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# Sequences with bounded l.c.m. of each pair of terms

by

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**0. Introduction.** Let $A_x$ be a set of positive integers with the least common multiple of each pair of terms not exceeding $x$ and $|A_x|$ being the largest. In 1951, P. Erdős [3] proposed the following problem: what is the value of $|A_x|$? It is known that

$$\sqrt{\tfrac{9}{8}x}+O(1)\le |A_x|\le \sqrt{4x}+O(1).$$

For a proof one may see Erdős [4]. The problem is problem E2 and a part of problem B26 in the well known problem book [5] of Guy. Choi [1] improved the upper bound to $1.638\sqrt{x}$, and later [2] to $1.43\sqrt{x}$.

In number theory, it is rare to give an asymptotic formula for such a problem. In this paper an asymptotic formula for $|A_x|$ is given. Further, let $B_x$ be the union of the set of positive integers not exceeding $\sqrt{x/2}$ and the set of even integers between $\sqrt{x/2}$ and $\sqrt{2x}$. It is clear that the least common multiple of each pair of terms of $B_x$ does not exceed $x$. We will show that $A_x$ is almost the same as $B_x$. That is,

THEOREM. *We have*

$$|A_x\setminus B_x|=o(\sqrt{x}).$$

*In particular*,

$$|A_x|=|B_x|+o(\sqrt{x})=\sqrt{\tfrac{9}{8}x}+o(\sqrt{x}).$$

*Note.* From the proof of the Theorem we will see that $o(\sqrt{x})$ can be given explicitly. By the Theorem we have

$$|A_x\cap B_x|=|A_x|-|A_x\setminus B_x|=\sqrt{\tfrac{9}{8}x}+o(\sqrt{x})$$

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and

$$|B_x \setminus A_x| = |B_x| - |A_x \cap B_x| = o(\sqrt{x}).$$

**1. Preliminary lemmas**

LEMMA 1. *Let $M$ be an integer with $M \geq 3$, and let $c_0$, $c_1$ and $c_2$ be real numbers with $c_1 \geq c_0 > 0$. Then there exists an $x_0 = x_0(M, c_0)$ such that if $x \geq x_0$ and $a_i$, $b_i$ ($1 \leq i \leq t \leq M/2$) are integers with $(a_i, b_i) = 1$ ($1 \leq i \leq t$) and with each prime factor of $\prod_{i=1}^{t}(a_i n + b_i)$ exceeding $M$ for any integer $n$, then there exists an integer $k \in (c_1 x^{1/2} + c_2, c_1(x^{1/2} + x^{1/4}) + c_2)$ such that each prime factor of $\prod_{i=1}^{t}(a_i k + b_i)$ exceeds*

$$\frac{1}{6\log M}\log x.$$

P r o o f. We employ the standard Eratosthenes–Legendre sieve. One may refer to [6], p. 31, Theorem 1.1. We take

$$\mathcal{A} = \Big\{\prod_{i=1}^{t}(a_i k + b_i) : k \in (c_1 x^{1/2} + c_2, c_1(x^{1/2} + x^{1/4}) + c_2)\Big\},$$

$$\mathcal{P} = \mathcal{P}_1, \qquad z = \frac{1}{6\log M}\log x, \qquad X = c_1 x^{1/4}, \qquad A_0 = \tfrac{1}{2}M,$$

$\omega(p)$ being the number of solutions of

$$\prod_{i=1}^{t}(a_i n + b_i) \equiv 0 \pmod{p}.$$

Noting that $\overline{\mathcal{P}} = \emptyset$ we have $|R_d| = |r_d| \leq \omega(d)$ if $\mu(d) \neq 0$. By Theorem 1.1 of [6], p. 31, we have

$$\begin{aligned} S(\mathcal{A}; \mathcal{P}, z) &= XW(z) + \theta(1 + A_0)^z \\ &= c_1 x^{1/4}\prod_{p\leq z}\Big(1 - \frac{\omega(p)}{p}\Big) + \theta\Big(1 + \frac{1}{2}M\Big)^z \\ &\geq c_0 x^{1/4}\prod_{M<p\leq z}\Big(1 - \frac{M}{2p}\Big) - M^z \gg \frac{x^{1/4}}{(\log\log x)^{M/2}}, \end{aligned}$$

where $|\theta| \leq 1$ and $\gg$ depends only on $M$ and $c_0$. From this we obtain the assertion of Lemma 1.

*Note.* $a_i$ and $b_i$ may depend on $x$, $c_0$, $c_1$ and $c_2$. $x_0(M, c_0)$ can be effectively computed. For a stronger result one should use Brun's sieve. Here the conclusion is sufficient for the present paper.

