type 1. If x is rational, then there exists such a sequence of type 2; if y is rational, then there exists a sequence of type 3; and if both x and y are rational, then there exists a sequence of type 4.*

P r o o f. If $x = y$, then $H(x)$ is a super Bernoulli sequence related to (x,y) of type 1. Let $x < y$. Let $\{N_i\}$ be an increasing sequence of natural numbers such that $\lim_{i \to \infty} N_i = \infty$. Let $z \in [x, y]$. Then, by Lemma 23, for

each N_i there exist $N_z^i \in \mathbb{N}$ and $\varepsilon_z^i > 0$ such that if $|\beta - z| < \varepsilon_z^i$ then each subword of $G(z)$ of length N_i is contained in every subword of $G(\beta)$ of length greater than N_z^i. Since $[x, y]$ is a compact set, there exist $z_0^i, z_1^i, \ldots, z_{m_i}^i$ such that

$$[x, y] \subset \bigcup_{j=0}^{m_i} U(z_{m_j}^i, \varepsilon_{z_{m_j}^i}^i),$$

where $U(z, \varepsilon) = \{\beta \in [0, 1] \mid |z - \beta| < \varepsilon\}$. We assume that

- $x \in U(z_0^i, \varepsilon_0^i)$,
- $y \in U(z_{m_i}^i, \varepsilon_{m_i}^i)$,
- $U(z_j^i, \varepsilon_{z_j^i}^i) \cap U(z_{j+1}^i, \varepsilon_{z_{j+1}^i}^i) \neq \emptyset$,

if i is even, and

- $y \in U(z_0^i, \varepsilon_0^i)$,
- $x \in U(z_{m_i}^i, \varepsilon_{m_i}^i)$,
- $U(z_j^i, \varepsilon_{z_j^i}^i) \cap U(z_{j+1}^i, \varepsilon_{z_{j+1}^i}^i) \neq \emptyset$,

if i is odd, for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i - 1$.

Choose numbers v_j^i for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i - 1$ such that

$$v_j^i \in U(z_j^i, \varepsilon_{z_j^i}^i) \cap U(z_{j+1}^i, \varepsilon_{z_{j+1}^i}^i).$$

Define a sequence of words P^i, A_j^i, R_j^i, S_j^i for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i$ as follows.

First, we define P^0, A_j^0, R_j^0, S_j^0 for $j = 0, 1, \ldots, m_0$. Put

$$P^0 = G(x, 1)G(x, 2) \ldots G(x, N_0)$$

and consider

$$P = G(x, N_0 + 1)G(x, N_0 + 2) \ldots G(x, N_0 + N_{z_0^0}^0).$$

By Lemma 23, there exists a subword R_0^0 of P such that $|R_0^0| = N_0$ and R_0^0 is also a subword of $G(z_0^0)$. Write

$$P = A_0^0 R_0^0 B_0^0.$$

Since R_0^0 is a subword of $G(z_0^0)$, there exists a word S_0^0 such that $|S_0^0| = N_0$ and $R_0^0 S_0^0$ is a subword of $G(z_0^0)$. Since $v_0^0 \in U(z_0^0, \varepsilon_{z_0^0}^0) \cap U(z_1^0, \varepsilon_{z_1^0}^0)$, by Lemma 23 there exist words A_1^0, R_1^0 such that $|R_1^0| = N_0$, $S_0^0 A_1^0 R_1^0$ is a subword of $G(v_0^0)$ and R_1^0 is a subword of $G(z_1^0)$. Since R_1^0 is a subword of $G(z_1^0)$, there exists a word S_1^0 such that $|S_1^0| = N_0$ and $R_1^0 S_1^0$ is a subword of $G(z_1^0)$.

