

On the theory of cubic residues and nonresidues

by

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1. Introduction. Let \mathbb{Z} be the set of integers, $\omega = (-1 + \sqrt{-3})/2$ and $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$. For $\pi = a + b\omega \in \mathbb{Z}[\omega]$ the *norm* of π is given by $N\pi = \pi\bar{\pi} = a^2 - ab + b^2$. When $\pi \equiv 2 \pmod{3}$ we say that π is *primary*.

If $\pi \in \mathbb{Z}[\omega]$, $N\pi > 1$ and $\pi \equiv \pm 2 \pmod{3}$ we may write $\pi = \pm\pi_1 \dots \pi_r$, where π_1, \dots, π_r are primary primes. For $\alpha \in \mathbb{Z}[\omega]$ the *cubic Jacobi symbol* $\left(\frac{\alpha}{\pi}\right)_3$ is defined by

$$\left(\frac{\alpha}{\pi}\right)_3 = \left(\frac{\alpha}{\pi_1}\right)_3 \cdots \left(\frac{\alpha}{\pi_r}\right)_3,$$

where $\left(\frac{\alpha}{\pi_t}\right)_3$ is the *cubic residue character* of α modulo π_t which is given by

$$\left(\frac{\alpha}{\pi_t}\right)_3 = \begin{cases} 0 & \text{if } \pi_t \mid \alpha, \\ \omega^i & \text{if } \alpha^{(N\pi_t-1)/3} \equiv \omega^i \pmod{\pi_t}. \end{cases}$$

According to [IR, pp. 135, 313] the cubic Jacobi symbol has the following properties:

(1.1) If $a, b \in \mathbb{Z}$ and $a + b\omega \equiv 2 \pmod{3}$ then $\left(\frac{\omega}{a+b\omega}\right)_3 = \omega^{(a+b+1)/3}$.

(1.2) If $a, b \in \mathbb{Z}$ and $a + b\omega \equiv 2 \pmod{3}$ then $\left(\frac{1-\omega}{a+b\omega}\right)_3 = \omega^{2(a+1)/3}$.

(1.3) If $\pi, \lambda \in \mathbb{Z}[\omega]$ and $\pi, \lambda \equiv \pm 2 \pmod{3}$ then $\left(\frac{\lambda}{\pi}\right)_3 = \left(\frac{\pi}{\lambda}\right)_3$.

The assertion (1.3) is now called the *general cubic reciprocity law*; it was first proved by G. Eisenstein.

Let p be a prime of the form $3n+1$. It is well known that there are unique integers L and $|M|$ such that $4p = L^2 + 27M^2$ with $L \equiv 1 \pmod{3}$. It follows that $\left(\frac{L}{3M}\right)^2 \equiv -3 \pmod{p}$ and therefore $m^{(p-1)/3} \equiv 1$, $(-1 - \frac{L}{3M})/2$ or $(-1 + \frac{L}{3M})/2 \pmod{p}$ for any integer $m \not\equiv 0 \pmod{p}$.

In 1827 Jacobi [J] established the following rational cubic reciprocity law.

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THEOREM 1.1 (Jacobi). *Let q be a prime of the form $3n + 1$, $q \neq p$ and $4q = L^2 + 27M^2$. Then q is a cubic residue modulo p if and only if $(LM' - L'M)/(LM' + L'M)$ is a cubic residue modulo q .*

In 1958, using the period equation of degree 3, E. Lehmer [L1] gave the following criterion for cubic residuacity.

THEOREM 1.2 (E. Lehmer). *If q is an odd prime different from p then q is a cubic residue of p if and only if either $LM \equiv 0 \pmod{q}$ or $L \equiv \mu M \pmod{q}$, where μ satisfies the congruence*

$$\mu^2 \equiv \frac{3u+1}{3u-3} \left(\frac{9}{2u+1} \right)^2 \pmod{q}$$

with $u \not\equiv 0, 1, -\frac{1}{2}, -\frac{1}{3} \pmod{q}$ and $\left(\frac{(3u+1)(3u-3)}{q} \right) = 1$. Here $\left(\frac{\cdot}{q} \right)$ is the Legendre symbol.

In 1975 K. S. Williams [W1] showed how to choose the sign of M so that $m^{(p-1)/3} \equiv \left(-1 - \frac{L}{3M} \right) / 2 \pmod{p}$ when m is a cubic nonresidue modulo p .

Let ε_d be the fundamental unit in the quadratic field $\mathbb{Q}(\sqrt{d})$. In 1970's E. Lehmer [L3], [L4] began to study criteria for ε_d to be a cubic residue modulo p , where p is a prime of the form $3n + 1$ satisfying $\left(\frac{d}{p} \right) = 1$.

Since the work of Euler, Gauss, Jacobi and Eisenstein (see [IR, p. 133]) it is known that cubic congruences are connected with binary quadratic forms. In 1992 B. K. Spearman and K. S. Williams [SW] showed that m is a cubic residue modulo p if and only if p can be represented by one of the third (composition) powers of primitive integral binary quadratic forms of discriminant $-27m^2$, where p is a prime greater than 3 for which $m \not\equiv 0 \pmod{p}$.

Let m be a positive integer, and \mathbb{Z}_m the set of those rational numbers whose denominator is prime to m . Inspired by the above work of Jacobi, Lehmer and Williams we introduce the subsets $C_0(m), C_1(m)$ and $C_2(m)$ of \mathbb{Z}_m for $m \not\equiv 0 \pmod{3}$, where

$$C_i(m) = \left\{ k \mid \left(\frac{k+1+2\omega}{m} \right)_3 = \omega^i, k \in \mathbb{Z}_m \right\} \quad \text{for } i = 0, 1, 2.$$

In Sections 2 and 3 we concentrate on the structure and properties of $C_0(m), C_1(m)$ and $C_2(m)$. Here are some typical results:

(1.4) Let p be a prime of the form $3n + 1$ and hence $4p = L^2 + 27M^2$ for some $L, M \in \mathbb{Z}$ and $L \equiv 1 \pmod{3}$. If q is a prime such that $M \not\equiv 0 \pmod{q}$ and $i \in \{0, 1, 2\}$ then $q^{(p-1)/3} \equiv \left(\left(-1 - \frac{L}{3M} \right) / 2 \right)^i \pmod{p}$ if and only if $L/(3M) \in C_i(q)$.

(1.5) Let p be a prime for which $p \equiv 1 \pmod{3}$, $t^2 \equiv -3 \pmod{p}$ ($t \in \mathbb{Z}$), $k \in \mathbb{Z}_p$, $k^2 + 3 \not\equiv 0 \pmod{p}$ and $i \in \{0, 1, 2\}$. Then $k \in C_i(p)$ if and only if

$$\left(\frac{k-t}{k+t}\right)^{(p-1)/3} \equiv \left(\frac{-1-t}{2}\right)^i \pmod{p}.$$

(1.6) Let p be a prime greater than 3, $k \in \mathbb{Z}_p$ and $k^2 + 3 \not\equiv 0 \pmod{p}$. Then $k \in C_0(p)$ if and only if

$$k \equiv \frac{x^3 - 9x}{3x^2 - 3} \pmod{p} \quad \text{for some integer } x.$$

If q is also a prime of the form $3n+1$ and $4q = L'^2 + 27M'^2$ ($L', M' \in \mathbb{Z}$) with $L' \equiv 1 \pmod{3}$, in view of (1.4) and (1.5) we see that

$$q^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2}\right)^i \pmod{p}$$

if and only if

$$\left(\frac{LM' - L'M}{LM' + L'M}\right)^{(q-1)/3} \equiv \left(\frac{-1 - L'/(3M')}{2}\right)^i \pmod{q}.$$

This generalizes Jacobi's result.

Combining (1.4) with (1.6) gives a simple criterion for cubic residuacity which improves Lehmer's result.

Section 4 is devoted to cubic congruences. Here are two main results:

(1.7) If $p > 3$ is a prime, $a, b \in \mathbb{Z}_p$, $p \nmid ab$ and $s^2 \equiv -3(b^2 - 4a) \pmod{p}$ for some $s \in \mathbb{Z}$, then the congruence $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is solvable if and only if $s/b \in C_0(p)$.

(1.8) Suppose that p is a prime greater than 3 and that N is the number of values of $x^3 + Ax^2 + Bx + C$ modulo p , where $A, B, C \in \mathbb{Z}$ and x runs over all integers. If $A^2 \not\equiv 3B \pmod{p}$ then $N = p - (p - (\frac{-3}{p}))/3$. If $A^2 \equiv 3B \pmod{p}$ then $N = (p+2)/3$ or p according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

In Section 5 the criteria for $s(d) \in C_i(p)$ ($i = 0, 1, 2$) are given in terms of binary quadratic forms, where $p > 3$ is a prime, $d \in \mathbb{Z}$, $p \nmid (d+3)$ and $(s(d))^2 \equiv d \pmod{p}$. In particular, sufficient and necessary conditions for $s(d) \in C_0(p)$ are described in the cases $d = -1, -2, -5, -6, -7$ and -15 . As a consequence we obtain criteria for $\varepsilon_6, \varepsilon_{15}, \varepsilon_{21}$ to be cubic residues modulo p .

In Section 6 we mainly determine $u_{(p-(\frac{-3}{p}))/3}(a, b)$ modulo p , where $p > 3$ is a prime and $\{u_n(a, b)\}$ is the Lucas sequence given by $u_0(a, b) = 0$, $u_1(a, b) = 1$ and $u_{n+1}(a, b) = bu_n(a, b) - au_{n-1}(a, b)$ ($n \geq 1$). In particular, we obtain $F_{(p-(\frac{-3}{p}))/3} \pmod{p}$ and $P_{(p-(\frac{-3}{p}))/3} \pmod{p}$, where $\{F_n\}$ and $\{P_n\}$ denote the Fibonacci sequence and Pell sequence respectively.

To illustrate the connections in the above work I state the following result:

(1.9) Let p be a prime for which $\left(\frac{-3}{p}\right) = \left(\frac{5}{p}\right) = 1$, and q a prime of the form $3n + 1$ satisfying $L^2 + 135M^2 \equiv 0 \pmod{p}$, where L and M are determined by $4q = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$). Then the following statements are equivalent:

- (a) p is a cubic residue modulo q .
- (b) $s(-15) \in C_0(p)$.
- (c) $\varepsilon_5 (= (1 + \sqrt{5})/2)$ is a cubic residue modulo p .
- (d) The congruence $x^3 + 3x + 1 \equiv 0 \pmod{p}$ is solvable.
- (e) $p \mid F_{(p-1)/3}$.
- (f) $p = x^2 + 135y^2$ for some integers x and y .

For later convenience we list the following notations:

$\omega = (-1 + \sqrt{-3})/2$, \mathbb{Z} —the set of integers, \mathbb{Z}^+ —the set of natural numbers, $\mathbb{Z}[\omega]$ —the set $\{a + b\omega \mid a, b \in \mathbb{Z}\}$, $N\pi$ —the norm of π , \mathbb{Q} —the set of rational numbers, \mathbb{Z}_m —the set of those rational numbers whose denominator is prime to m , $[x]$ —the greatest integer not exceeding x , $[x]_p$ —the set $\{k \mid k \equiv x \pmod{p}, k \in \mathbb{Z}_p\}$, (a, b) —the greatest common divisor of a and b , $[a, b]$ —the least common multiple of a and b , $m \mid n$ — m divides n , $m \nmid n$ — m does not divide n , $\left(\frac{a}{p}\right)$ —the Legendre symbol, $\left(\frac{\alpha}{\pi}\right)_3$ —the cubic Jacobi symbol.

2. Basic properties of $C_i(m)$. Let $m \in \mathbb{Z}^+$ and $m \not\equiv 0 \pmod{3}$. For $a, b \in \mathbb{Z}_m$ it is clear that there are unique integers $a_0, b_0 \in \{0, 1, \dots, m-1\}$ satisfying $a \equiv a_0 \pmod{m}$ and $b \equiv b_0 \pmod{m}$. From this we may define

$$(a, m) = (a_0, m) \quad \text{and} \quad \left(\frac{a + b\omega}{m}\right)_3 = \left(\frac{a_0 + b_0\omega}{m}\right)_3 \quad \text{for } m > 1.$$

When $m = 1$ define

$$(a, m) = 1 \quad \text{and} \quad \left(\frac{a + b\omega}{m}\right)_3 = 1.$$

One can easily verify the following facts:

(2.1) If $a, b, c, d \in \mathbb{Z}_m$ then

$$\left(\frac{a + b\omega}{m}\right)_3 \left(\frac{c + d\omega}{m}\right)_3 = \left(\frac{(a + b\omega)(c + d\omega)}{m}\right)_3.$$

(2.2) If $n \in \mathbb{Z}_m$ and $(m, n) = 1$ then $\left(\frac{n}{m}\right)_3 = 1$.

(2.3) If $a, b \in \mathbb{Z}_{m_1 m_2}$ then

$$\left(\frac{a + b\omega}{m_1 m_2}\right)_3 = \left(\frac{a + b\omega}{m_1}\right)_3 \left(\frac{a + b\omega}{m_2}\right)_3.$$

DEFINITION 2.1. Suppose $m \in \mathbb{Z}^+$ and $m \not\equiv 0 \pmod{3}$. For $i = 0, 1, 2$ define

$$C_i(m) = \left\{ k \mid \left(\frac{k+1+2\omega}{m} \right)_3 = \omega^i, k \in \mathbb{Z}_m \right\}.$$

From the above definition it is easy to prove the following results:

(2.4) $C_0(m) \cup C_1(m) \cup C_2(m) = \{k \mid (k^2 + 3, m) = 1, k \in \mathbb{Z}_m\}$.

(2.5) $k \in C_0(m)$ if and only if $-k \in C_0(m)$.

(2.6) $k \in C_1(m)$ if and only if $-k \in C_2(m)$.

EXAMPLE 2.1. Set $C_i^*(m) = C_i(m) \cap \{k \mid -m/2 < k \leq m/2, k \in \mathbb{Z}\}$ for $i = 0, 1, 2$. Then

$$\begin{aligned} C_0^*(5) &= \{0\}, & C_1^*(5) &= \{1, 2\}; \\ C_0^*(7) &= \{0\}, & C_1^*(7) &= \{-1, 3\}; \\ C_0^*(11) &= \{0, 5, -5\}, & C_1^*(11) &= \{-1, -2, 3, -4\}; \\ C_0^*(13) &= \{0, 4, -4\}, & C_1^*(13) &= \{1, -2, -3, -5\}; \\ C_0^*(17) &= \{0, 1, -1, 3, -3\}, & C_1^*(17) &= \{2, 4, -5, -6, 7, -8\}; \\ C_0^*(19) &= \{0, 1, -1, 3, -3\}, & C_1^*(19) &= \{-2, 5, -6, 7, -8, -9\}. \end{aligned}$$

PROPOSITION 2.1. Suppose $m \in \mathbb{Z}^+$ and $m \not\equiv 0 \pmod{3}$. Then $0 \in C_0(m)$.

Proof. Since

$$\left(\frac{1+2\omega}{m} \right)_3 = \left(\frac{1+2\omega}{m} \right)_3^4 = \left(\frac{(1+2\omega)^4}{m} \right)_3 = \left(\frac{9}{m} \right)_3 = 1$$

we see that $0 \in C_0(m)$.

LEMMA 2.1. Suppose that $m \in \mathbb{Z}^+, m \not\equiv 0 \pmod{3}, k_1, k_2 \in \mathbb{Z}_m, ((k_1^2 + 3)(k_2^2 + 3), m) = 1$, and m' is the greatest divisor of m for which $(m', k_1 + k_2) = 1$. Then

$$\left(\frac{k_1+1+2\omega}{m} \right)_3 \left(\frac{k_2+1+2\omega}{m} \right)_3 = \left(\frac{\frac{k_1k_2-3}{k_1+k_2} + 1 + 2\omega}{m'} \right)_3.$$

Proof. Since $(k_1+1+2\omega)(k_2+1+2\omega) = k_1k_2 - 3 + (k_1+k_2)(1+2\omega)$ it is seen that

$$\begin{aligned} &\left(\frac{k_1+1+2\omega}{m} \right)_3 \left(\frac{k_2+1+2\omega}{m} \right)_3 \\ &= \left(\frac{k_1k_2 - 3 + (k_1+k_2)(1+2\omega)}{m} \right)_3 \\ &= \left(\frac{\frac{k_1k_2-3}{k_1+k_2} + 1 + 2\omega}{m'} \right)_3 \left(\frac{k_1k_2 - 3 + (k_1+k_2)(1+2\omega)}{m/m'} \right)_3. \end{aligned}$$

When $m = m'$, we have

$$\left(\frac{k_1 k_2 - 3 + (k_1 + k_2)(1 + 2\omega)}{m/m'} \right)_3 = 1.$$

Now assume that $m > m'$ and that p is a prime divisor of m/m' . It is clear that $k_1 + k_2 \equiv 0 \pmod{p}$ and therefore that

$$\left(\frac{k_1 k_2 - 3 + (k_1 + k_2)(1 + 2\omega)}{p} \right)_3 = \left(\frac{k_1 k_2 - 3}{p} \right)_3 = \left(\frac{-k_1^2 - 3}{p} \right)_3 = 1.$$

Thus,

$$\begin{aligned} & \left(\frac{k_1 k_2 - 3 + (k_1 + k_2)(1 + 2\omega)}{m/m'} \right)_3 \\ &= \prod_{p|m/m'} \left(\frac{k_1 k_2 - 3 + (k_1 + k_2)(1 + 2\omega)}{p} \right)_3 = 1. \end{aligned}$$

This completes the proof.

PROPOSITION 2.2. *Let m be a positive integer not divisible by 3, and $i \in \{0, 1, 2\}$.*

(i) *If $k, k' \in \mathbb{Z}_m$ and $kk' \equiv -3 \pmod{m}$ then $k \in C_i(m)$ if and only if $k' \in C_i(m)$.*

(ii) *If $k_1, k_2 \in C_i(m)$ and $(k_1 + k_2, m) = 1$ then $(3 - k_1 k_2)/(k_1 + k_2) \in C_i(m)$.*

Proof. Since $(k, m) = 1$, by Proposition 2.1 we have

$$\begin{aligned} \left(\frac{k' + 1 + 2\omega}{m} \right)_3 &= \left(\frac{k}{m} \right)_3 \left(\frac{k' + 1 + 2\omega}{m} \right)_3 = \left(\frac{-3 + k + 2k\omega}{m} \right)_3 \\ &= \left(\frac{1 + 2\omega}{m} \right)_3 \left(\frac{k + 1 + 2\omega}{m} \right)_3 = \left(\frac{k + 1 + 2\omega}{m} \right)_3. \end{aligned}$$

So (i) follows.

To prove (ii), we note that

$$\begin{aligned} \left(\frac{\frac{3 - k_1 k_2}{k_1 + k_2} + 1 + 2\omega}{m} \right)_3 &= \left(\frac{\frac{k_1 k_2 - 3}{k_1 + k_2} - 1 - 2\omega}{m} \right)_3 \\ &= \overline{\left(\frac{\frac{k_1 k_2 - 3}{k_1 + k_2} + 1 + 2\omega}{m} \right)_3} = \overline{\left(\frac{\frac{k_1 k_2 - 3}{k_1 + k_2} + 1 + 2\omega}{m} \right)_3} \\ &= \overline{\left(\frac{k_1 + 1 + 2\omega}{m} \right)_3 \left(\frac{k_2 + 1 + 2\omega}{m} \right)_3} \\ &= \overline{\omega^i} \cdot \overline{\omega^i} = \omega^i. \end{aligned} \quad \text{(by Lemma 2.1)}$$

PROPOSITION 2.3. Let $m \in \mathbb{Z}^+$ with $m \not\equiv 0 \pmod{3}$, and $k \in \mathbb{Z}_m$ with $((k^2 - 1)(k^2 + 3), m) = 1$. Then $(k^3 - 9k)/(3k^2 - 3) \in C_0(m)$.

Proof. Clearly,

$$(k+1+2\omega)^3 = (k+1+2\omega)(k^2 - 3 + 2k(1+2\omega)) = k^3 - 9k + (3k^2 - 3)(1+2\omega).$$

Thus,

$$\begin{aligned} \left(\frac{\frac{k^3-9k}{3k^2-3} + 1 + 2\omega}{m}\right)_3 &= \left(\frac{k^3 - 9k + (3k^2 - 3)(1 + 2\omega)}{m}\right)_3 \\ &= \left(\frac{(k + 1 + 2\omega)^3}{m}\right)_3 = 1. \end{aligned}$$

The proof is now complete.

PROPOSITION 2.4. Let $m_1, m_2 \in \mathbb{Z}^+$ be such that $m_1 m_2 \not\equiv 0 \pmod{3}$, $k \in \mathbb{Z}$ and $i \in \{0, 1, 2\}$. If $m_1 \equiv m_2 \pmod{[9, k^2 + 3]}$ then $k \in C_i(m_1)$ if and only if $k \in C_i(m_2)$.

Proof. Write $k + 1 + 2\omega = (-1)^j \omega^s (1 - \omega)^t \pi_1 \dots \pi_r$, where π_1, \dots, π_r are primary primes in $\mathbb{Z}[\omega]$. Since $(k + 1 + 2\omega)(k + 1 + 2\omega^2) = k^2 + 3$ it is seen that $k^2 + 3 \equiv 0 \pmod{\pi_i}$ ($i = 1, \dots, r$). Using Proposition 2.1 and (1.1) we find

$$\left(\frac{\omega(1 - \omega)}{m_1}\right)_3 = \left(\frac{\omega(1 - \omega)}{m_2}\right)_3 = 1 \quad \text{and} \quad \left(\frac{\omega}{m_1}\right)_3 = \left(\frac{\omega}{m_2}\right)_3.$$

Hence,

$$\begin{aligned} \left(\frac{k + 1 + 2\omega}{m_1}\right)_3 &= \left(\frac{(-1)^j \omega^{s-t}}{m_1}\right)_3 \left(\frac{\omega(1 - \omega)}{m_1}\right)_3^t \prod_{i=1}^r \left(\frac{\pi_i}{m_1}\right)_3 \\ &= \left(\frac{\omega}{m_1}\right)_3^{s-t} \prod_{i=1}^r \left(\frac{m_1}{\pi_i}\right)_3 = \left(\frac{\omega}{m_2}\right)_3^{s-t} \prod_{i=1}^r \left(\frac{m_2}{\pi_i}\right)_3 \\ &= \left(\frac{k + 1 + 2\omega}{m_2}\right)_3. \end{aligned}$$

This proves the result.

Now we point out the connections between $C_i(m)$ ($i \in \{0, 1, 2\}$) and cubic residues.

THEOREM 2.1. Let $p \equiv 1 \pmod{3}$ be a prime, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$) with $L \equiv 1 \pmod{3}$, and $m = 2^\alpha 3^\beta m' m'' \in \mathbb{Z}^+$ with $m' = \max\{d \mid d \mid m, (d, 6M) = 1\}$ and $(6, m'') = 1$. Then, for $i = 0, 1, 2$,

$$m^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2}\right)^i \pmod{p}$$

if and only if $L/(3M) \in C_{i'}(m')$, where $i' \in \{0, 1, 2\}$ is determined by

$$i' \equiv \begin{cases} i + \beta M \pmod{3} & \text{if } 3 \mid \alpha \text{ or } 2 \mid M, \\ i + \beta M + (-1)^{r+s} \pmod{3} & \\ & \text{if } \alpha \equiv (-1)^r \pmod{3} \text{ and } L \equiv (-1)^{s-1}M \pmod{4}. \end{cases}$$

Proof. Set $\pi = (L + 3M)/2 + 3M\omega$. Then $\pi \in \mathbb{Z}[\omega]$. Clearly $\pi \equiv 2 \pmod{3}$ and $N\pi = p$. Thus,

$$\begin{aligned} m^{(p-1)/3} &\equiv \left(\frac{-1 - L/(3M)}{2} \right)^i \pmod{p} \\ &\Leftrightarrow m^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2} \right)^i \equiv \omega^i \pmod{\pi} \\ &\Leftrightarrow \left(\frac{2^\alpha m' m''}{\pi} \right)_3 \left(\frac{\omega(1-\omega)}{\pi} \right)_3^{2\beta} = \left(\frac{m}{\pi} \right)_3 = \omega^i \\ &\Leftrightarrow \left(\frac{\pi}{2^\alpha m' m''} \right)_3 = \left(\frac{2^\alpha m' m''}{\pi} \right)_3 = \omega^{i-2\beta M} = \omega^{i+\beta M} \end{aligned}$$

(by (1.1), (1.2) and (1.3)).

Now let us calculate $\left(\frac{\pi}{m''}\right)_3$. Obviously $\left(\frac{\pi}{m''}\right)_3 = 1$ for $m'' = 1$. Assume that $m'' > 1$ and that q is a prime divisor of m'' . It is clear that $q \mid M$ and so that $q \nmid L$. Thus,

$$\left(\frac{\pi}{m''} \right)_3 = \prod_{q \mid m''} \left(\frac{(L + 3M)/2 + 3M\omega}{q} \right)_3 = \prod_{q \mid m''} \left(\frac{L/2}{q} \right)_3 = 1.$$

On the other hand,

$$\left(\frac{(L + 3M)/2 + 3M\omega}{2} \right)_3 = \begin{cases} \left(\frac{(L + 3M)/2}{2} \right)_3 & \text{if } 2 \mid M, \\ \left(\frac{3M\omega}{2} \right)_3 = \left(\frac{\omega}{2} \right)_3 & \\ & \text{if } 2 \nmid M \text{ and } L \equiv M \pmod{4}, \\ \left(\frac{1 + \omega}{2} \right)_3 = \left(\frac{\omega}{2} \right)_3^2 = \omega^{-1} & \\ & \text{if } 2 \nmid M \text{ and } L \equiv -M \pmod{4}. \end{cases}$$

So

$$\left(\frac{\pi}{2^\alpha} \right)_3 = \left(\frac{\pi}{2} \right)_3^\alpha = \begin{cases} 1 & \text{if } 3 \mid \alpha \text{ or } 2 \mid M, \\ (\omega^{(-1)^{s-1}})^\alpha = \omega^{-(-1)^{s+r}} & \\ & \text{if } 3 \mid (\alpha - (-1)^r) \text{ and } 4 \mid (L - (-1)^{s-1}M). \end{cases}$$

Putting the above together we see that

$$m^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2} \right)^i \pmod{p} \Leftrightarrow \left(\frac{\pi}{m'} \right)_3 = \omega^{i+\beta M} \left(\frac{\pi}{2^\alpha} \right)_3^{-1} = \omega^{i'}$$

This concludes the proof.

COROLLARY 2.1. *Let p and q be distinct primes greater than 3, $p \equiv 1 \pmod{3}$ and $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$) with $L \equiv 1 \pmod{3}$. If $q \mid M$ then $q^{(p-1)/3} \equiv 1 \pmod{p}$. If $q \nmid M$ and $i \in \{0, 1, 2\}$ then*

$$q^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2} \right)^i \pmod{p}$$

if and only if $L/(3M) \in C_i(q)$.

REMARK 2.1. According to Theorem 2.1 the value of $m^{(p-1)/3} \pmod{p}$ can be completely determined. The special cases $m = 2, 3$ were treated by E. Lehmer [L2] and K. S. Williams [W1] respectively. When m is a prime for which $m \neq 2, 3, p$, it follows from Corollary 2.1 that $m^{(p-1)/3} \pmod{p}$ depends only on $L/(3M) \pmod{m}$. This important fact was first observed by Jacobi [J], and proved by E. Lehmer [L1] and K. S. Williams [W1].

LEMMA 2.2. *Let $p \neq 3$ be a prime and $k \in \mathbb{Z}_p$.*

(i) *If $p \equiv 1 \pmod{3}$ and so $p = \lambda\bar{\lambda}$ with $\lambda \in \mathbb{Z}[\omega]$ and $\lambda \equiv 2 \pmod{3}$ then*

$$\left(\frac{k + 1 + 2\omega}{p} \right)_3 = \left(\frac{(k^2 + 3)(k - 1 - 2\omega)}{\lambda} \right)_3$$

(ii) *If $p \equiv 2 \pmod{3}$ then*

$$\left(\frac{k + 1 + 2\omega}{p} \right)_3 \equiv (k^2 + 3)^{(p-2)/3} (k + 1 + 2\omega)^{(p+1)/3} \pmod{p}.$$

Proof. Suppose $p = \lambda\bar{\lambda} \equiv 1 \pmod{3}$ with $\lambda \in \mathbb{Z}[\omega]$ and $\lambda \equiv 2 \pmod{3}$. From the properties of the cubic residue character it is seen that

$$\begin{aligned} \left(\frac{k + 1 + 2\omega}{p} \right)_3 &= \left(\frac{k + 1 + 2\omega}{\lambda} \right)_3 \left(\frac{k + 1 + 2\omega}{\bar{\lambda}} \right)_3 \\ &= \left(\frac{k + 1 + 2\omega}{\lambda} \right)_3 \overline{\left(\frac{k - 1 - 2\omega}{\lambda} \right)_3} \\ &= \left(\frac{k + 1 + 2\omega}{\lambda} \right)_3 \left(\frac{k - 1 - 2\omega}{\lambda} \right)_3^2 \\ &= \left(\frac{(k^2 + 3)(k - 1 - 2\omega)}{\lambda} \right)_3 \end{aligned}$$

For (ii), we note that

$$(k + 1 + 2\omega)^p \equiv (k + 1)^p + 2^p \omega^p \equiv k + 1 + 2\omega^2 = k - 1 - 2\omega \pmod{p}$$

and so

$$\begin{aligned} & \left(\frac{k+1+2\omega}{p} \right)_3 \\ & \equiv (k+1+2\omega)^{(p^2-1)/3} = (k+1+2\omega)^{\frac{p(p-2)}{3} + \frac{p-2}{3} + \frac{p+1}{3}} \\ & \equiv (k-1-2\omega)^{(p-2)/3} (k+1+2\omega)^{(p-2)/3} (k+1+2\omega)^{(p+1)/3} \\ & = (k^2+3)^{(p-2)/3} (k+1+2\omega)^{(p+1)/3} \pmod{p}. \end{aligned}$$

Now we are ready to give

THEOREM 2.2. *Let $p \neq 3$ be a prime, $i \in \{0, 1, 2\}$ and $k \in \mathbb{Z}_p$ with $k^2 + 3 \not\equiv 0 \pmod{p}$.*

(i) *If $p \equiv 1 \pmod{3}$ and so $t^2 \equiv -3 \pmod{p}$ for some $t \in \mathbb{Z}_p$ then $k \in C_i(p)$ if and only if*

$$\left(\frac{k-t}{k+t} \right)^{(p-1)/3} \equiv \left(\frac{-1-t}{2} \right)^i \pmod{p}.$$

(ii) *If $p \equiv 2 \pmod{3}$ then $k \in C_i(p)$ if and only if*

$$\left(\frac{k-1-2\omega}{k+1+2\omega} \right)^{(p+1)/3} \equiv \omega^i \pmod{p}.$$

Proof. Suppose $p \equiv 1 \pmod{3}$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$) and $L \equiv 1 \pmod{3}$. Since $(L/(3M))^2 \equiv t^2 \equiv -3 \pmod{p}$ we may choose M so that $L \equiv 3Mt \pmod{p}$. Set $\lambda = (L+3M)/2 + 3M\omega$. Then $\lambda \in \mathbb{Z}[\omega]$ and $\lambda \equiv 2 \pmod{3}$. Clearly $N\lambda = p$ and

$$\omega \equiv \frac{-1 - L/(3M)}{2} \pmod{\lambda}.$$

Thus, by Lemma 2.2 we have

$$\begin{aligned} \left(\frac{k+1+2\omega}{p} \right)_3 &= \left(\frac{(k^2+3)(k-1-2\omega)}{\lambda} \right)_3 \\ &\equiv ((k^2+3)(k-1-2\omega))^{(p-1)/3} \\ &\equiv \left((k^2+3) \left(k + \frac{L}{3M} \right) \right)^{(p-1)/3} \\ &\equiv ((k+t)^2(k-t))^{(p-1)/3} \pmod{\lambda}. \end{aligned}$$

It then follows that

$$\begin{aligned} k \in C_i(p) &\Leftrightarrow \left(\frac{k+1+2\omega}{p} \right)_3 = \omega^i \\ &\Leftrightarrow ((k+t)^2(k-t))^{(p-1)/3} \equiv \left(\frac{-1-t}{2} \right)^i \pmod{\lambda} \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow ((k+t)^2(k-t))^{(p-1)/3} \equiv \left(\frac{-1-t}{2}\right)^i \pmod{p} \\ &\Leftrightarrow \left(\frac{k-t}{k+t}\right)^{(p-1)/3} \equiv \left(\frac{-1-t}{2}\right)^i \pmod{p}. \end{aligned}$$

This proves (i).

Now consider (ii). Note that $(k+1+2\omega)^p \equiv k-1-2\omega \pmod{p}$. Using Lemma 2.2 we see that

$$\begin{aligned} \left(\frac{k+1+2\omega}{p}\right)_3 &\equiv (k-1-2\omega)^{(p-2)/3}(k+1+2\omega)^{(p-2)/3}(k+1+2\omega)^{(p+1)/3} \\ &= \left(\frac{k-1-2\omega}{k+1+2\omega}\right)^{(p-2)/3} (k+1+2\omega)^{\frac{2(p-2)}{3} + \frac{p+1}{3}} \\ &= \left(\frac{k-1-2\omega}{k+1+2\omega}\right)^{(p+1)/3} \frac{k+1+2\omega}{k-1-2\omega} (k+1+2\omega)^{p-1} \\ &\equiv \left(\frac{k-1-2\omega}{k+1+2\omega}\right)^{(p+1)/3} \pmod{p}. \end{aligned}$$

This completes the proof.

From Theorem 2.2 we have the following rational cubic reciprocity law.

COROLLARY 2.2. *Let p and q be distinct primes, $p \equiv 1 \pmod{3}$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$), $L \equiv 1 \pmod{3}$, $q > 3$ and $i \in \{0, 1, 2\}$.*

(i) *If $q \equiv 1 \pmod{3}$ and hence $4q = L'^2 + 27M'^2$ ($L', M' \in \mathbb{Z}$) then*

$$q^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2}\right)^i \pmod{p}$$

if and only if

$$\left(\frac{LM' - L'M}{LM' + L'M}\right)^{(q-1)/3} \equiv \left(\frac{-1 - L'/(3M')}{2}\right)^i \pmod{q}.$$

(ii) *If $q \equiv 2 \pmod{3}$ then*

$$q^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2}\right)^i \pmod{p}$$

if and only if

$$\left(\frac{L - 3M - 6M\omega}{L + 3M + 6M\omega}\right)^{(q+1)/3} \equiv \omega^i \pmod{q}.$$

Proof. If $q \mid M$, it follows from Corollary 2.1 that $q^{(p-1)/3} \equiv 1 \pmod{p}$. If $q \nmid M$, using Corollary 2.1 and Theorem 2.2 we see that

$$q^{(p-1)/3} \equiv \left(\frac{-1 - L/(3M)}{2}\right)^i \pmod{p} \Leftrightarrow \frac{L}{3M} \in C_i(q)$$

$$\Leftrightarrow \begin{cases} \left(\frac{L/(3M) - L'/(3M')}{L/(3M) + L'/(3M')}\right)^{(q-1)/3} \equiv \left(\frac{-1 - L'/(3M')}{2}\right)^i \pmod{q} & \text{if } q \equiv 1 \pmod{3}, \\ \left(\frac{L/(3M) - 1 - 2\omega}{L/(3M) + 1 + 2\omega}\right)^{(q+1)/3} \equiv \omega^i \pmod{q} & \text{if } q \equiv 2 \pmod{3}. \end{cases}$$

This completes the proof.