Define $A_{k+1}^0, R_{k+1}^0, S_{k+1}^0$ ($1 \leq k \leq m_0 - 1$) inductively as follows: Suppose that $|R_k^0| = |S_k^0| = N_0$ and $R_k^0 S_k^0$ is a subword of $G(z_k^0)$ and $S_{k-1}^0 A_k^0 R_k^0$ is a subword of $G(v_{k-1}^0)$. From $v_k^0 \in U(z_k^0, \varepsilon_{z_k^0}^0) \cap U(z_{k+1}^0, \varepsilon_{z_{k+1}^0}^0)$ and Lemma 23

there exist words A_{k+1}^0 and R_{k+1}^0 such that $|R_{k+1}^0| = N_0$, $S_k^0 A_{k+1}^0 R_{k+1}^0$ is a subword of $G(v_k^0)$ and R_{k+1}^0 is a subword of $G(z_{k+1}^0)$. Since R_{k+1}^0 is a subword of $G(z_{k+1}^0)$, there exists a word S_{k+1}^0 such that $|S_{k+1}^0| = N_0$ and $R_{k+1}^0 S_{k+1}^0$ is a subword of $G(z_{k+1}^0)$.

Similarly, we can construct P^i, A_j^i, R_j^i, S_j^i for $i > 0$ and $j = 0, 1, \ldots, m_i$ such that

- $|P^i| = |R_j^i| = |S_j^i| = N_i$,
- $S_{m_i}^{i-1} P_i$ is a subword of $G(x)$ if i is even, and of $G(y)$ if i is odd,
- $S_{k-1}^i A_k^i R_k^i$ is a subword of $G(v_{k-1}^i)$ for $k = 1, \ldots, m_i$, and $R_k^i S_k^i$ is a subword of $G(z_k^i)$,
- $P^i A_0^i R_k^i$ is a subword of $G(x)$ if i is even, and of $G(y)$ if i is odd.

For $i = 0, 1, \ldots$ define

$$U_i = P^i A_0^i R_0^i S_0^i A_1^i R_1^i S_1^i \ldots R_{m_i}^i S_{m_i}^i \quad \text{and} \quad U = U_0 U_1 U_2 \ldots$$

By construction, we have

$$(20) \quad D_{U_i}(N_i)$$

$$\subset D_{G(x)}(N_i) \cup D_{G(y)}(N_i) \cup \bigcup_{j=0}^{m_i} D_{G(z_j^i)}(N_i) \cup \bigcup_{j=0}^{m_i-1} D_{G(v_j^i)}(N_i).$$

Let us show that

$$(21) \qquad\qquad D_U'(N) = \bigcup_{z \in [x,y]} D_{G(z)}(N) \quad \text{for all } N \in \mathbb{N}.$$

By (20), we have immediately

$$D_U'(N) \subset \bigcup_{z \in [x,y]} D_{G(z)}(N).$$

Let $z \in [x, y]$. Then, by Lemma 23 there exist $\varepsilon > 0$ and N' such that if $\beta \in U(z, \varepsilon)$, then each subword of $G(z)$ of length N is contained in every subword of $G(\beta)$ of length N'. Since S_j^i is a subword of $G(z_j^i)$ and of $G(v_j^i)$, and R_{j+1}^i is a subword of $G(z_{j+1}^i)$ and of $G(v_j^i)$ for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, m_i - 1$, by Lemma 21 we have

$$|z_j^i - v_j^i| \leq 2/N_i, \qquad |z_{j+1}^i - v_j^i| \leq 2/N_i.$$

Consequently, there exist $i, j \in \mathbb{N} \cup \{0\}$ with $j \leq m_i$ such that $z_j^i \in U(z, \varepsilon)$ and $N_i > N'$. Therefore, any subword of $G(z)$ of length N is a subword of R_j^i. Hence, we get (21) and U is a super Bernoulli sequence related to (x, y).