REMARK 2.2. In the case $i = 0$ Corollary 2.2(i) was first observed by Jacobi [J], and Corollary 2.2(ii) can be deduced from [W1]. Inspired by K. Burde’s rational biquadratic reciprocity, H. von Lienen (see [Li], [Bu]) established the first rational cubic reciprocity law.

THEOREM 2.3. Let $p > 3$ be a prime, $k \in \mathbb{Z}_p$ and $k^2 + 3 \not\equiv 0 \pmod{p}$.

(i) If $p \equiv 1 \pmod{3}$ and so $t^2 \equiv -3 \pmod{p}$ for some $t \in \mathbb{Z}$ then $k \in C_0(p)$ if and only if $(k^2 + 3)(k + t)$ is a cubic residue \pmod{p} .

(ii) If $p \equiv 2 \pmod{3}$ then $k \in C_0(p)$ if and only if

$$\sum_{r \equiv (p+1)/3 \pmod{3}} \binom{(p+1)/3}{r} \left(\frac{k+1}{2}\right)^r \equiv \frac{1}{3} \left(\frac{k+3}{2}\right)^{(p+1)/3} + \frac{1}{3} (2(k^2 + 3))^{-(p-2)/3} \pmod{p}.$$

Proof. If $p \equiv 1 \pmod{3}$, it follows from Theorem 2.2 that

$$k \in C_0(p) \Leftrightarrow \left(\frac{k-t}{k+t}\right)^{(p-1)/3} \equiv 1 \pmod{p}$$

$$\Leftrightarrow ((k^2 + 3)(k + t))^{(p-1)/3} \equiv 1 \pmod{p}$$

$$\Leftrightarrow (k^2 + 3)(k + t) \text{ is a cubic residue } \pmod{p}.$$

This proves (i).

Now consider (ii). For $i = 0, 1, 2$ set

$$A_i = \sum_{r \equiv i \pmod{3}} \binom{(p+1)/3}{r} \left(\frac{k+1}{2}\right)^{(p+1)/3-r}.$$

Then $A_0 + A_1 + A_2 = (1 + (k + 1)/2)^{(p+1)/3}$ and hence

$$\left(\frac{k+1}{2} + \omega\right)^{(p+1)/3} = A_0 + A_1\omega + A_2\omega^2 = A_0 - A_2 + (A_1 - A_2)\omega$$

$$\begin{aligned}
&= 2A_0 + A_1 - \left(\frac{k+3}{2}\right)^{(p+1)/3} \\
&\quad + \left(A_0 + 2A_1 - \left(\frac{k+3}{2}\right)^{(p+1)/3}\right)\omega.
\end{aligned}$$

In view of Lemma 2.2(i) we obtain

$$\begin{aligned}
&2^{-(p+1)/3}(k^2+3)^{-(p-2)/3}\left(\frac{k+1+2\omega}{p}\right)_3 \\
&\equiv 2A_0 + A_1 - \left(\frac{k+3}{2}\right)^{(p+1)/3} + \left(A_0 + 2A_1 - \left(\frac{k+3}{2}\right)^{(p+1)/3}\right)\omega \pmod{p}.
\end{aligned}$$

If $\left(\frac{k+1+2\omega}{p}\right)_3 = 1$, it is clear that

$$\begin{cases} 2A_0 + A_1 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} + 2^{-(p+1)/3}(k^2+3)^{-(p-2)/3} \pmod{p}, \\ A_0 + 2A_1 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} \pmod{p} \end{cases}$$

and therefore that

$$3A_0 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} + (2(k^2+3))^{-(p-2)/3} \pmod{p}.$$

If $\left(\frac{k+1+2\omega}{p}\right)_3 = \omega$, then we have

$$\begin{cases} 2A_0 + A_1 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} \pmod{p}, \\ A_0 + 2A_1 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} + 2^{-(p+1)/3}(k^2+3)^{-(p-2)/3} \pmod{p} \end{cases}$$

and hence

$$(2.7) \quad 3A_0 \equiv \left(\frac{k+3}{2}\right)^{(p+1)/3} - \frac{1}{2}(2(k^2+3))^{-(p-2)/3} \pmod{p}.$$

If $\left(\frac{k+1+2\omega}{p}\right)_3 = \omega^2$, one can similarly prove that (2.7) holds.

Now, by the above, (ii) follows and the proof is complete.

COROLLARY 2.3. *Let m be the product of primes of the form $3n+1$, and hence $t^2 \equiv -3 \pmod{m}$ for some $t \in \mathbb{Z}$. If $x \in \mathbb{Z}$ and $(x(x^3-1), m) = 1$ then $\frac{x^3+1}{x^3-1}t \in C_0(m)$.*

Proof. Write $m = p_1 \dots p_r$, where p_1, \dots, p_r are primes of the form $3n + 1$. For $i = 1, \dots, r$ it is clear that $t^2 \equiv -3 \pmod{p_i}$. Thus,

$$\left(\frac{x^3 + 1}{x^3 - 1}t\right)^2 + 3 \equiv 3\left(1 - \left(\frac{x^3 + 1}{x^3 - 1}\right)^2\right) \equiv -\frac{12x^3}{(x^3 - 1)^2} \not\equiv 0 \pmod{p_i}$$

and so

$$\begin{aligned} \left(\left(\frac{x^3 + 1}{x^3 - 1}t\right)^2 + 3\right)\left(\frac{x^3 + 1}{x^3 - 1}t + t\right) &\equiv -\frac{12x^3}{(x^3 - 1)^2} \cdot \frac{2x^3t}{x^3 - 1} \\ &\equiv \left(\frac{2x^2t}{x^3 - 1}\right)^3 \pmod{p_i}. \end{aligned}$$

Applying Theorem 2.3(i) we find $\frac{x^3+1}{x^3-1}t \in C_0(p_i)$ and hence

$$\left(\frac{\frac{x^3+1}{x^3-1}t + 1 + 2\omega}{m}\right)_3 = \prod_{i=1}^r \left(\frac{\frac{x^3+1}{x^3-1}t + 1 + 2\omega}{p_i}\right)_3 = 1.$$

This is the result.

3. The structure of $C'_i(p)$. In this section we introduce the sets $C'_0(p)$, $C'_1(p)$ and $C'_2(p)$, and study their group structure. As an application we confirm a conjecture due to K. S. Williams [W1].

DEFINITION 3.1. Let $p \neq 3$ be a prime, $k \in \mathbb{Z}_p$, $[k]_p = \{x \mid x \equiv k \pmod{p}, x \in \mathbb{Z}_p\}$ and $[\infty]_p = \{n/m \mid m, n \in \mathbb{Z}, p \mid m, p \nmid n\}$. Define

$$\begin{aligned} C'_0(p) &= \{[k]_p \mid k \in C_0(p)\} \cup \{[\infty]_p\}, \\ C'_1(p) &= \{[k]_p \mid k \in C_1(p)\} \quad \text{and} \quad C'_2(p) = \{[k]_p \mid k \in C_2(p)\}. \end{aligned}$$

As an example, taking $p = 5$ we have $C'_0(5) = \{[0]_5, [\infty]_5\}$, $C'_1(5) = \{[1]_5, [2]_5\}$ and $C'_2(5) = \{[-1]_5, [-2]_5\}$.

Let p be a prime greater than 3,

$$D_p = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } p \equiv 1 \pmod{3}, \\ \mathbb{Z}[\omega]/p\mathbb{Z}[\omega] & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

be the residue class ring modulo p , and U_p the multiplicative group of D_p . It is well known that U_p is a cyclic group of order $p^{(3 - (\frac{-3}{p})) / 2} - 1$. Denote the unique subgroup of order $p - (\frac{-3}{p})$ of U_p by G_p . Then G_p is also a cyclic group. So

$$S_p = \begin{cases} \{g \mid g^{p-1} \equiv 1 \pmod{p}, g^n \not\equiv 1 \pmod{p} \ (n = 1, \dots, p-2), g \in \mathbb{Z}\} & \text{if } p \equiv 1 \pmod{3}, \\ \{g \mid g^{p+1} \equiv 1 \pmod{p}, g^n \not\equiv 1 \pmod{p} \ (n = 1, \dots, p), g \in \mathbb{Z}[\omega]\} & \text{if } p \equiv 2 \pmod{3} \end{cases} \neq \emptyset.$$

We are now ready to give

THEOREM 3.1. *Let p be a prime greater than 3 and $g \in S_p$. For $i = 0, 1, 2$ we have*

$$(i) \quad |C'_i(p)| = \frac{p - \left(\frac{-3}{p}\right)}{3}.$$

$$(ii) \quad C'_i(p) = \left\{ \left[\left(\frac{-3}{p}\right) (1 + 2g^{(p - \left(\frac{-3}{p}\right))/3}) \frac{g^{3r+i} + 1}{g^{3r+i} - 1} \right]_p \mid r = 0, 1, \dots, (p - \left(\frac{-3}{p}\right))/3 - 1 \right\}.$$

Proof. Suppose $k \in \mathbb{Z}_p$ with $k^2 + 3 \not\equiv 0 \pmod{p}$. If $p \equiv 1 \pmod{3}$, it is clear that $(-1 - 2g^{(p-1)/3})^2 \equiv -3 \pmod{p}$. For $i \in \{0, 1, 2\}$ it follows from Theorem 2.2 that

$$\begin{aligned} k \in C_i(p) &\Leftrightarrow \left(\frac{k + 1 + 2g^{(p-1)/3}}{k - 1 - 2g^{(p-1)/3}} \right)^{(p-1)/3} \equiv g^{i(p-1)/3} \pmod{p} \\ &\Leftrightarrow \frac{k + 1 + 2g^{(p-1)/3}}{k - 1 - 2g^{(p-1)/3}} \equiv g^{3r+i} \pmod{p} \\ &\quad \text{for some } r \in \{0, 1, \dots, (p-1)/3 - 1\} \\ &\Leftrightarrow k \equiv (1 + 2g^{(p-1)/3}) \frac{g^{3r+i} + 1}{g^{3r+i} - 1} \pmod{p} \\ &\quad \text{for some } r \in \{0, 1, \dots, (p-4)/3\}. \end{aligned}$$

If $p \equiv 2 \pmod{3}$, it is clear that $g^{(p+1)/3} \equiv \omega$ or $\omega^2 \pmod{p}$. For $i \in \{0, 1, 2\}$ it follows from Theorem 2.2 that

$$\begin{aligned} k \in C_i(p) &\Leftrightarrow \left(\frac{k - 1 - 2g^{(p+1)/3}}{k + 1 + 2g^{(p+1)/3}} \right)^{(p+1)/3} \equiv g^{i(p+1)/3} \pmod{p} \\ &\Leftrightarrow \frac{k - 1 - 2g^{(p+1)/3}}{k + 1 + 2g^{(p+1)/3}} \equiv g^{3r+i} \pmod{p} \\ &\quad \text{for some } r \in \{0, 1, \dots, (p+1)/3 - 1\} \\ &\Leftrightarrow k \equiv -(1 + 2g^{(p+1)/3}) \frac{g^{3r+i} + 1}{g^{3r+i} - 1} \pmod{p} \\ &\quad \text{for some } r \in \{0, 1, \dots, (p-2)/3\}. \end{aligned}$$

To conclude the proof, we note that

$$\left[\left(\frac{-3}{p}\right) (1 + 2g^{(p - \left(\frac{-3}{p}\right))/3}) \frac{g^{3 \cdot 0 + 0} + 1}{g^{3 \cdot 0 + 0} - 1} \right]_p = [\infty]_p$$

and that

$$\frac{g^{3r_1+i} + 1}{g^{3r_1+i} - 1} = 1 + \frac{2}{g^{3r_1+i} - 1} \not\equiv 1 + \frac{2}{g^{3r_2+i} - 1} = \frac{g^{3r_2+i} + 1}{g^{3r_2+i} - 1} \pmod{p}$$

provided $r_1 \not\equiv r_2 \pmod{(p - \left(\frac{-3}{p}\right))/3}$.

COROLLARY 3.1. *Let $p > 3$ be a prime, and R_p a complete residue system modulo p . Then*

$$\sum_{k \in C_1(p) \cap R_p} k \equiv -\frac{1}{3} \pmod{p}.$$

Proof. Let $g \in S_p$ and $m = (p - (\frac{-3}{p}))/3$. It follows from Theorem 3.1 that

$$\begin{aligned} \sum_{k \in C_1(p) \cap R_p} k &\equiv \left(\frac{-3}{p}\right) (1 + 2g^m) \sum_{r=0}^{m-1} \frac{g^{3r+1} + 1}{g^{3r+1} - 1} \\ &= \left(\frac{-3}{p}\right) (1 + 2g^m) \left(m + \sum_{r=0}^{m-1} \frac{2}{g^{3r+1} - 1}\right) \pmod{p}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{r=0}^{m-1} \frac{1}{g^{3r+1} - 1} &= \sum_{r=0}^{m-1} \frac{1}{(g^{3r+1})^m - 1} \sum_{s=0}^{m-1} (g^{3r+1})^s \\ &\equiv \sum_{r=0}^{m-1} \frac{1}{g^m - 1} \sum_{s=0}^{m-1} g^s \cdot g^{3sr} = \frac{1}{g^m - 1} \sum_{s=0}^{m-1} g^s \sum_{r=0}^{m-1} g^{3sr} \\ &= \frac{1}{g^m - 1} \left(m + \sum_{s=1}^{m-1} g^s \frac{1 - g^{3sm}}{1 - g^{3s}}\right) \equiv \frac{m}{g^m - 1} \pmod{p}, \end{aligned}$$

we find

$$\begin{aligned} \sum_{k \in C_1(p) \cap R_p} k &\equiv \left(\frac{-3}{p}\right) (1 + 2g^m) \left(m + \frac{2m}{g^m - 1}\right) \\ &\equiv \left(\frac{-3}{p}\right) m (g^m - g^{2m}) \frac{-g^{2m}}{g^m - 1} \end{aligned}$$

(Note that $1 + g^m + g^{2m} = (g^{3m} - 1)/(g^m - 1) \equiv 0 \pmod{p}$.)

$$= \left(\frac{-3}{p}\right) m g^{3m} \equiv -\frac{1}{3} \pmod{p}.$$

We are done.

REMARK 3.1. Corollary 3.1 is equivalent to a result conjectured by K. S. Williams [W1].

COROLLARY 3.2. *Let $p > 3$ be a prime, and R_p a complete residue system modulo p . Then*

$$\begin{aligned} &\left| \left\{ k \mid k \in C_1(p) \cap R_p, \left(\frac{k^2 + 3}{p}\right) = 1 \right\} \right| \\ &= \left| \left\{ k \mid k \in C_1(p) \cap R_p, \left(\frac{k^2 + 3}{p}\right) = -1 \right\} \right| = \frac{p - (\frac{-3}{p})}{6}. \end{aligned}$$

Proof. Let $g \in S_p$. In view of Theorem 3.1 we can write

$$C_1(p) \cap R_p = \{k_r \mid r = 0, 1, \dots, (p - \left(\frac{-3}{p}\right))/3 - 1\},$$

where

$$k_r \equiv \left(\frac{-3}{p}\right) (1 + 2g^{(p - \left(\frac{-3}{p}\right))/3}) \frac{g^{3r+1} + 1}{g^{3r+1} - 1} \pmod{p}.$$

From this it follows that

$$\begin{aligned} k_r^2 &\equiv (1 + 4g^{(p - \left(\frac{-3}{p}\right))/3} + 4g^{(2(p - \left(\frac{-3}{p}\right))/3)}) \left(1 + \frac{2}{g^{3r+1} - 1}\right)^2 \\ &\equiv -3 \left(1 + \frac{2}{g^{3r+1} - 1} \left(2 + \frac{2}{g^{3r+1} - 1}\right)\right) = -3 - \frac{3 \cdot 4 \cdot g^{3r+1}}{(g^{3r+1} - 1)^2} \pmod{p} \end{aligned}$$

and so

$$\begin{aligned} \left(\frac{k_r^2 + 3}{p}\right) &\equiv (k_r^2 + 3)^{(p-1)/2} \equiv (-3 \cdot 4)^{(p-1)/2} \cdot g^{\frac{p-1}{2}(3r+1)} \cdot \frac{g^{3r+1} - 1}{(g^{3r+1} - 1)^p} \\ &\equiv \left(\frac{-3}{p}\right) g^{\frac{p - \left(\frac{-3}{p}\right)}{2}(3r+1)} \cdot g^{\frac{\left(\frac{-3}{p}\right) - 1}{2}(3r+1)} \cdot \frac{g^{3r+1} - 1}{g^{(3r+1)p} - 1} \\ &\equiv \left(\frac{-3}{p}\right) (-1)^{3r+1} g^{\frac{\left(\frac{-3}{p}\right) - 1}{2}(3r+1)} \frac{g^{3r+1} - 1}{g^{\left(\frac{-3}{p}\right)(3r+1)} - 1} \\ &= (-1)^{r+1} \pmod{p}. \end{aligned}$$

Thus,

$$\left(\frac{k_{2n}^2 + 3}{p}\right) = -\left(\frac{k_{2n+1}^2 + 3}{p}\right) = -1 \quad \text{for } n = 0, 1, \dots, (p - \left(\frac{-3}{p}\right))/6 - 1.$$

This proves the corollary.

THEOREM 3.2. *Let p be a prime greater than 3. For $[k]_p, [k']_p \in C'_0(p) \cup C'_1(p) \cup C'_2(p)$ define*

$$[k]_p * [k']_p = \left[\frac{kk' - 3}{k + k'}\right]_p \quad ([k]_p * [\infty]_p = [\infty]_p * [k]_p = [k]_p).$$

Then $C'_0(p) \cup C'_1(p) \cup C'_2(p)$ forms a cyclic group of order $p - \left(\frac{-3}{p}\right)$, and $C'_0(p)$ is a subgroup of order $(p - \left(\frac{-3}{p}\right))/3$. Moreover, $C'_0(p)$, $C'_1(p)$ and $C'_2(p)$ are the three distinct cosets of $C'_0(p)$.

Proof. Suppose $g \in S_p$. From Theorem 3.1 we know that

$$C'_0(p) \cup C'_1(p) \cup C'_2(p) = \left\{ [k_r]_p \mid r = 0, 1, \dots, p - \left(\frac{-3}{p}\right) - 1 \right\},$$

where

$$[k_r]_p = \left[\left(\frac{-3}{p}\right) (1 + 2g^{(p - \left(\frac{-3}{p}\right))/3}) \frac{g^r + 1}{g^r - 1} \right]_p.$$

Since

$$\begin{aligned} \left[\frac{k_i k_j - 3}{k_i + k_j} \right]_p &= \left[\frac{\left(\left(\frac{-3}{p} \right) (1 + 2g^{(p - (\frac{-3}{p})) / 3}) \right)^2 \cdot \frac{g^i + 1}{g^i - 1} \cdot \frac{g^j + 1}{g^j - 1} - 3}{\left(\frac{-3}{p} \right) (1 + 2g^{(p - (\frac{-3}{p})) / 3}) \left(\frac{g^i + 1}{g^i - 1} + \frac{g^j + 1}{g^j - 1} \right)} \right]_p \\ &= \left[\left(\frac{-3}{p} \right) (1 + 2g^{(p - (\frac{-3}{p})) / 3}) \frac{(g^i + 1)(g^j + 1) + (g^i - 1)(g^j - 1)}{(g^i + 1)(g^j - 1) + (g^i - 1)(g^j + 1)} \right]_p \\ &= \left[\left(\frac{-3}{p} \right) (1 + 2g^{(p - (\frac{-3}{p})) / 3}) \frac{g^{i+j} + 1}{g^{i+j} - 1} \right]_p, \end{aligned}$$

we see that

$$[k_i]_p * [k_j]_p = \left[\frac{k_i k_j - 3}{k_i + k_j} \right]_p = [k_{\langle i+j \rangle}]_p,$$

where $\langle x \rangle$ denotes the least nonnegative residue of x modulo $p - (\frac{-3}{p})$.

By the above, $C'_0(p) \cup C'_1(p) \cup C'_2(p)$ is a cyclic group generated by $[k_1]_p$. Applying Theorem 3.1 we see that $C'_0(p)$ is a cyclic group generated by $[k_3]_p$, and that $C'_0(p)$, $C'_1(p)$ and $C'_2(p)$ are the three cosets of $C'_0(p)$. The proof is now complete.

COROLLARY 3.3. *Let p be a prime greater than 3. Then*

$$C'_0(p) = \left\{ \left[\frac{x^3 - 9x}{3x^2 - 3} \right]_p \mid x \in \{0, 1, \dots, p - 1\}, x^2 \not\equiv -3 \pmod{p} \right\}.$$

Proof. Clearly

$$\left[\frac{1^3 - 9 \cdot 1}{3 \cdot 1^2 - 3} \right]_p = [\infty]_p \in C'_0(p).$$

Suppose $k \in \mathbb{Z}_p$. It follows from Theorem 3.2 that

$$\begin{aligned} [k]_p \in C'_0(p) &\Leftrightarrow [k]_p = [x]_p * [x]_p * [x]_p \\ &\text{for some } [x]_p \in C'_0(p) \cup C'_1(p) \cup C'_2(p) \\ &\Leftrightarrow [k]_p = \left[\frac{x^2 - 3}{2x} \right]_p * [x]_p = \left[\frac{x^3 - 9x}{3x^2 - 3} \right]_p \text{ for some integer } x \\ &\text{satisfying } x^2 + 3 \not\equiv 0 \pmod{p} \text{ and } x \in \{0, 1, \dots, p - 1\}. \end{aligned}$$

So the result follows.

COROLLARY 3.4. *Let $p > 3$ be a prime, $i \in \{0, 1, 2\}$ and $[k_i]_p \in C'_i(p)$. For $[k]_p \in C'_0(p)$ define*

$$\varphi([k]_p) = \left[\frac{k k_i - 3}{k + k_i} \right]_p \quad (\varphi([\infty]_p) = [k_i]_p).$$

Then φ is a one-to-one correspondence from $C'_0(p)$ to $C'_i(p)$.

Proof. In view of Theorem 3.2,

$$C'_i(p) = [k_i]_p C'_0(p) = \{\varphi([k]_p) \mid [k]_p \in C'_0(p)\}.$$

So the result follows.

REMARK 3.2. Corollaries 3.3 and 3.4 provide a simple method of calculating $C'_0(p)$, $C'_1(p)$ and $C'_2(p)$ for any prime $p > 3$.

4. Cubic congruences. Let p be a prime greater than 3. In this section we consider the general cubic congruence $x^3 + Ax^2 + Bx + C \equiv 0 \pmod{p}$, where $A, B, C \in \mathbb{Z}_p$.

In [St] Stickelberger showed that the number of solutions of $x^3 + Ax^2 + Bx + C \equiv 0 \pmod{p}$ is given by

$$N = \begin{cases} 0 \text{ or } 3 & \text{if } \left(\frac{D}{p}\right) = 1, \\ 1 & \text{if } \left(\frac{D}{p}\right) = -1, \end{cases}$$

where $D = A^2B^2 - 4B^3 - 4A^3C - 27C^2 + 18ABC$.

Since

$$x^3 + Ax^2 + Bx + C = \left(x + \frac{A}{3}\right)^3 - 3 \cdot \frac{A^2 - 3B}{9} \left(x + \frac{A}{3}\right) + \frac{2A^3 - 9AB + 27C}{27},$$

it is enough to discuss the congruence $x^3 - 3ax - ab \equiv 0 \pmod{p}$ ($a, b \in \mathbb{Z}_p$).

LEMMA 4.1. Assume that p is a prime greater than 3, $k \in \mathbb{Z}_p$ and $k^2 + 3 \not\equiv 0 \pmod{p}$. Then $k \in C_0(p)$ if and only if the congruence $x^3 - 9(k^2 + 3)x - 18(k^2 + 3) \equiv 0 \pmod{p}$ is solvable. Moreover, if $k \in C_0(p)$ then the solutions of the above congruence are given by

$$x \equiv \begin{cases} (-3 + kt)u(1 - u) \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ (k - 3 + 2k\omega)u(1 - u) \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where t and u satisfy $t^2 \equiv -3 \pmod{p}$ ($t \in \mathbb{Z}$) and

$$u^3 \equiv \begin{cases} \frac{k - t}{k + t} \pmod{p} \ (u \in \mathbb{Z}) & \text{if } p \equiv 1 \pmod{3}, \\ \frac{k - 1 - 2\omega}{k + 1 + 2\omega} \pmod{p} \ (u \in \mathbb{Z}[\omega]) & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. If $k \equiv 0 \pmod{p}$, then $k \in C_0(p)$. Clearly, the congruence $x^3 - 9(k^2 + 3)x - 18(k^2 + 3) \equiv 0 \pmod{p}$ has the solutions $x \equiv 6, -3, -3 \pmod{p}$. So the result is true when $p \mid k$.

Now assume $k \not\equiv 0 \pmod{p}$. It follows from Corollary 3.3 that

$$\begin{aligned} k \in C_0(p) &\Leftrightarrow k \equiv \frac{s^3 - 9s}{3s^2 - 3} \pmod{p} \text{ for some } s \in \{0, 1, \dots, p - 1\} \\ &\Leftrightarrow s^3 - 3ks^2 - 9s + 3k \equiv 0 \pmod{p} \text{ is solvable.} \end{aligned}$$

Set $x \equiv \frac{3(k-s)}{s} \pmod{p}$. Then

$$\begin{aligned} \frac{9k^2}{s^3}(s^3 - 3ks^2 - 9s + 3k) &= 9k^2 - 9k^2 \cdot \frac{3k}{s} - 9\left(\frac{3k}{s}\right)^2 + \left(\frac{3k}{s}\right)^3 \\ &\equiv 9k^2 - 9k^2(x+3) - 9(x+3)^2 + (x+3)^3 \\ &= x^3 - 9(k^2+3)x - 18(k^2+3) \pmod{p}. \end{aligned}$$

So $k \in C_0(p)$ if and only if $x^3 - 9(k^2+3)x - 18(k^2+3) \equiv 0 \pmod{p}$ is solvable.

Let $k \in C_0(p)$ and $r = t$ or $1+2\omega$ according as $p \equiv 1$ or $2 \pmod{3}$. From Theorem 2.2 we know that

$$\left(\frac{k-r}{k+r}\right)^{(p-\frac{-3}{p})/3} \equiv 1 \pmod{p}.$$

So the congruence

$$u^3 \equiv \frac{k-r}{k+r} \pmod{p}$$

is solvable. Suppose $u^3 \equiv \frac{k-r}{k+r} \pmod{p}$ and $x \equiv (-3+kr)u(1-u) \pmod{p}$. Then

$$\begin{aligned} u^3(1-u)^3 &= u^3(1-3u+3u^2-u^3) \\ &\equiv \frac{k-r}{k+r} \left(\frac{2r}{k+r} - 3u + 3u^2 \right) \pmod{p} \end{aligned}$$

and hence

$$\begin{aligned} x^3 - 9(k^2+3)x &\equiv (-3+kr)^3 u^3(1-u)^3 - 9(k^2+3)(-3+kr)(u-u^2) \\ &\equiv r^3(k+r)^3 \frac{k-r}{k+r} \left(\frac{2r}{k+r} - 3u + 3u^2 \right) \\ &\quad - 9r(k+r)^2(k-r)(u-u^2) \\ &\equiv 18(k^2+3) \pmod{p}. \end{aligned}$$

When $p \equiv 2 \pmod{3}$ it is easily seen that $u^3 \bar{u}^3 \equiv 1 \pmod{p}$ and so that $\bar{u} \equiv u^{-1} \pmod{p}$. Hence,

$$\begin{aligned} \overline{(k-3+2k\omega)u(1-u)} &= \overline{(1+2\omega)(k+1+2\omega)} \bar{u}(1-\bar{u}) \\ &\equiv (-1-2\omega)(k-1-2\omega) \frac{1}{u} \left(1 - \frac{1}{u} \right) \\ &= (1+2\omega)(k-1-2\omega) \frac{u(1-u)}{u^3} \\ &\equiv (k-3+2k\omega)u(1-u) \pmod{p}. \end{aligned}$$

This shows that $(k-3+2k\omega)u(1-u)$ is congruent to an integer modulo p .

By the above, the lemma is proved.

THEOREM 4.1. *Let $p > 3$ be a prime, $a, b, s \in \mathbb{Z}_p$, $ab \not\equiv 0 \pmod{p}$ and $s^2 \equiv -3(b^2 - 4a) \pmod{p}$. Then the congruence $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is solvable if and only if $s/b \in C_0(p)$. Moreover, if $s/b \in C_0(p)$ then the solutions of the above congruence are given by*

$$x \equiv \begin{cases} \frac{1}{6}(st - 3b)u(1 - u) \pmod{p} & \text{if } p \equiv 1 \pmod{3}, \\ \frac{1}{6}(s - 3b + 2s\omega)u(1 - u) \pmod{p} & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where t and u satisfy $t^2 \equiv -3 \pmod{p}$ ($t \in \mathbb{Z}$) and

$$u^3 \equiv \begin{cases} \frac{s - bt}{s + bt} \pmod{p} \quad (u \in \mathbb{Z}) & \text{if } p \equiv 1 \pmod{3}, \\ \frac{s - b(1 + 2\omega)}{s + b(1 + 2\omega)} \pmod{p} \quad (u \in \mathbb{Z}[\omega]) & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Proof. Set $k = s/b$. Then $k^2 + 3 \equiv 12a/b^2 \not\equiv 0 \pmod{p}$. It is clear that

$$\begin{aligned} x^3 - 9(k^2 + 3)x - 18(k^2 + 3) &\equiv x^3 - \frac{108a}{b^2}x - \frac{216a}{b^2} \\ &= \left(\frac{6}{b}\right)^3 \left(\left(\frac{b}{6}x\right)^3 - 3a \cdot \frac{b}{6}x - ab \right) \pmod{p}. \end{aligned}$$

So the result follows from Lemma 4.1.

COROLLARY 4.1. *Let $p > 3$ be a prime and $a, b \in \mathbb{Z}_p$. Then the congruence $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is unsolvable if and only if $-3(b^2 - 4a) \equiv k^2b^2 \pmod{p}$ for some $k \in C_1(p)$.*

Proof. If $ab \equiv 0 \pmod{p}$ then $0^3 - 3a \cdot 0 - ab \equiv 0 \pmod{p}$. If $b^2 - 4a \equiv 0 \pmod{p}$ then $b^3 - 3ab - ab \equiv 0 \pmod{p}$. So $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is solvable when $ab(b^2 - 4a) \equiv 0 \pmod{p}$.

Now assume $ab(b^2 - 4a) \not\equiv 0 \pmod{p}$. Since $-4(-3a)^3 - 27(-ab)^2 = -3(b^2 - 4a) \cdot 9a^2$, using Stickelberger's result we see that $x^3 - 3ax - ab \equiv 0 \pmod{p}$ has one solution if $\left(\frac{-3(b^2 - 4a)}{p}\right) = -1$.

If $\left(\frac{-3(b^2 - 4a)}{p}\right) = 1$, there is an integer k such that $k^2 \equiv -3(b^2 - 4a)/b^2 \pmod{p}$. Since $k^2 + 3 \equiv 12a/b^2 \not\equiv 0 \pmod{p}$ we have $k \in C_0(p) \cup C_1(p) \cup C_2(p)$. Applying Theorem 4.1 we see that $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is solvable if and only if $k \in C_0(p)$. So $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is unsolvable if and only if $-3(b^2 - 4a) \equiv k^2b^2 \pmod{p}$ for some $k \in C_1(p) \cup C_2(p)$.

Since $k \in C_2(p)$ if and only if $-k \in C_1(p)$, by the above the corollary is proved.

REMARK 4.1. If p is a prime greater than 3, $a, b \in \mathbb{Z}_p$ and $\left(\frac{-3(b^2 - 4a)}{p}\right) = -1$, one can easily check that the unique solution of $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is given by

$$x \equiv a^{(p - (\frac{-3}{p})) / 3} v_{(p + 2(\frac{-3}{p})) / 3} \pmod{p},$$

where $\{v_n\}$ is defined by $v_0 = 2, v_1 = b$ and $v_{n+1} = bv_n - av_{n-1}$ ($n \geq 1$).

As applications of Theorem 4.1 we have

THEOREM 4.2. *Let $p > 3$ be a prime, $m, n \in \mathbb{Z}_p$ and $mn \not\equiv 0 \pmod{p}$. Then*

$$\left| \left\{ \left[\frac{x^3}{mx+n} \right]_p \mid x \not\equiv -\frac{n}{m} \pmod{p}, x \in \mathbb{Z}_p \right\} \right| = p - \frac{p - (\frac{-3}{p})}{3}.$$

Proof. Set $b = 3n/m$. Then $b \in \mathbb{Z}_p$ and $b \not\equiv 0 \pmod{p}$. From Corollary 4.1 and Theorem 3.1(i) we see that

$$\begin{aligned} & \left| \left\{ \left[\frac{x^3}{mx+n} \right]_p \mid x \in \mathbb{Z}_p, x \not\equiv -\frac{n}{m} \pmod{p} \right\} \right| \\ &= \left| \left\{ \left[\frac{x^3}{3x+b} \right]_p \mid x \in \mathbb{Z}_p, x \not\equiv -\frac{b}{3} \pmod{p} \right\} \right| \\ &= |\{[a]_p \mid x^3 - 3ax - ab \equiv 0 \pmod{p} \text{ is solvable}\}| \\ &= p - |\{[a]_p \mid x^3 - 3ax - ab \equiv 0 \pmod{p} \text{ is unsolvable}\}| \\ &= p - |\{[a]_p \mid -3(b^2 - 4a) \equiv k^2b^2 \pmod{p} \text{ for some } k \in C_1(p)\}| \\ &= p - \left| \left\{ \left[\frac{(k^2 + 3)b^2}{12} \right]_p \mid k \in C_1(p) \right\} \right| = p - \frac{p - (\frac{-3}{p})}{3}. \end{aligned}$$

We are done.