Let $x > 0$ be irrational. We can construct a super Bernoulli sequence related to (x, y) of type 2 similarly to the above construction. We outline the proof. Let $x_0, x_1, \ldots \in [0, 1]$ be an increasing sequence $x_i < x$ for $i = 0, 1, \ldots$ and $\lim_{i \to \infty} x_i = x$. Let $N_0, N_1, \ldots$ be an increasing sequence of natural

numbers. For each $k \in \mathbb{N} \cup \{0\}$ define words $P^i(x_k), A^i_j(x_k), R^i_j(x_k), S^i_j(x_k),$ $U^i(x_i)$ for $i \geq 0$ and $j = 0, 1, \ldots, m_i$ as in the previous discussion using x_k instead of x. Then it is not difficult to show that there exist words H_k for $k = 0, 1, \ldots$ such that $S^{2k}_{2k}(x_k) H_k P^{2k+1}(x_{k+1})$ is a subword of $G(y)$. Let

$$U' = U_0(x_0) H_0 U_1(x_1) U_2(x_1) H_1 U_3(x_2) \ldots U_{2k}(x_k) H_k U_{2k+1}(x_{k+1}) \ldots$$

Then U' is a super Bernoulli sequence related to (x, y) of type 2. Other cases are similar. ∎

For a one-sided infinite word $S \in W(0, 1)$, we define $P_S : \mathbb{N} \to \mathbb{N}$ and $P^*_S : \mathbb{N} \to \mathbb{N}$ by

$$P_S(N) = \sharp D_S(N), \quad P^*_S(N) = \sharp D'_S(N).$$

$P_S(\ )$ is called the *complexity* of S and $P^*_S(\ )$ is called the *modified complexity*; the latter was introduced in [8]. In Proposition 3 below we give a representation of super Bernoulli sequences in a specific case. A constructive representation of super Bernoulli sequences related to (x, x) for $x \in [0, 1]$ is mentioned in [8]. But we have no idea of representation of super Bernoulli sequences in general cases.

PROPOSITION 3. *Let $x \in [0, 1]$ be rational and let S be a super Bernoulli sequence related to (x, x) of type* 1. *Then S coincides with $H(x)$ except for a finite number of letters.*

Proof. Let $x = p/q$, where $p \geq 0$, $q > 0$ are integers, $p \leq q$ and $(p, q) = 1$. It is not difficult to see that $H(x)$ is periodic with minimal period q. Therefore, for all N,

$$(22) \qquad\qquad P_{H(x)}(N) \leq q.$$

Let us show that $P_{H(x)}(q) = q$. Suppose that $P_{H(x)}(q) < q$. Then there exist integers i, j such that $1 \leq i < j \leq q$ and $G(x, i + n) = G(x, j + n)$ for $n = 0, \ldots, q-1$. Since $H(x)$ is periodic with period q, we have $G(x, j-i+n) = G(x, n)$ for $n = 1, 2, \ldots$ Therefore,

$$H(x) = G(x, 1) \ldots G(x, j - i) H(x) = (G(x, 1) \ldots G(x, j - i))_\infty.$$

But this contradicts the fact that $H(x)$ is periodic with minimal period q. Therefore, $P_{H(x)}(q) = q$ and from (22) we see that for all $N \geq q$, $P_{H(x)}(N) = q$.

We set $S = s_0 s_1 \ldots$, where $s_0, s_1, \ldots \in \{0, 1\}$. There exists an integer $m > 0$ such that any subword of $s_m s_{m+1} \ldots$ of length $q + 1$ occurs infinitely many times in $s_m s_{m+1} \ldots$ From the hypothesis, $s_m s_{m+1} \ldots s_{m+q} \in D_{G(x)}(q + 1)$. Since $G(x)$ is also periodic with the same period as $H(x)$,

$$(23) \qquad\qquad s_m s_{m+1} \ldots s_{m+q} \in D_{H(x)}(q + 1).$$

Since $H(x)$ is periodic with period q, $s_m = s_{m+q}$. In the same way, we have

$$s_{m+n} = s_{m+n+q} \quad \text{for } n = 0, 1, \ldots$$

Therefore, $s_m s_{m+1} \ldots$ is periodic with period q. From (23) we get $s_m s_{m+1} \ldots$
$\ldots s_{m+q-1} \in D_{H(x)}(q)$. Therefore, S coincides with $H(x)$ except for a finite
number of letters. ∎

It is known that for all irrational x and natural N, $P_{G(x)}(N) = N+1$ ([7]).
It seems difficult to calculate the complexity of super Bernoulli sequences.
But we can calculate their modified complexity.