THEOREM 4.3. *Let $p > 3$ be a prime and $A, B, C \in \mathbb{Z}_p$. Then*

$$\begin{aligned} & |\{[x^3 + Ax^2 + Bx + C]_p \mid x \in \mathbb{Z}_p\}| \\ &= |\{[x^3 + Ax^2 + Bx + C]_p \mid x \in \{0, 1, \dots, p-1\}\}| \\ &= \begin{cases} \frac{p+2}{3} & \text{if } p \equiv 1 \pmod{3} \text{ and } A^2 \equiv 3B \pmod{p}, \\ p & \text{if } p \equiv 2 \pmod{3} \text{ and } A^2 \equiv 3B \pmod{p}, \\ p - \frac{p - (\frac{-3}{p})}{3} & \text{if } A^2 \not\equiv 3B \pmod{p}. \end{cases} \end{aligned}$$

Proof. Since

$$x^3 + Ax^2 + Bx + C = \left(x + \frac{A}{3}\right)^3 - \frac{A^2 - 3B}{3} \left(x + \frac{A}{3}\right) + \frac{2A^3 - 9AB + 27C}{27}$$

we see that

$$\begin{aligned}
& |\{[x^3 + Ax^2 + Bx + C]_p \mid x \in \mathbb{Z}_p\}| \\
&= \left| \left\{ \left[x^3 - \frac{A^2 - 3B}{3}x + \frac{2A^3 - 9AB + 27C}{27} \right]_p \mid x \in \mathbb{Z}_p \right\} \right| \\
&= \left| \left\{ \left[x^3 - \frac{A^2 - 3B}{3}x \right]_p \mid x \in \mathbb{Z}_p \right\} \right| \\
&= \left| \left\{ [t]_p \mid x^3 - \frac{A^2 - 3B}{3}x \equiv t \pmod{p} \text{ is solvable, } t \in \mathbb{Z}_p \right\} \right| \\
&= \begin{cases} 1 + \frac{p-1}{3} & \text{if } p \equiv 1 \pmod{3} \text{ and } A^2 \equiv 3B \pmod{p}, \\ p & \text{if } p \equiv 2 \pmod{3} \text{ and } A^2 \equiv 3B \pmod{p}, \\ \left| \left\{ [b]_p \mid x^3 - 3 \cdot \frac{A^2 - 3B}{9}x - \frac{A^2 - 3B}{9}b \equiv 0 \pmod{p} \text{ is solvable} \right\} \right| & \text{if } A^2 \not\equiv 3B \pmod{p}. \end{cases}
\end{aligned}$$

Now suppose $A^2 \not\equiv 3B \pmod{p}$ and $a = (A^2 - 3B)/9$. By Corollaries 4.1 and 3.2 we get

$$\begin{aligned}
& |\{[b]_p \mid x^3 - 3ax - ab \equiv 0 \pmod{p} \text{ is solvable}\}| \\
&= p - |\{[b]_p \mid x^3 - 3ax - ab \equiv 0 \pmod{p} \text{ is unsolvable}\}| \\
&= p - |\{[b]_p \mid -3(b^2 - 4a) \equiv k^2b^2 \pmod{p} \text{ for some } k \in C_1(p)\}| \\
&= p - \left| \left\{ [b]_p \mid b^2 \equiv \frac{12a}{k^2 + 3} \pmod{p} \text{ for some } k \in C_1(p) \right\} \right| \\
&= p - 2 \left| \left\{ \left[\frac{12a}{k^2 + 3} \right]_p \mid \left(\frac{k^2 + 3}{p} \right) = \left(\frac{12a}{p} \right), k \in C_1(p) \right\} \right| \\
&= p - 2 \cdot \frac{p - \left(\frac{-3}{p} \right)}{6}.
\end{aligned}$$

Putting the above together yields the result.

5. Connections with binary quadratic forms. Let d be a squarefree integer, and p a prime greater than 3 satisfying $\left(\frac{d}{p}\right) = 1$. In this section we obtain a criterion for $s(d) \in C_i(p)$ ($i \in \{0, 1, 2\}$) in terms of the binary quadratic forms of discriminant $4d$, where $s(d)$ satisfies $(s(d))^2 \equiv d \pmod{p}$.

THEOREM 5.1. *Let p be a prime greater than 3 and $p = ax^2 + 2bxy + cy^2$ with $a, b, c, x, y \in \mathbb{Z}$. If $d = b^2 - ac$, $a = 2^\alpha 3^r a_1$ ($2 \nmid a_1, 3 \nmid a_1$), $d+3 = 2^\beta 3^s d_1$ ($2 \nmid d_1, 3 \nmid d_1$), $(a, d+3) = 1$ and $a(d+3) \not\equiv 0 \pmod{p}$, then*

$$\left(\frac{ax + (b + 1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \begin{cases} \omega^{(1-s)\left(\frac{-3}{ax}\right)^{\frac{y}{3}}} & \text{if } y \equiv 0 \pmod{3}, \\ \omega^{f_1(u)} & \text{if } a \equiv 0 \pmod{3} \text{ and } x \equiv uy \pmod{9}, \\ \omega^{f_2(u)} & \text{if } a(ax + by) \not\equiv 0 \pmod{3} \text{ and } x \equiv uy \pmod{9}, \\ 1 & \text{if } ax + by \equiv 0 \pmod{9}, \\ \omega^{\pm\left(\frac{-3}{a}\right)} & \text{if } ax + by \equiv \pm 3y \pmod{9}, \end{cases}$$

where

$$f_1(u) = \frac{1}{3} \left(\frac{-3}{b}\right) \left((2bu + c) \left(\frac{-3}{2bu + c}\right) - 1 + (r - 1) \left(1 - b \left(\frac{-3}{b}\right)\right) + ac + 2^\alpha a_1 \left(\frac{-3}{2^\alpha a_1}\right) - 4\right),$$

$$f_2(u) = \frac{1}{3} \left(\frac{-3}{au + b}\right) \left(\left(s - \left(\frac{-3}{1-d}\right)\right) \left((au + b) \left(\frac{-3}{au + b}\right) - 1\right) + (1 - d) \left(\frac{-3}{1-d}\right) - 2^\beta d_1 \left(\frac{-3}{2^\beta d_1}\right)\right)$$

and

$$\left(\frac{-3}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{3}, \\ -1 & \text{if } m \equiv -1 \pmod{3}. \end{cases}$$

Proof. For later use we first point out the following facts:

(i) $y \not\equiv 0 \pmod{p}$. Indeed, if $p \mid y$ then $p \mid ax^2$ and so $p \mid x$. Thus, $p = ax^2 + 2bxy + cy^2 \equiv 0 \pmod{p^2}$. This is a contradiction.

(ii) If $\pi = ax + by + y + 2y\omega$ then $(N\pi, a) = (N\pi, d + 3) = 1$. Indeed, clearly $N\pi = \pi\bar{\pi} = (ax + by)^2 + 3y^2 = ap + (d + 3)y^2$. Thus, $(N\pi, a) = ((d + 3)y^2, a) = 1$ and $(N\pi, d + 3) = (ap, d + 3) = 1$.

(iii) If $A + B\omega \in \mathbb{Z}[\omega]$ and $A + B\omega \equiv 2 \pmod{3}$ then $\left(\frac{3}{A+B\omega}\right)_3 = \omega^{-B/3}$. Indeed, since $3 = -\omega^2(1 - \omega)^2$, it follows from (1.1) and (1.2) that

$$\begin{aligned} \left(\frac{3}{A + B\omega}\right)_3 &= \left(\frac{\omega}{A + B\omega}\right)_3^2 \left(\frac{1 - \omega}{A + B\omega}\right)_3^2 \\ &= \omega^{2(A+B+1)/3} \cdot \omega^{4(A+1)/3} = \omega^{-B/3}. \end{aligned}$$

Now let $\pi = ax + (b + 1)y + 2y\omega$. Since $N(1 - \omega) = 3$ and $N\pi = (ax + by)^2 + 3y^2 \not\equiv 0 \pmod{9}$, there are integers $i, k \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ such that $\pi = (-1)^i \omega^j (1 - \omega)^k \pi'$, where $\pi' \in \mathbb{Z}[\omega]$ and $\pi' \equiv 2 \pmod{3}$.

Assume $y = 3^t y_0$ ($3 \nmid y_0$) and $\pi' = A + B\omega$. Then we have

$$\begin{aligned}
 \left(\frac{ax + (b+1)y + 2y\omega}{p}\right)_3 &= \left(\frac{(-1)^i \omega^j (1-\omega)^k \pi'}{p}\right)_3 \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{\omega(1-\omega)}{p}\right)_3^k \left(\frac{\pi'}{p}\right)_3 = \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{1+2\omega}{p}\right)_3^k \left(\frac{p}{\pi'}\right)_3 \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{ap}{\pi'}\right)_3 \left(\frac{a^2}{\pi'}\right)_3 = \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{-(d+3)y^2}{\pi'}\right)_3 \left(\frac{a^2}{\pi'}\right)_3 \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{3^{2r+s+2t}}{\pi'}\right)_3 \left(\frac{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}{\pi'}\right)_3 \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \left(\frac{3}{\pi'}\right)_3^{s-r-t} \left(\frac{\pi'}{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}\right)_3 \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \omega^{-(s-r-t)B/3} \left(\frac{\pi}{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}\right)_3 \left(\frac{(-1)^i \omega^{j-k} (1+2\omega)^k}{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}\right)_3^{-1} \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \omega^{(r+t-s)B/3} \left(\frac{\omega}{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}\right)_3^{k-j} \left(\frac{\pi}{a_1^2 d_1}\right)_3 \left(\frac{\pi}{2}\right)_3^{2\alpha+\beta} \left(\frac{\pi}{y_0}\right)_3^2 \\
 &\hspace{15em} \text{(by Proposition 2.1)} \\
 &= \left(\frac{\omega}{p}\right)_3^{j-k} \omega^{(r+t-s)B/3} \left(\frac{\omega}{2^{2\alpha+\beta} a_1^2 d_1 y_0^2}\right)_3^{2(j-k)} \left(\frac{\pi}{a_1^2 d_1}\right)_3 \\
 &\hspace{15em} \text{(Note that } \left(\frac{\pi}{2}\right)_3 = \left(\frac{\pi}{y_0}\right)_3 = 1.) \\
 &= \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y_0}\right)_3^{j-k} \omega^{(r+t-s)B/3} \left(\frac{\pi}{a_1 d_1^2}\right)_3^{-1}.
 \end{aligned}$$

That is,

$$(5.1) \quad \left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \omega^{(r+t-s)B/3} \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y_0}\right)_3^{j-k}.$$

Let us consider five cases.

CASE 1: $y \equiv 0 \pmod{3}$. Since $p = ax^2 + 2bxy + cy^2 \equiv ax^2 \pmod{3}$ we have $ax \not\equiv 0 \pmod{3}$ and so $r = 0$. Clearly $\pi' = -\left(\frac{-3}{ax}\right)\pi$. Hence $j = k = 0$ and $B = -2\left(\frac{-3}{ax}\right)y$. From (5.1) we see that

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \omega^{-\left(\frac{-3}{ax}\right)\frac{2y}{3}(t-s)} = \omega^{(1-s)\left(\frac{-3}{ax}\right)\frac{y}{3}}.$$

CASE 2: $a \equiv 0 \pmod{3}$. In this case, $y \not\equiv 0 \pmod{3}$. Since $(a, d+3) = 1$ we have $b^2 = ac - 3 + (d+3) \not\equiv 0 \pmod{3}$ and so $ax + by \not\equiv 0 \pmod{3}$.

If $y \not\equiv 0 \pmod{3}$, $x \equiv uy \pmod{9}$ and $ax + by \not\equiv 0 \pmod{3}$, then clearly

$$\pi' = \begin{cases} -\left(\frac{-3}{y}\right)\omega\pi = \left(\frac{-3}{y}\right)(2y + (-ax - by + y)\omega) & \text{if } au + b \equiv 1 \pmod{3}, \\ \left(\frac{-3}{y}\right)\omega^2\pi = \left(\frac{-3}{y}\right)(-ax - by + y - (ax + by + y)\omega) & \text{if } au + b \equiv -1 \pmod{3}. \end{cases}$$

From this and (5.1) it follows that

$$\begin{aligned} & \left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 \\ &= \begin{cases} \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3^2 \omega^{\left(\frac{-3}{y}\right)(-ax-by+y)(r-s)/3} & \text{if } au + b \equiv 1 \pmod{3}, \\ \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3 \omega^{-\left(\frac{-3}{y}\right)(ax+by+y)(r-s)/3} & \text{if } au + b \equiv -1 \pmod{3} \end{cases} \\ &= \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3^{-\left(\frac{-3}{au+b}\right)} \omega^{(au+b-\left(\frac{-3}{au+b}\right))(s-r)/3}. \end{aligned}$$

Observing that

$$\begin{aligned} py &= \left(a\left(\frac{x}{y}\right)^2 + 2b\frac{x}{y} + c\right) \left(3 \cdot \frac{y - \left(\frac{-3}{y}\right)}{3} + \left(\frac{-3}{y}\right)\right)^3 \\ &\equiv (au^2 + 2bu + c) \left(\frac{-3}{y}\right) \pmod{9}, \end{aligned}$$

we get

$$(5.2) \quad \left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \omega^{(au+b-\left(\frac{-3}{au+b}\right))(s-r)/3} \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 (au^2 + 2bu + c)}\right)_3^{-\left(\frac{-3}{au+b}\right)}.$$

Since $a \equiv 0 \pmod{3}$ we must have $d+3 \not\equiv 0 \pmod{3}$ and so $d = b^2 - ac \equiv 1 \pmod{3}$. Hence,

$$\left(\frac{\omega}{2^{2\beta} d_1^2}\right)_3 = \left(\frac{\omega}{d+3}\right)_3^2 = \omega^{2(1-d-3)/3} = \omega^{(d-1)/3+1}$$

and

$$\begin{aligned} \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 (au^2 + 2bu + c)}\right)_3 &= \left(\frac{\omega}{2^\alpha a_1}\right)_3 \left(\frac{\omega}{2^{2\beta} d_1^2}\right)_3 \left(\frac{\omega}{au^2 + 2bu + c}\right)_3 \\ &= \omega^{\frac{1}{3}(1-\left(\frac{-3}{2^\alpha a_1}\right)2^\alpha a_1)} \cdot \omega^{\frac{d-1}{3}+1} \cdot \omega^{\frac{1}{3}(1-\left(\frac{-3}{2bu+c}\right)(au^2+2bu+c))}. \end{aligned}$$

In view of (5.2),

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p} \right)_3 = \omega^{n_1},$$

where

$$\begin{aligned} n_1 &= -r \frac{au + b - \left(\frac{-3}{au+b}\right)}{3} - \left(\frac{-3}{au+b}\right) \left(\frac{1}{3} \left(1 - \left(\frac{-3}{2^\alpha a_1}\right) 2^\alpha a_1 \right) \right. \\ &\quad \left. + \frac{d-1}{3} + 1 + \frac{1}{3} \left(1 - \left(\frac{-3}{2bu+c}\right) (au^2 + 2bu + c) \right) \right) \\ &= -\frac{1}{3} \left(r \left(au + b - \left(\frac{-3}{b}\right) \right) + \left(\frac{-3}{b}\right) \left(d + 3 - \left(\frac{-3}{2^\alpha a_1}\right) 2^\alpha a_1 \right) \right. \\ &\quad \left. + \left(\frac{-3}{b}\right) \left(1 - \left(\frac{-3}{2bu+c}\right) (au^2 + 2bu + c) \right) \right). \end{aligned}$$

Since

$$\begin{aligned} rau - \left(\frac{-3}{b}\right) \left(\frac{-3}{2bu+c}\right) au^2 &= au \left(r - \left(\frac{-3}{bc-u}\right) u \right) \\ &\equiv \begin{cases} 0 \pmod{9} & \text{if } a \equiv 0 \pmod{9} \text{ or } u \equiv 0 \pmod{3}, \\ a(-bc) \left(1 - \left(\frac{-3}{2bc}\right) (-bc) \right) \equiv 0 \pmod{9} \\ & \text{if } a \equiv \pm 3 \pmod{9} \text{ and } u \equiv -bc \pmod{3} \end{cases} \end{aligned}$$

and

$$d + 3 = b^2 - 1 - ac + 4 \equiv 1 - b \left(\frac{-3}{b}\right) - ac + 4 \pmod{9},$$

we see that $n_1 \equiv f_1(u) \pmod{3}$ and so

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p} \right)_3 = \omega^{f_1(u)}.$$

CASE 3: $a(ax + by) \not\equiv 0 \pmod{3}$. In this case, $r = 0$. Suppose $x \equiv uy \pmod{9}$. Then

$$\begin{aligned} 2^\alpha a_1 (au^2 + 2bu + c) &= a^2 u^2 + 2abu + ac \\ &= \left(3 \cdot \frac{au + b - \left(\frac{-3}{au+b}\right)}{3} + \left(\frac{-3}{au+b}\right) \right)^2 - d \\ &\equiv 1 - d - \left((au + b) \left(\frac{-3}{au+b}\right) - 1 \right) \pmod{9}. \end{aligned}$$

From this and (5.2) it follows that

$$\begin{aligned} & \left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p} \right)_3 \\ &= \left(\frac{\omega}{2^{2\beta} d_1^2 (1-d - ((au+b)\left(\frac{-3}{au+b}\right) - 1))} \right)_3^{-\left(\frac{-3}{au+b}\right)} \omega^{s(au+b - (\frac{-3}{au+b}))/3} = \omega^{n_2}, \end{aligned}$$

where

$$\begin{aligned} n_2 &= -\frac{1}{3} \left(\frac{-3}{au+b} \right) \left(1 - \left(\frac{-3}{1-d} \right) 2^{2\beta} d_1^2 \right. \\ &\quad \left. \times \left(1 - d - \left((au+b) \left(\frac{-3}{au+b} \right) - 1 \right) \right) \right) + \frac{s(au+b - (\frac{-3}{au+b}))}{3} \\ &= \frac{1}{3} \left(\frac{-3}{au+b} \right) \left(\left((au+b) \left(\frac{-3}{au+b} \right) - 1 \right) \right. \\ &\quad \left. \times \left(s - \left(\frac{-3}{1-d} \right) \right) + (2^\beta d_1)^2 (1-d) \left(\frac{-3}{1-d} \right) - 1 \right). \end{aligned}$$

Since

$$\begin{aligned} & (2^\beta d_1)^2 (1-d) \left(\frac{-3}{1-d} \right) - 1 \\ &= ((2^\beta d_1)^2 - 1) \left((1-d) \left(\frac{-3}{1-d} \right) - 1 \right) + (1-d) \left(\frac{-3}{1-d} \right) + (2^\beta d_1)^2 - 2 \\ &\equiv (1-d) \left(\frac{-3}{1-d} \right) - 2^\beta d_1 \left(\frac{-3}{2^\beta d_1} \right) \pmod{9}, \end{aligned}$$

we obtain

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p} \right)_3 = \omega^{n_2} = \omega^{f_2(u)}.$$

CASE 4: $ax + by \equiv 0 \pmod{9}$. Since $ax + by \equiv 0 \pmod{3}$ we have $ap = (ax + by)^2 - dy^2 \equiv -dy^2 \pmod{9}$. We claim that $ady \not\equiv 0 \pmod{3}$ and so that $r = s = t = 0$.

If $y \equiv 0 \pmod{3}$ then

$$p = ax^2 + 2bxy + cy^2 = (ax + by)x + (bx + cy)y \equiv 0 \pmod{3}.$$

Thus, $y \not\equiv 0 \pmod{3}$.

If $a \equiv 0 \pmod{3}$ then $dy^2 \equiv -ap \equiv 0 \pmod{3}$. Since $y \not\equiv 0 \pmod{3}$ we have $d \equiv 0 \pmod{3}$ and so $d + 3 \equiv 0 \pmod{3}$. This contradicts the assumption $(a, d + 3) = 1$. Hence, $a \not\equiv 0 \pmod{3}$.

By the above, $ay \not\equiv 0 \pmod{3}$ and $dy^2 \equiv -ap \pmod{9}$. So $d \not\equiv 0 \pmod{3}$.

This proves the assertion.

Now, it is easy to check that

$$\pi' = -\left(\frac{-3}{y} \right) \frac{\pi}{\omega(1-\omega)} = \left(\frac{-3}{y} \right) \left(-y + \frac{ax + by}{3} + \frac{2(ax + by)}{3} \omega \right).$$

So we have

$$r = s = t = 0, \quad j = k = 1 \quad \text{and} \quad B = \left(\frac{-3}{y}\right) \frac{2(ax + by)}{3}.$$

This together with (5.1) gives

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \omega^0 = 1.$$

CASE 5: $ax + by \equiv \pm 3y \pmod{9}$. In this case, $ax + by \equiv 0 \pmod{3}$. By the above claim we have $r = s = t = 0$. It is clear that

$$\pi' = \begin{cases} -\left(\frac{-3}{y}\right) \frac{\pi}{\omega^2(1-\omega)} = \left(\frac{-3}{y}\right) \left(\frac{ax+by}{3} + y + \left(y - \frac{ax+by}{3}\right)\omega\right) \\ \quad \text{if } ax + by \equiv 3y \pmod{9}, \\ -\left(\frac{-3}{y}\right) \frac{\pi}{1-\omega} = -\left(\frac{-3}{y}\right) \left(\frac{2(ax+by)}{3} + \left(\frac{ax+by}{3} + y\right)\omega\right) \\ \quad \text{if } ax + by \equiv -3y \pmod{9}. \end{cases}$$

Thus, by (5.1),

$$\begin{aligned} & \left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 \\ &= \begin{cases} \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3^{2-1} & \text{if } ax + by \equiv 3y \pmod{9}, \\ \left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3^{0-1} & \text{if } ax + by \equiv -3y \pmod{9}. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} 2^\alpha a_1 (2^\beta d_1)^2 p y &= a(d+3)^2 (ax^2 + 2bxy + cy^2)y \\ &= (d+3)^2 ((ax+by)^2 - dy^2)y \equiv (d^2 - 3d)(-d)y^3 \\ &= (-d^3 + 3d^2)y^3 \equiv \left(-\left(\frac{-3}{d}\right) + 3\right) \left(\frac{-3}{y}\right) \pmod{9}, \end{aligned}$$

we obtain

$$\left(\frac{\omega}{2^{\alpha+2\beta} a_1 d_1^2 p y}\right)_3 = \left(\frac{\omega}{3 - \left(\frac{-3}{d}\right)}\right)_3 = \omega^{\frac{1}{3}(1 - (3 - \left(\frac{-3}{d}\right))\left(\frac{-3}{3 - \left(\frac{-3}{d}\right)}\right))} = \omega^{\left(\frac{-3}{d}\right)}$$

and hence

$$\left(\frac{ax + (b+1)y + 2y\omega}{a_1 d_1^2 p}\right)_3 = \omega^{\pm\left(\frac{-3}{d}\right)}.$$

This completes the proof.

From Theorem 5.1 we have

THEOREM 5.2. *Let p be a prime greater than 3, $d \in \{-1, -2, -5, -6, -7, -15\}$, $\left(\frac{d}{p}\right) = 1$ and $(s(d))^2 \equiv d \pmod{p}$. Then $s(d) \in C_0(p)$ if and only if p can be represented by one of the corresponding binary quadratic forms in Table 5.1.*

Table 5.1

d	Binary quadratic forms
-1	$x^2 + 81y^2, 2x^2 + 2xy + 41y^2$
-2	$x^2 + 162y^2, 2x^2 + 81y^2$
-5	$x^2 + 405y^2, 5x^2 + 81y^2, 10x^2 + 10xy + 43y^2, 2x^2 + 2xy + 203y^2$
-6	$x^2 + 54y^2, 2x^2 + 27y^2$
-7	$x^2 + 567y^2, 7x^2 + 81y^2, 23x^2 + 20xy + 29y^2$
-15	$x^2 + 135y^2, 5x^2 + 27y^2$

Proof. If $d = -1$ then $\left(\frac{-1}{p}\right) = 1$ and so $p = x^2 + y^2$ for some $x, y \in \mathbb{Z}$. Setting $a = 1, b = 0$ and $c = 1$ in Theorem 5.1 we get

$$\left(\frac{x + y + 2y\omega}{p}\right)_3 = \begin{cases} \omega^{\left(\frac{-3}{x}\right)\frac{y}{3}} & \text{if } 3 \mid y, \\ \omega^{(u - \left(\frac{-3}{u}\right))/3} & \text{if } 3 \nmid x \text{ and } x \equiv uy \pmod{9}, \\ 1 & \text{if } 9 \mid x, \\ \omega^{\mp 1} & \text{if } x \equiv \pm 3y \pmod{9}. \end{cases}$$

Since $y \not\equiv 0 \pmod{p}$ and $s(-1) \equiv \pm x/y \pmod{p}$ we see that

$$\begin{aligned} s(-1) \in C_0(p) &\Leftrightarrow x/y \in C_0(p) \\ &\Leftrightarrow \left(\frac{x + y + 2y\omega}{p}\right)_3 = \left(\frac{x/y + 1 + 2\omega}{p}\right)_3 = 1 \\ &\Leftrightarrow 9 \mid x, 9 \mid y \text{ or } x \equiv \left(\frac{-3}{xy}\right)y \pmod{9}. \end{aligned}$$

Clearly, $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$ and $9 \mid xy$ if and only if $p = x_1^2 + 81y_1^2$ for some $x_1, y_1 \in \mathbb{Z}$. If $p = x^2 + y^2$ with $x \equiv \left(\frac{-3}{xy}\right)y \pmod{9}$ then $p = 2x_1^2 + 2x_1y_1 + 41y_1^2$ for $x_1 = \frac{1}{9}(4x + 5\left(\frac{-3}{xy}\right)y)$ and $y_1 = \frac{1}{9}(x - \left(\frac{-3}{xy}\right)y)$. Conversely, if $p = 2x_1^2 + 2x_1y_1 + 41y_1^2$ with $x_1, y_1 \in \mathbb{Z}$ then $p = x^2 + y^2$ for $x = x_1 + 5y_1$ and $y = x_1 - 4y_1$. Also, $x \equiv \left(\frac{-3}{xy}\right)y \pmod{9}$.

By the above, $s(-1) \in C_0(p)$ if and only if $p = x^2 + 81y^2$ or $p = 2x^2 + 2xy + 41y^2$ for some $x, y \in \mathbb{Z}$.

If $d = -2$ then $\left(\frac{-2}{p}\right) = 1$ and so $p = x^2 + 2y^2$ for some $x, y \in \mathbb{Z}$. Using the fact that $s(-2) \equiv \pm x/y \pmod{p}$ and Theorem 5.1 we see that

$$\begin{aligned} s(-2) \in C_0(p) &\Leftrightarrow \left(\frac{x + y + 2y\omega}{p}\right)_3 = 1 \Leftrightarrow 9 \mid x \text{ or } 9 \mid y \\ &\Leftrightarrow p = x_1^2 + 162y_1^2 \text{ or } p = 2x_1^2 + 81y_1^2 \quad (x_1, y_1 \in \mathbb{Z}). \end{aligned}$$

If $d = -5$ then $\left(\frac{-5}{p}\right) = 1$ and so $p = x^2 + 5y^2$ or $p = 3x^2 + 2xy + 2y^2$ for some $x, y \in \mathbb{Z}$. Using Theorem 5.1 we see that

$$s(-5) \in C_0(p) \Leftrightarrow \begin{cases} x/y \in C_0(p) \Leftrightarrow 9 \mid x \text{ or } 9 \mid y \\ \quad \text{if } p = x^2 + 5y^2, \\ 1 + 3x/y \in C_0(p) \Leftrightarrow 9 \mid x \text{ or } x \equiv -2y \pmod{9} \\ \quad \text{if } p = 3x^2 + 2xy + 2y^2. \end{cases}$$

This yields the result.

If $d = -6$ then $\left(\frac{-6}{p}\right) = 1$ and so $p = x^2 + 6y^2$ or $p = 2x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. Applying Theorem 5.1 we get

$$s(-6) \in C_0(p) \Leftrightarrow \begin{cases} x/y \in C_0(p) \Leftrightarrow 3 \mid y & \text{if } p = x^2 + 6y^2, \\ 2x/y \in C_0(p) \Leftrightarrow 3 \mid y & \text{if } p = 2x^2 + 3y^2. \end{cases}$$

This gives the result.

If $d = -7$ then $\left(\frac{-7}{p}\right) = 1$ and so $p = x^2 + 7y^2$ for some $x, y \in \mathbb{Z}$. Applying Theorem 5.1 we see that

$$s(-7) \in C_0(p) \Leftrightarrow x/y \in C_0(p) \Leftrightarrow 9 \mid x, 9 \mid y \text{ or } x \equiv 4 \left(\frac{-3}{xy}\right) y \pmod{9}.$$

This yields the desired result.

If $d = -15$ then $\left(\frac{-15}{p}\right) = 1$ and hence $p = x^2 + 15y^2$ or $p = 5x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. In view of Theorem 5.1 we get

$$s(-15) \in C_0(p) \Leftrightarrow \begin{cases} x/y \in C_0(p) \Leftrightarrow 3 \mid y & \text{if } p = x^2 + 15y^2, \\ 5x/y \in C_0(p) \Leftrightarrow 3 \mid y & \text{if } p = 5x^2 + 3y^2. \end{cases}$$

This deduces the result.

Combining the above we prove the theorem.

COROLLARY 5.1. *Let p be a prime greater than 3.*

(i) *If $\left(\frac{-1}{p}\right) = 1$ then $x^3 + 6x + 4 \equiv 0 \pmod{p}$ is solvable if and only if $p = x^2 + 81y^2$ or $p = 2x^2 + 2xy + 41y^2$ for some $x, y \in \mathbb{Z}$.*

(ii) *If $\left(\frac{-2}{p}\right) = 1$ then $x^3 - 9x - 18 \equiv 0 \pmod{p}$ is solvable if and only if $p = x^2 + 162y^2$ or $p = 2x^2 + 81y^2$ for some $x, y \in \mathbb{Z}$.*

(iii) *If $\left(\frac{-6}{p}\right) = 1$ then $x^3 + 3x + 2 \equiv 0 \pmod{p}$ is solvable if and only if $p = x^2 + 54y^2$ or $p = 2x^2 + 27y^2$ for some $x, y \in \mathbb{Z}$.*

(iv) *If $\left(\frac{-15}{p}\right) = 1$ then $x^3 + 3x + 1 \equiv 0 \pmod{p}$ is solvable if and only if $p = x^2 + 135y^2$ or $p = 5x^2 + 27y^2$ for some $x, y \in \mathbb{Z}$.*

Proof. If $\left(\frac{-1}{p}\right) = 1$, then $(s(-1))^2 \equiv -1 \pmod{p}$ for some $s(-1) \in \mathbb{Z}$. Set $a = -2$ and $b = 2$. Then $(6s(-1))^2 \equiv -3(b^2 - 4a) \pmod{p}$. From Theorems 4.1 and 5.2 we see that

$$\begin{aligned}
x^3 + 6x + 4 \equiv 0 \pmod{p} \text{ is solvable} &\Leftrightarrow 6s(-1)/2 \in C_0(p) \\
&\Leftrightarrow s(-1) \in C_0(p) \text{ (by Proposition 2.2(i))} \\
&\Leftrightarrow p = x^2 + 81y^2 \text{ or } p = 2x^2 + 2xy + 41y^2 \text{ } (x, y \in \mathbb{Z}).
\end{aligned}$$

This proves (i).

Similarly, by using Theorems 4.1 and 5.2 one can prove (ii)–(iv).

REMARK 5.1. Kronecker [K] showed that $x^3 + x + 1 \equiv 0 \pmod{p}$ is solvable for prime p satisfying $\left(\frac{-31}{p}\right) = 1$ if and only if $p = x^2 + 31y^2$ for some integers x and y . In 1973, E. Lehmer [L3] proved Corollary 5.1(iv) in the case $p \equiv 1 \pmod{3}$. For recent important papers along this line one may consult [WH] and [SW].

COROLLARY 5.2. *Let p be a prime of the form $3n + 1$, and ε_d denote the fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{d})$.*

(i) *If $d \in \{2, 3, 5\}$ and $\left(\frac{d}{p}\right) = 1$ then ε_d is a cubic residue (mod p) if and only if $p = x^2 + 27dy^2$ for some integers x and y .*

(ii) *If $\left(\frac{6}{p}\right) = 1$ then $\varepsilon_6 (= 5 + 2\sqrt{6})$ is a cubic residue (mod p) if and only if $p = x^2 + 162y^2$ for some integers x and y .*

(iii) *If $\left(\frac{15}{p}\right) = 1$ then $\varepsilon_{15} (= 4 + \sqrt{15})$ is a cubic residue (mod p) if and only if $p = x^2 + 405y^2$ or $p = 10x^2 + 10xy + 43y^2$ for some integers x and y .*

(iv) *If $\left(\frac{21}{p}\right) = 1$ then $\varepsilon_{21} (= \frac{1}{2}(5 + \sqrt{21}))$ is a cubic residue (mod p) if and only if $p = x^2 + 567y^2$ or $p = 7x^2 + 81y^2$ for some integers x and y .*

PROOF. Suppose $t^2 \equiv -3 \pmod{p}$, $\left(\frac{-d}{p}\right) = 1$ and $(s(-d))^2 \equiv -d \pmod{p}$. By Theorem 2.2(i), $s(-d) \in C_0(p)$ if and only if $(s(-d) - t)/(s(-d) + t)$ is a cubic residue (mod p). Observing that

$$\frac{s(-d) - t}{s(-d) + t} = \frac{(s(-d))^2 - 2s(-d)t + t^2}{(s(-d))^2 - t^2} \equiv \frac{d + 3 + 2s(-d)t}{d - 3} \pmod{p}$$

and that $(s(-d)t)^2 \equiv 3d \pmod{p}$ we find that

$$(5.3) \quad s(-d) \in C_0(p) \Leftrightarrow \frac{d + 3 + 2\sqrt{3d}}{d - 3} \text{ is a cubic residue (mod } p).$$

Clearly,

$$\begin{aligned}
\varepsilon_2 &= 1 + \sqrt{2} = \frac{(1 + \sqrt{2})^3}{3 + 2\sqrt{2}}, & \varepsilon_3 &= 2 + \sqrt{3}, \\
\varepsilon_5 &= \frac{1 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^3 \bigg/ \left(\frac{3 + \sqrt{5}}{2}\right), & \varepsilon_6 &= 5 + 2\sqrt{6}, \\
\varepsilon_{15} &= 4 + \sqrt{15}, & \varepsilon_{21} &= \frac{1}{2}(5 + \sqrt{21}).
\end{aligned}$$

Hence, combining Theorem 5.2 with (5.3) in the cases $d = 6, 1, 15, 2, 5, 7$ gives the result.

REMARK 5.2. Corollary 5.2(i) was known by E. Lehmer [L3], and the rest of Corollary 5.2 is new. For a general result on the cubic character of quadratic units one may consult [We].

THEOREM 5.3. *Let p be a prime of the form $3n + 1$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$), and $q(d)$ ($q(d) > 3$) a prime divisor of $L^2 - 9dM^2$ or $-dL^2 + 81M^2$.*

(i) *If $k \in \mathbb{Z}$ then $q(-3k^2)$ is a cubic residue (mod p) if and only if $(k - 1)(k^2 - 1)$ is a cubic residue (mod $q(-3k^2)$).*

(ii) *If $d \in \{-1, -2, -5, -6, -7, -15\}$ then $q(d)$ is a cubic residue (mod p) if and only if $q(d)$ can be represented by one of the corresponding binary quadratic forms in Table 5.1.*

PROOF. Suppose $(s(d))^2 \equiv d \pmod{q(d)}$. We first claim that

$$(5.4) \quad q(d) \text{ is a cubic residue (mod } p) \Leftrightarrow s(d) \in C_0(q(d)).$$

If $q(d) \mid d$ then $q(d) \mid LM$ and $q(d) \mid s(d)$. From Proposition 2.1 and Corollary 2.1 we see that $s(d) \in C_0(q(d))$ and that $q(d)$ is a cubic residue (mod p).

If $q(d) \nmid d$ then $q(d) \nmid LM$. (Otherwise, $4p = L^2 + 27M^2 \equiv 0 \pmod{(q(d))^2}$.) Since

$$\left(\frac{L}{3M}\right)^2 \equiv d \pmod{q(d)} \quad \text{or} \quad \left(\frac{9M}{L}\right)^2 \equiv d \pmod{q(d)}$$

we have

$$s(d) \equiv \pm \frac{L}{3M} \quad \text{or} \quad s(d) \equiv \pm \frac{9M}{L} \pmod{q(d)}.$$

Now, applying Corollary 2.1 and Proposition 2.2(i) we see that

$$q(d) \text{ is a cubic residue (mod } p) \Leftrightarrow \frac{L}{3M} \in C_0(q(d)) \Leftrightarrow s(d) \in C_0(q(d)).$$

This proves the assertion.