PROPOSITION 4. *Let $x, y \in [0, 1]$ and $x \leq y$. Let S be a super Bernoulli
sequence related to (x, y) of type 1. Then, for all N,*

$$(24) \qquad P_S^*(N) = \begin{cases} N + 1 + \sum_{i=1}^N F(x, y; i) & \text{if } x < y, \\ N + 1 & \text{if } x = y \text{ is irrational,} \\ \begin{cases} N + 1 & \text{if } N \leq m - 1, \\ m & \text{if } N \geq m \end{cases} & \text{if } x = y \text{ is rational,} \end{cases}$$

where

$$F(x, y; i) = \sharp\{q \in \mathbb{Q} \mid x < q < y, \text{ and the denominator of } q \text{ is } \leq i\},$$

and m is the denominator of x.

Proof. Let $x < y$. Fix $N > 1$. By Lemma 20,

$P_S^*(N) = $ the number of connected components of $\Delta_N \cap (x, y) \times S^1$.

Define $\tau_n' : I \to I \times S^1$ for $n = 0, 1, \ldots$ by

$$\tau_n'(u) = (u, nu) \quad \text{for } u \in I,$$

and set

$$\Delta_N' = I \times S^1 - \bigcup_{n=0}^N \tau_n'(I).$$

Define

$$g(u, v) = (u, -v) \quad \text{for } (u, v) \in I \times S^1.$$

Then g is a bijective mapping $\Delta_N \cap (x, y) \times S^1 \to \Delta_N' \cap (x, y) \times S^1$. Therefore,

$$\Diamond(\Delta_N \cap (x, y) \times S^1) = \Diamond(\Delta_N' \cap (x, y) \times S^1),$$

where $\Diamond(\Theta)$ is the number of connected components of the topological
space Θ.

From geometrical considerations,

$$\Diamond(\Delta_N' \cap (x, y) \times S^1) - \Diamond(\Delta_{N-1}' \cap (x, y) \times S^1)$$

$$= \sharp\left(\tau_N'(I) \cap \bigcup_{n=0}^{N-1} \tau_n'(I) \cap (x, y) \times S^1\right) + 1.$$

We calculate the right side of the above equation. Notice that for each natural k,

$$\tau'_k(I) = \bigcup_{i=0}^{k-1} \{(u,v) \mid v = ku - i, \ i/k \le u < (i+1)/k\},$$

and the solution of

$$\begin{cases} v = Nu - m, \\ v = nu - n', \end{cases}$$

where $m, n, n' \in \mathbb{N} \cup \{0\}$, $0 \le m \le N - 1$, $n < N$ and $0 \le n' \le n - 1$, is given by

$$u = \frac{m - n'}{N - n}, \qquad v = \frac{Nn' - nm}{n - N}.$$

Therefore, the number of points of $\tau'_N(I) \cap \bigcup_{n=0}^{N} \tau'_n(I) \cap (x,y) \times S^1$ is the number of elements of the set

$$(25) \quad \left\{ \frac{m - n'}{N - n} \ \middle| \ \frac{m}{N} \le \frac{m - n'}{N - n} < \frac{m + 1}{N}, \right.$$
$$\left. x < \frac{m - n'}{N - n} < y, \ 0 \le m, n < N, \ 0 \le n' < n \right\}.$$

It is not difficult to show that the set (25) is equal to

$$\{q \in \mathbb{Q} \mid x < q < y, \ \text{and the denominator of } q \text{ is } \le N\},$$

and we have immediately

$$\diamondsuit(\Delta'_1 \cap (x,y) \times S^1) = 2.$$

This yields (24). Other cases are easy. ∎

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