Now let us consider (i). Suppose $d = -3k^2$ for some $k \in \mathbb{Z}$. If $k \equiv \pm 1 \pmod{q(d)}$ then $d \equiv -3 \pmod{q(d)}$ and so $4p = L^2 + 27M^2 \equiv 0 \pmod{q(d)}$. This implies $q(d) = p$. So $q(d)$ is a cubic residue (mod p). If $k \not\equiv \pm 1 \pmod{q(d)}$, by (5.3) and (5.4) we see that

$$\begin{aligned} q(d) \text{ is a cubic residue (mod } p) &\Leftrightarrow s(d) \in C_0(q(d)) \\ &\Leftrightarrow \frac{3k^2 + 3 + 2 \cdot 3k}{3k^2 - 3} \left(= \frac{k + 1}{k - 1} \right) \text{ is a cubic residue (mod } q(d)) \\ &\Leftrightarrow (k - 1)(k^2 - 1) \text{ is a cubic residue (mod } q(d)). \end{aligned}$$

This proves (i).

(ii) follows from (5.4) and Theorem 5.2.

REMARK 5.3. When $q(d)$ is a prime divisor of $L^2 + 9dM^2$ Federighi and Roll [FR] conjectured Theorem 5.3(ii) in the cases $d = 6, 15$. Ph. Barkan [Ba] showed how to prove their conjecture about primes $q(d) \equiv 1 \pmod{3}$.

In 1992, using class field theory Spearman and Williams [SW] proved the following important result:

(5.5) Suppose $p > 3$ is a prime and $x^3 + Ax^2 + Bx + C$ ($A, B, C \in \mathbb{Z}$) is irreducible over the rational field \mathbb{Q} . If the discriminant $D = A^2B^2 - 4B^3 - 4A^3C - 27C^2 + 18ABC$ is not a perfect square such that $\left(\frac{D}{p}\right) = 1$, and $H(D)$ is the form class group of classes of primitive, integral binary quadratic forms of discriminant D , then the cubic congruence $x^3 + Ax^2 + Bx + C \equiv 0 \pmod{p}$ is solvable if and only if p can be represented by one of the third (composition) powers of forms in $H(D)$.

From (5.5) we have

LEMMA 5.1. Assume that $p > 3$ is a prime, $m, n \in \mathbb{Z}$, $2 \mid mn$, $p \nmid mn$, $m^3n \neq -2, 4, 64, 108, 250$, $m^3n/2 - 27 \notin \{k^2 \mid k \in \mathbb{Z}\}$ and $\left(\frac{m^3n/2 - 27}{p}\right) = 1$. Then the cubic congruence $x^3 - \frac{mn}{2}x - n \equiv 0 \pmod{p}$ is solvable if and only if p can be represented by one of the third powers of primitive integral binary quadratic forms of discriminant $(m^3n/2 - 27)n^2$.

Proof. Clearly the discriminant of $x^3 - \frac{mn}{2}x - n$ is given by

$$D = -4\left(-\frac{mn}{2}\right)^3 - 27(-n)^2 = \left(\frac{m^3n}{2} - 27\right)n^2.$$

Since D is not a square, by (5.5) it is sufficient to prove that $x^3 - \frac{mn}{2}x - n \neq 0$ for any integer x .

If $t \in \mathbb{Z}$ and $t^3 - \frac{mn}{2}t - n = 0$, then $n = st$ for some $s \in \mathbb{Z}$. Since $n \not\equiv 0 \pmod{p}$ we have $st \neq 0$ and so $t^2 - \frac{mst}{2} - s = 0$. This implies $t \mid 2s$. Write $2s = rt$. Then $t^2 - \frac{mr}{4}t^2 - \frac{rt}{2} = 0$. Namely, $4t - mrt - 2r = 0$. It then follows that $4t = kr$ for some $k \in \mathbb{Z}$. Observing that $r = 2s/t \neq 0$ we find $k(4 - mr) = 8$ and hence $k \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. Since

$$m^3n = m^3 \cdot \frac{r}{2} \cdot \left(\frac{kr}{4}\right)^2 = \frac{(mr)^3}{32}k^2 = \frac{k^2}{32}\left(4 - \frac{8}{k}\right)^3 = \frac{2(k-2)^3}{k}$$

and $k \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$ we get $m^3n \in \{-2, 54, 0, 64, 4, 108, 250\}$. This contradicts the assumption. Thus, $x^3 - \frac{mn}{2}x - n \neq 0$ for any $x \in \mathbb{Z}$. This completes the proof.

Now we can give

THEOREM 5.4. If $p > 3$ is a prime, $d \in \mathbb{Z}$, $d \neq 3$, $d \not\equiv -3 \pmod{p}$, $d \notin \{k^2 \mid k \in \mathbb{Z}\}$, $\left(\frac{d}{p}\right) = 1$, $(s(d))^2 \equiv d \pmod{p}$ and $18(d+3) = m^3n$ with $m, n \in \mathbb{Z}$, then $s(d) \in C_0(p)$ if and only if p can be represented by one of

the third powers of primitive integral binary quadratic forms of discriminant $9dn^2$.

Proof. It follows from Lemma 4.1 that

$$\begin{aligned} s(d) \in C_0(p) & \\ \Leftrightarrow x^3 - 9((s(d))^2 + 3)x - 18((s(d))^2 + 3) &\equiv 0 \pmod{p} \text{ is solvable} \\ \Leftrightarrow x^3 - 9(d + 3)x - 18(d + 3) &\equiv 0 \pmod{p} \text{ is solvable} \\ \Leftrightarrow (my)^3 - \frac{m^3n}{2}my - m^3n &\equiv 0 \pmod{p} \text{ is solvable} \\ \Leftrightarrow y^3 - \frac{mn}{2}y - n &\equiv 0 \pmod{p} \text{ is solvable.} \end{aligned}$$

Since $(m^3n/2 - 27)n^2 = 9dn^2$ and $m^3n = 18(d + 3) \neq -2, 4, 64, 108, 250$, applying Lemma 5.1 we obtain the result.

COROLLARY 5.3. *If p is a prime, $p \equiv 1 \pmod{3}$, $k \in \mathbb{Z}$, $k \neq 0, \pm 1 \pmod{p}$ and $2(k^2 - 1) = m^3n$ ($m, n \in \mathbb{Z}$), then $\frac{k+1}{k-1}$ is a cubic residue \pmod{p} if and only if p can be represented by one of the third powers of primitive integral binary quadratic forms of discriminant $-27k^2n^2$.*

Proof. Suppose $d = -3k^2$ and $t^2 \equiv -3 \pmod{p}$. Clearly, $\left(\frac{d}{p}\right) = 1$, $d \not\equiv -3 \pmod{p}$ and $(-kt)^2 \equiv d \pmod{p}$. By Theorem 2.2(i), $-kt \in C_0(p)$ if and only if $\frac{k+1}{k-1}$ ($= \frac{-kt-t}{-kt+t}$) is a cubic residue \pmod{p} . Also, $18(d + 3) = 18(3 - 3k^2) = (-3m)^3n$ and $9dn^2 = -27k^2n^2$. So the result follows from Theorem 5.4.

COROLLARY 5.4. *Let p be a prime of the form $3n + 1$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$), $d \in \mathbb{Z}$, $d \notin \{k^2 \mid k \in \mathbb{Z}\}$, $d \neq 3$, $d \not\equiv -3 \pmod{p}$ and $18(d + 3) = m^3n$ ($m, n \in \mathbb{Z}$). If $q(d)$ is a prime divisor of $L^2 - 9dM^2$ or $-dL^2 + 81M^2$ satisfying $q(d) \neq 2, 3$ and $q(d) \nmid d$, then $q(d)$ is a cubic residue \pmod{p} if and only if $q(d)$ can be represented by one of the third powers of primitive integral binary quadratic forms of discriminant $9dn^2$.*

Proof. This is immediate from (5.4) and Theorem 5.4.

6. Applications to Lucas series. Let a and b be two real numbers. The Lucas sequences $\{u_n(a, b)\}$ and $\{v_n(a, b)\}$ are defined as follows:

$$(6.1) \quad \begin{aligned} u_0(a, b) &= 0, & u_1(a, b) &= 1, \\ u_{n+1}(a, b) &= bu_n(a, b) - au_{n-1}(a, b) & (n \geq 1); \end{aligned}$$

$$(6.2) \quad \begin{aligned} v_0(a, b) &= 2, & v_1(a, b) &= b, \\ v_{n+1}(a, b) &= bv_n(a, b) - av_{n-1}(a, b) & (n \geq 1). \end{aligned}$$

It is well known that

$$(6.3) \quad u_n(a, b) = \frac{1}{\sqrt{b^2 - 4a}} \left(\left(\frac{b + \sqrt{b^2 - 4a}}{2} \right)^n - \left(\frac{b - \sqrt{b^2 - 4a}}{2} \right)^n \right) \quad (b^2 - 4a \neq 0)$$

and that

$$(6.4) \quad v_n(a, b) = \left(\frac{b + \sqrt{b^2 - 4a}}{2} \right)^n + \left(\frac{b - \sqrt{b^2 - 4a}}{2} \right)^n.$$

Suppose that p is a prime greater than 3. It is the purpose of this section to determine $u_{(p - (\frac{-3}{p})) / 3}(a, b) \pmod{p}$ and $v_{(p - (\frac{-3}{p})) / 3}(a, b) \pmod{p}$.

THEOREM 6.1. *Let $p > 3$ be a prime, $a, b \in \mathbb{Z}_p, p \nmid ab, (\frac{-3(b^2 - 4a)}{p}) = 1$ and $s^2 \equiv -3(b^2 - 4a) \pmod{p}$. Then*

$$u_{(p - (\frac{-3}{p})) / 3}(a, b) \equiv \begin{cases} 0 \pmod{p} & \text{if } s/b \in C_0(p), \\ \pm \frac{3}{s} (-a)^{-[p/3]} \pmod{p} & \text{if } \pm s/b \in C_1(p) \end{cases}$$

and

$$v_{(p - (\frac{-3}{p})) / 3}(a, b) \equiv \begin{cases} 2a^{-[p/3]} \pmod{p} & \text{if } s/b \in C_0(p), \\ -a^{-[p/3]} \pmod{p} & \text{if } s/b \notin C_0(p). \end{cases}$$

Proof. Set $k = -3b/s$. For $n \in \mathbb{Z}^+$ it is clear that

$$\begin{aligned} u_n(a, b) &= \frac{1}{\sqrt{b^2 - 4a}} \left(\left(\frac{b + \sqrt{b^2 - 4a}}{2} \right)^n - \left(\frac{b - \sqrt{b^2 - 4a}}{2} \right)^n \right) \\ &= \frac{2}{2^n \sqrt{b^2 - 4a}} \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} b^{n-2r-1} (\sqrt{b^2 - 4a})^{2r+1} \\ &= \frac{2}{2^n} \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} b^{n-2r-1} (b^2 - 4a)^r \\ &\equiv \frac{2}{2^n} \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} b^{n-2r-1} \left(\frac{s(1+2\omega)}{-3} \right)^{2r+1} \frac{-3}{s(1+2\omega)} \\ &= \frac{-3}{s(1+2\omega)} \left(\left(\frac{b + s(1+2\omega)/(-3)}{2} \right)^n - \left(\frac{b - s(1+2\omega)/(-3)}{2} \right)^n \right) \\ &= \frac{\omega(1-\omega)}{s} \left(-\frac{s}{6} \right)^n ((k+1+2\omega)^n - (k-1-2\omega)^n) \pmod{p}. \end{aligned}$$

Similarly,

$$\begin{aligned} v_n(a, b) &= \frac{2}{2^n} \sum_{r=0}^{[n/2]} \binom{n}{2r} b^{n-2r} (b^2 - 4a)^r \\ &\equiv \frac{2}{2^n} \sum_{r=0}^{[n/2]} \binom{n}{2r} b^{n-2r} \left(\frac{s(1+2\omega)}{-3} \right)^{2r} \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{b + s(1 + 2\omega)/(-3)}{2} \right)^n + \left(\frac{b - s(1 + 2\omega)/(-3)}{2} \right)^n \\
 &= \left(-\frac{s}{6} \right)^n ((k + 1 + 2\omega)^n + (k - 1 - 2\omega)^n) \pmod{p}.
 \end{aligned}$$

If $p \equiv 1 \pmod{3}$, we may write $p = \lambda\bar{\lambda}$ with $\lambda \in \mathbb{Z}[\omega]$ and $\lambda \equiv 2 \pmod{3}$.
 By Lemma 2.2(i),

$$\begin{aligned}
 &(k^2 + 3)^{(p-1)/3} ((k - 1 - 2\omega)^{(p-1)/3} \pm (k + 1 + 2\omega)^{(p-1)/3}) \\
 &\equiv \left(\frac{(k^2 + 3)(k - 1 - 2\omega)}{\lambda} \right)_3 \pm \left(\frac{(k^2 + 3)(k + 1 + 2\omega)}{\lambda} \right)_3 \\
 &= \left(\frac{k + 1 + 2\omega}{p} \right)_3 \pm \left(\frac{k + 1 + 2\omega}{p} \right)_3^{-1} \pmod{\lambda}.
 \end{aligned}$$

Hence, by the above and Fermat's little theorem we get

$$\begin{aligned}
 u_{(p-1)/3}(a, b) &\equiv -\frac{\omega(1 - \omega)}{s} \left(-\frac{s}{6} \right)^{(p-1)/3} (k^2 + 3)^{-(p-1)/3} \\
 &\quad \times \left(\left(\frac{k + 1 + 2\omega}{p} \right)_3 - \left(\frac{k + 1 + 2\omega}{p} \right)_3^{-1} \right) \\
 &\equiv \begin{cases} 0 \pmod{\lambda} & \text{if } k \in C_0(p), \\ \pm \frac{3}{s} (-a)^{(p-1)/3} \pmod{\lambda} & \text{if } \pm k \in C_1(p) \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 v_{(p-1)/3}(a, b) &\equiv \left(-\frac{s}{6} \right)^{(p-1)/3} (k^2 + 3)^{-(p-1)/3} \left(\left(\frac{k + 1 + 2\omega}{p} \right)_3 + \left(\frac{k + 1 + 2\omega}{p} \right)_3^{-1} \right) \\
 &\equiv \begin{cases} 2a^{-(p-1)/3} \pmod{\lambda} & \text{if } k \in C_0(p), \\ -a^{-(p-1)/3} \pmod{\lambda} & \text{if } k \in C_1(p) \cup C_2(p). \end{cases}
 \end{aligned}$$

Since both sides of the above congruences are rational, the congruences are also true when λ is replaced by $p (= N\lambda)$.

If $p \equiv 2 \pmod{3}$, it follows from Lemma 2.2(ii) that

$$\begin{aligned}
 &(k + 1 + 2\omega)^{(p+1)/3} \pm (k - 1 - 2\omega)^{(p+1)/3} \\
 &\equiv (k^2 + 3)^{-(p-2)/3} \left(\left(\frac{k + 1 + 2\omega}{p} \right)_3 \pm \left(\frac{k + 1 + 2\omega}{p} \right)_3^{-1} \right) \pmod{p}.
 \end{aligned}$$

From this and the above it follows that

$$\begin{aligned}
 u_{(p+1)/3}(a, b) &\equiv \frac{\omega(1 - \omega)}{s} \left(-\frac{s}{6} \right)^{(p+1)/3} (k^2 + 3)^{-(p-2)/3} \\
 &\quad \times \left(\left(\frac{k + 1 + 2\omega}{p} \right)_3 - \left(\frac{k + 1 + 2\omega}{p} \right)_3^{-1} \right)
 \end{aligned}$$

$$\equiv \begin{cases} 0 \pmod{p} & \text{if } k \in C_0(p), \\ \pm \frac{3}{s}(-a)^{-(p-2)/3} \pmod{p} & \text{if } \pm k \in C_1(p) \end{cases}$$

and

$$\begin{aligned} v_{(p+1)/3}(a, b) &\equiv \left(-\frac{s}{6}\right)^{(p+1)/3} (k^2 + 3)^{-(p-2)/3} \left(\left(\frac{k+1+2\omega}{p}\right)_3 + \left(\frac{k+1+2\omega}{p}\right)_3^{-1} \right) \\ &\equiv \begin{cases} 2a^{-(p-2)/3} \pmod{p} & \text{if } k \in C_0(p), \\ -a^{-(p-2)/3} \pmod{p} & \text{if } k \in C_1(p) \cup C_2(p). \end{cases} \end{aligned}$$

To complete the proof, we note that

$$k^2 + 3 \equiv \frac{36a}{-3(b^2 - 4a)} \not\equiv 0 \pmod{p}$$

and $k \in C_i(p)$ if and only if $s/b \in C_i(p)$ by Proposition 2.2(i).

COROLLARY 6.1. *Let $p > 3$ be a prime, $k \in \mathbb{Z}_p$ and $k(k^2 + 3) \not\equiv 0 \pmod{p}$. Then*

$$u_{(p-(\frac{-3}{p}))/3}(3k^2 + 9, 6) \equiv \begin{cases} 0 \pmod{p} & \text{if } k \in C_0(p), \\ \frac{1}{2k}(-3k^2 - 9)^{-[p/3]} \pmod{p} & \text{if } k \in C_1(p), \\ -\frac{1}{2k}(-3k^2 - 9)^{-[p/3]} \pmod{p} & \text{if } k \in C_2(p) \end{cases}$$

and

$$v_{(p-(\frac{-3}{p}))/3}(3k^2 + 9, 6) \equiv \begin{cases} 2(3k^2 + 9)^{-[p/3]} \pmod{p} & \text{if } k \in C_0(p), \\ -(3k^2 + 9)^{-[p/3]} \pmod{p} & \text{if } k \notin C_0(p). \end{cases}$$

COROLLARY 6.2. *Let $p > 3$ be a prime, $d \in \mathbb{Z}_p, d \not\equiv -3 \pmod{p}, (\frac{d}{p}) = 1$ and $(s(d))^2 \equiv d \pmod{p}$. Then $s(d) \in C_0(p)$ if and only if $u_{(p-(\frac{-3}{p}))/3}(3d + 9, 6) \equiv 0 \pmod{p}$.*

Proof. Set $k = s(d)$. Then $u_n(3k^2 + 9, 6) \equiv u_n(3d + 9, 6) \pmod{p}$ by (6.1). Hence the result follows from Corollary 6.1.

COROLLARY 6.3. *Let $p > 3$ be a prime, $a, b \in \mathbb{Z}_p, p \nmid ab$ and $(\frac{-3(b^2-4a)}{p}) = 1$. Then the congruence $x^3 - 3ax - ab \equiv 0 \pmod{p}$ is solvable if and only if $u_{(p-(\frac{-3}{p}))/3}(a, b) \equiv 0 \pmod{p}$.*

Proof. Suppose $s^2 \equiv -3(b^2 - 4a) \pmod{p}$. It then follows from Theorem 6.1 that $s/b \in C_0(p)$ if and only if $u_{(p-(\frac{-3}{p}))/3}(a, b) \equiv 0 \pmod{p}$. This together with Theorem 4.1 gives the result.

COROLLARY 6.4. *Let $p > 3$ be a prime, $a, b \in \mathbb{Z}, p \nmid ab$ and $(\frac{-3(b^2-4a)}{p}) = 1$. If $-3(b^2 - 4a)$ is not a square and $x^3 - 3ax - ab$ is irreducible over*

\mathbb{Q} , then $p \mid u_{(p - (\frac{-3}{p})) / 3}(a, b)$ if and only if p can be represented by one of the third powers of primitive integral binary quadratic forms of discriminant $-27a^2(b^2 - 4a)$.

Proof. Since the discriminant of $x^3 - 3ax - ab$ is $-27a^2(b^2 - 4a)$ the result follows from (5.5) and Corollary 6.3.

Let $\{F_n\}$ and $\{L_n\}$ be defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \geq 1)$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \quad (n \geq 1).$$

It is well known that $\{F_n\}$ is the Fibonacci sequence and that $\{L_n\}$ is the Lucas sequence.

From Theorems 5.1 and 6.1 we have

THEOREM 6.2. *Let $p > 5$ be a prime for which $(\frac{-15}{p}) = 1$ and hence $p = x^2 + 15y^2$ or $p = 5x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$ according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$. Then*

$$F_{(p - (\frac{-3}{p})) / 3} \equiv \begin{cases} 0 \pmod{p} & \text{if } y \equiv 0 \pmod{3}, \\ -\frac{x}{(2 + 3(\frac{-3}{p}))y} \pmod{p} & \text{if } y \equiv x \pmod{3} \end{cases}$$

and

$$L_{(p - (\frac{-3}{p})) / 3} \equiv \begin{cases} 2\left(\frac{-3}{p}\right) \pmod{p} & \text{if } y \equiv 0 \pmod{3}, \\ -\left(\frac{-3}{p}\right) \pmod{p} & \text{if } y \not\equiv 0 \pmod{3}. \end{cases}$$

Proof. Suppose $s = (3 - 2(\frac{-3}{p}))\frac{x}{y}$. Then $s^2 \equiv -15 \pmod{p}$. Since $F_n = u_n(-1, 1)$ and $L_n = v_n(-1, 1)$, it follows from Theorem 6.1 that

$$F_{(p - (\frac{-3}{p})) / 3} \equiv \begin{cases} 0 \pmod{p} & \text{if } s \in C_0(p), \\ \pm \frac{3}{s} \pmod{p} & \text{if } \pm s \in C_1(p) \end{cases}$$

and that

$$L_{(p - (\frac{-3}{p})) / 3} \equiv \begin{cases} 2(-1)^{-[p/3]} = 2\left(\frac{-3}{p}\right) \pmod{p} & \text{if } s \in C_0(p), \\ -(-1)^{-[p/3]} = \left(\frac{-3}{p}\right) \pmod{p} & \text{if } s \notin C_0(p). \end{cases}$$

From Theorem 5.1 we know that

$$\left(\frac{s+1+2\omega}{p}\right)_3 = \left(\frac{sy+y+2y\omega}{p}\right)_3 = \begin{cases} 1 & \text{if } y \equiv 0 \pmod{3}, \\ \omega & \text{if } x \equiv \left(\frac{-3}{p}\right)y \pmod{3}, \\ \omega^2 & \text{if } x \equiv -\left(\frac{-3}{p}\right)y \pmod{3}. \end{cases}$$

Hence, $s \in C_0(p)$ if and only if $y \equiv 0 \pmod{3}$.

If $y \not\equiv 0 \pmod{3}$ then $x \equiv \pm\left(\frac{-3}{p}\right)y \pmod{3}$ and so $\pm s \in C_1(p)$ by the above. Since

$$\frac{3}{s} = \frac{3y}{\left(3-2\left(\frac{-3}{p}\right)\right)x} \equiv -\frac{x}{\left(3+2\left(\frac{-3}{p}\right)\right)y} = -\left(\frac{-3}{p}\right)\frac{x}{\left(3\left(\frac{-3}{p}\right)+2\right)y} \pmod{p}$$

we obtain

$$F_{p-\left(\frac{-3}{p}\right)/3} \equiv \begin{cases} -\frac{x}{5y} \pmod{p} & \text{if } p \equiv 1 \pmod{3} \text{ and } x \equiv y \pmod{p}, \\ \frac{x}{y} \pmod{p} & \text{if } p \equiv 2 \pmod{3} \text{ and } x \equiv y \pmod{p}. \end{cases}$$

This completes the proof.

LEMMA 6.1. *Let p be a prime greater than 3, $a, b \in \mathbb{Z}$, $u_n = u_n(a, b)$, $v_n = v_n(a, b)$ and $ab(b^2 - 4a) \not\equiv 0 \pmod{p}$. Then*

- (a) $p \mid u_{p-\left(\frac{-3}{p}\right)}$ if and only if $\left(\frac{-3(b^2-4a)}{p}\right) = 1$.
- (b) $p \mid u_n$ if and only if $v_{2n} \equiv 2a^n \pmod{p}$.

Proof. From [D] and [R] we know that

$$(6.5) \quad u_{p-\left(\frac{b^2-4a}{p}\right)} \equiv 0 \pmod{p}, \quad u_p \equiv \left(\frac{b^2-4a}{p}\right) \pmod{p}.$$

Thus,

$$u_{p+\left(\frac{b^2-4a}{p}\right)} = \begin{cases} bu_p - au_{p-1} \equiv b \pmod{p} & \text{if } \left(\frac{b^2-4a}{p}\right) = 1, \\ \frac{1}{a}(bu_p - u_{p+1}) \equiv -\frac{b}{a} \pmod{p} & \text{if } \left(\frac{b^2-4a}{p}\right) = -1 \end{cases} \\ \not\equiv 0 \pmod{p}.$$

It then follows that

$$p \mid u_{p-\left(\frac{-3}{p}\right)} \Leftrightarrow \left(\frac{-3}{p}\right) = \left(\frac{b^2-4a}{p}\right) \Leftrightarrow \left(\frac{-3(b^2-4a)}{p}\right) = 1.$$

This proves part (a).

Now consider part (b). According to [D] and [R] we have

$$(6.6) \quad u_{2n} = u_n v_n, \quad v_{2n} = v_n^2 - 2a^n,$$

$$(6.7) \quad v_n^2 - (b^2 - 4a)u_n^2 = 4a^n.$$

Thus,

$$p \mid u_n \Leftrightarrow v_n^2 \equiv 4a^n \pmod{p} \Leftrightarrow v_{2n} \equiv 2a^n \pmod{p}.$$

This concludes the proof.

Using Lemma 6.1 and Theorem 6.2 we have

COROLLARY 6.5. *Let p be a prime greater than 5. Then*

(i) $p \mid F_{(p - (\frac{-3}{p})) / 3}$ if and only if p can be represented by $x^2 + 135y^2$ or $5x^2 + 27y^2$ according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

(ii) $p \mid F_{(p - (\frac{-3}{p})) / 6}$ if and only if p can be represented by $x^2 + 540y^2$ or $5x^2 + 108y^2$ according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

Proof. It is well known that (see [D], [SS], [R]) $F_n \mid F_{mn}$ for $m, n \in \mathbb{Z}^+$. Thus $F_{(p - (\frac{-3}{p})) / 3} \mid F_{p - (\frac{-3}{p})}$. If $p \mid F_{(p - (\frac{-3}{p})) / 3}$ then $p \mid F_{p - (\frac{-3}{p})}$. Applying Lemma 6.1 we find $(\frac{-15}{p}) = 1$ and so $p = A^2 + 15B^2$ or $p = 5A^2 + 3B^2$ for some $A, B \in \mathbb{Z}$. It then follows from Theorem 6.2 that $3 \mid B$. Hence $p = x^2 + 135y^2$ or $p = 5x^2 + 27y^2$ for some $x, y \in \mathbb{Z}$.

Conversely, if p is represented by $x^2 + 135y^2$ or $5x^2 + 27y^2$, then $(\frac{-15}{p}) = 1$. Applying Theorem 6.2 we find $p \mid F_{(p - (\frac{-3}{p})) / 3}$. This proves (i).

Let us consider (ii). If $p \mid F_{(p - (\frac{-3}{p})) / 6}$ then $p \mid F_{p - (\frac{-3}{p})}$ and so $(\frac{-15}{p}) = 1$ by Lemma 6.1. If p is represented by $x^2 + 540y^2$ or $5x^2 + 108y^2$, we also have $(\frac{-15}{p}) = 1$. Hence, we may assume $(\frac{-15}{p}) = 1$ and so $p = x^2 + 15y^2$ or $p = 5x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$. It then follows from Lemma 6.1 and Theorem 6.2 that

$$\begin{aligned} p \mid F_{(p - (\frac{-3}{p})) / 6} &\Leftrightarrow L_{(p - (\frac{-3}{p})) / 3} \equiv 2(-1)^{(p - (\frac{-3}{p})) / 6} \pmod{p} \\ &\Leftrightarrow (-1)^{(p - (\frac{-3}{p})) / 6} = \left(\frac{-3}{p}\right) \text{ and } 3 \mid y \\ &\Leftrightarrow p = \begin{cases} A^2 + 135B^2 \equiv 1 \pmod{12} & (A, B \in \mathbb{Z}) \text{ if } p \equiv 1 \pmod{3}, \\ 5A^2 + 27B^2 \equiv 5 \pmod{12} & (A, B \in \mathbb{Z}) \text{ if } p \equiv 2 \pmod{3} \end{cases} \\ &\Leftrightarrow p = A^2 + 135B^2 \text{ or } p = 5A^2 + 27B^2 \text{ with } B \equiv 0 \pmod{2} \\ &\Leftrightarrow p = x^2 + 540y^2 \text{ or } p = 5x^2 + 108y^2 \text{ for some } x, y \in \mathbb{Z}. \end{aligned}$$

This completes the proof.

REMARK 6.1. In [L3], [L4] E. Lehmer proved Corollary 6.5(i) in the case $p \equiv 1 \pmod{12}$. For the criteria for $p \mid F_{(p-1)/4}$ (if $p \equiv 1 \pmod{4}$ is a prime) one may consult [L6], [SS].

Now we point out similar results for the Pell sequence. The *Pell sequence* $\{P_n\}$ and its companion $\{Q_n\}$ are given by

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1} \quad (n \geq 1)$$

and

$$Q_0 = 2, \quad Q_1 = 2, \quad Q_{n+1} = 2Q_n + Q_{n-1} \quad (n \geq 1).$$

Clearly, $P_n = u_n(-1, 2)$ and $Q_n = v_n(-1, 2)$.

Using Theorems 6.1 and 5.1 one can similarly prove

THEOREM 6.3. *Let $p > 3$ be a prime such that $\left(\frac{-6}{p}\right) = 1$ and hence $p = x^2 + 6y^2$ or $p = 2x^2 + 3y^2$ for some $x, y \in \mathbb{Z}$ according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$. Then*

$$P_{(p - (\frac{-3}{p}))/3} \equiv \begin{cases} 0 \pmod{p} & \text{if } y \equiv 0 \pmod{3}, \\ \frac{x}{(1 + 3(\frac{-3}{p}))y} \pmod{p} & \text{if } y \equiv x \pmod{3} \end{cases}$$

and

$$Q_{(p - (\frac{-3}{p}))/3} \equiv \begin{cases} 2\left(\frac{-3}{p}\right) \pmod{p} & \text{if } y \equiv 0 \pmod{3}, \\ -\left(\frac{-3}{p}\right) \pmod{p} & \text{if } y \not\equiv 0 \pmod{3}. \end{cases}$$

REMARK 6.2. For the values of $P_{(p-1)/2} \pmod{p}$ and $P_{(p+1)/2} \pmod{p}$ one may consult [S1].

COROLLARY 6.6. *Let p be a prime greater than 3. Then*

(i) $p \mid P_{(p - (\frac{-3}{p}))/3}$ if and only if $p = x^2 + 54y^2$ or $p = 2x^2 + 27y^2$ for some integers x and y according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

(ii) $p \mid P_{(p - (\frac{-3}{p}))/6}$ if and only if $p = x^2 + 216y^2$ or $p = 8x^2 + 8xy + 29y^2$ for some integers x and y according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.

The proof of Corollary 6.6 is similar to the proof of Corollary 6.5.

REMARK 6.3. Let $p \equiv 1 \pmod{4}$ be a prime. From [L4], [S1] we know that $p \mid P_{(p-1)/4}$ if and only if $p = x^2 + 32y^2$ for some integers x and y .

Finally, we discuss the Lucas sequence $\{u_n(1, 4)\}$.

THEOREM 6.4. *Let $p \equiv 1 \pmod{4}$ be a prime and hence $p = x^2 + y^2$ for some integers x and y . Then*

$$\begin{aligned}
 & u_{(p - (\frac{-3}{p})) / 6}(1, 4) \\
 &= 2^{-(p - (\frac{-3}{p})) / 6} u_{(p - (\frac{-3}{p})) / 3}(-2, 2) \\
 &\equiv \begin{cases} 0 \pmod{p} & \text{if } 9 \mid xy(x^2 - y^2), \\ -\frac{1}{2} \left(\frac{2}{p}\right) \frac{x}{y} \pmod{p} & \text{if } x \equiv 2y, -3y, 4y \pmod{9} \text{ or } y \equiv 3x \pmod{9} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 v_{(p - (\frac{-3}{p})) / 6}(1, 4) &= 2^{-(p - (\frac{-3}{p})) / 6} v_{(p - (\frac{-3}{p})) / 3}(-2, 2) \\
 &\equiv \begin{cases} 2 \left(\frac{-6}{p}\right) \pmod{p} & \text{if } 9 \mid xy(x^2 - y^2), \\ -\left(\frac{-6}{p}\right) \pmod{p} & \text{if } 9 \nmid xy(x^2 - y^2). \end{cases}
 \end{aligned}$$

Proof. By (6.3) and (6.4) we have

$$\begin{aligned}
 2^n u_n(1, 4) &= \frac{1}{2\sqrt{3}} ((4 + 2\sqrt{3})^n - (4 - 2\sqrt{3})^n) \\
 &= \frac{1}{2\sqrt{3}} ((1 + \sqrt{3})^{2n} - (1 - \sqrt{3})^{2n}) = u_{2n}(-2, 2)
 \end{aligned}$$

and

$$\begin{aligned}
 2^n v_n(1, 4) &= (4 + 2\sqrt{3})^n + (4 - 2\sqrt{3})^n \\
 &= (1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n} = v_{2n}(-2, 2).
 \end{aligned}$$

Since $\frac{3x}{y} \cdot \frac{x}{y} \equiv -3 \pmod{p}$ it follows from Proposition 2.2(i) that $x/y \in C_i(p)$ if and only if $3x/y \in C_i(p)$. Thus, from $(6x/y)^2 \equiv -3(2^2 - 4(-2)) \pmod{p}$ and Theorem 6.1 we get

$$u_{(p - (\frac{-3}{p})) / 3}(-2, 2) \equiv \begin{cases} 0 \pmod{p} & \text{if } x/y \in C_0(p), \\ \pm \frac{y}{2x} \cdot 2^{-[p/3]} \pmod{p} & \text{if } \pm x/y \in C_1(p) \end{cases}$$

and

$$v_{(p - (\frac{-3}{p})) / 3}(-2, 2) \equiv \begin{cases} 2 \cdot (-2)^{-[p/3]} \pmod{p} & \text{if } x/y \in C_0(p), \\ -(-2)^{-[p/3]} \pmod{p} & \text{if } x/y \notin C_0(p). \end{cases}$$

From the proof of Theorem 5.2 we see that

$$x/y \in C_0(p) \Leftrightarrow 9 \mid xy(x^2 - y^2)$$

and that

$$x/y \in C_1(p) \Leftrightarrow x \equiv 2y, -3y, 4y \pmod{9} \text{ or } y \equiv 3x \pmod{9}.$$

Now, combining the above with the facts that

$$(-1)^{[p/3]} = \left(\frac{-3}{p}\right) \quad \text{and} \quad 2^{-(p-(\frac{-3}{p}))/6} \cdot 2^{-[p/3]} = 2^{-(p-1)/2} \equiv \left(\frac{2}{p}\right) \pmod{p}$$

yields the desired result.

COROLLARY 6.7. *Let $p > 3$ be a prime. Then $p \mid u_{(p-(\frac{-3}{p}))/6}(1, 4)$ (or $p \mid u_{(p-(\frac{-3}{p}))/3}(-2, 2)$) if and only if p can be represented by $x^2 + 81y^2$ or $2x^2 + 2xy + 41y^2$ according as $p \equiv 1 \pmod{3}$ or $p \equiv 2 \pmod{3}$.*

REMARK 6.4. Let $p > 3$ be a prime. Using the method in the proof of Corollary 6.5(ii) one can similarly prove that $p \mid u_{(p-(\frac{-3}{p}))/6}(-2, 2)$ if and only if p can be represented by $16x^2 + 81y^2$, $x^2 + 1296y^2$, $8x^2 + 8xy + 41y^2$ or $32x^2 - 8xy + 41y^2$.

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