LEMMA 2. *Let $c_i$ ($3 \leq i \leq 6$) be nonnegative real numbers with $c_4 > c_3$. Let $D$ and $M$ be integers with $|D| \leq c_5 x^{c_6}$ and with each prime factor of $D$*

*exceeding*

$$\frac{1}{6\log M}\log x.$$

*Then the number of a with* $(a,D)>1$, $a\in[c_3x^{1/2},c_4x^{1/2}]$ *is* $O(\sqrt{x}/\log\log x)$, *where* $O$ *depends only on* $M$ *and* $c_i$ $(3\le i\le 6)$*.*

Proof. If $D=0$, then $x\le M^{12}$ and the conclusion is trivial. Now we assume that $D\ne 0$. Let $|D|=p_1^{l_1}p_2^{l_2}\ldots p_r^{l_r}$ be the standard factorization of $|D|$. Then

$$r\log\left(\frac{1}{6\log M}\log x\right)\le\sum_{i=1}^{r}\log p_i\le\log|D|\ll\log x.$$

Thus

$$r\ll\frac{\log x}{\log\log x}.$$

Hence

$$\sum_{\substack{a\in[c_3\sqrt{x},c_4\sqrt{x}]\\(a,D)>1}}1\le\sum_{i=1}^{r}\sum_{\substack{a\in[c_3\sqrt{x},c_4\sqrt{x}]\\p_i\mid a}}1\le\sum_{i=1}^{r}\left(\frac{(c_4-c_3)\sqrt{x}}{p_i}+1\right)$$
$$\le\frac{(c_4-c_3)r\sqrt{x}}{\log x}6\log M+r\ll\frac{\sqrt{x}}{\log\log x}.$$

This completes the proof of Lemma 2.

**2. General lemmas.** For an interval $I=(a,b]$, let

$$|I\sqrt{x}\cap A_x|=\alpha(I)|I|\sqrt{x},$$
$$|I\sqrt{x}\cap A_x\cap(2\mathbb{Z})|=\alpha^{(0)}(I)|I|\sqrt{x},$$
$$|I\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1)|=\alpha^{(1)}(I)|I|\sqrt{x},$$

where $|X|$ denotes the number of elements of $X$ or the length of an interval $X$, and $2\mathbb{Z}$ and $2\mathbb{Z}+1$ denote the sets of all even integers and all odd integers respectively. Let $\mathcal{I}=\{I_1,\ldots,I_l\}$ be a set of pairwise disjoint intervals with $I_i=(a_i,b_i]$ and $0<a_0<a_1<\ldots<a_l$. Let

$$\alpha_i=\alpha(I_i),\quad \alpha_i^{(0)}=\alpha^{(0)}(I_i),\quad \alpha_i^{(1)}=\alpha^{(1)}(I_i),\quad M=4(1+[a_l^2]),$$

where $[a_l^2]$ denotes the integral part of $a_l^2$. It is clear that $\alpha_i=\alpha_i^{(0)}+\alpha_i^{(1)}$.

Lemma 3. *Let* $r_{ij}$ $(j=1,\ldots,k_i;\ i=1,\ldots,l)$ *be distinct integers with*

$$|r_{ij}-r_{uv}|\le g(r_{ij},r_{uv})a_ia_u,$$

*where* $g(a,b)=1+\frac{1}{4}(1-(-1)^a)(1-(-1)^b)$. *Let*

$$k_i^{(0)}=|\{r_{ij}:2\,|\,r_{ij},j=1,\ldots,k_i\}|,$$
$$k_i^{(1)}=k_i-k_i^{(0)},\qquad i=1,\ldots,l.$$

*Then*

$$\sum_{i=1}^{l}(k_i^{(0)}\alpha_i^{(0)}+k_i^{(1)}\alpha_i^{(1)})\le\frac{1}{2}+O((\log\log x)^{-1/2}),$$

*where* $O$ *depends only on* $\mathcal{I}$.

Proof. Let $K=\sum_{i=1}^{l}k_i$. If $K=0$ or 1, then by the definitions of $\alpha_i^{(0)}$ and $\alpha_i^{(1)}$ the assertion of Lemma 3 is true. In the following we assume that $K\ge 2$. Let $\delta$ be a small positive number which will be determined later, and let

$$I_i(t)=(a_i+t\delta,a_i+(t+1)\delta].$$

For the (index) set

$$\{t_{ij}:0\le t_{ij}\le|I_i|/\delta-1,\ t_{ij}\in\mathbb{Z},\ j=1,\ldots,k_i;\ i=1,\ldots,l\}$$

we first show that

$$\Big|\bigcup_{i,j}(I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big|\le\frac{1}{2}\delta\sqrt{x}+O\bigg(\frac{\sqrt{x}}{\log\log x}\bigg),$$

where $O$ depends only on $\mathcal{I}$. To do this we consider the set

$$\Delta(a)=\bigcup_{i,j}\{M!l_{ij}+r_{ij}+2a\},$$

where $l_{ij}$ are integers which will be determined later such that

$$(1)\qquad (a_i+t_{ij}\delta)\sqrt{x}\le M!l_{ij}+r_{ij}\le(a_i+t_{ij}\delta)(\sqrt{x}+x^{1/4})$$

hold for $j=1,\ldots,k_i$; $i=1,\ldots,l$. For convenience we rewrite $\Delta(0)$ as

$$\Delta(0)=\{M!l_1+r_1,M!l_2+r_2,\ldots,M!l_K+r_K\}.$$

Since $a_ia_u<M/4$ $(i,u=1,\ldots,l)$, by the conditions of Lemma 3 we have

$$|r_i-r_j|<M/2,\qquad i,j=1,\ldots,K,$$

whence $K\le M/2$. Now we take $l_1$ satisfying (1). Suppose that we have chosen $l_1,\ldots,l_u$ $(u<K)$. By Lemma 1 for $x\ge x_0(M,a_0/M!)$ there exists a $l_{u+1}$ satisfying (1) such that each prime factor of

$$\prod_{i=1}^{u}\bigg(\frac{M!}{r_{u+1}-r_i}l_{u+1}-\frac{M!}{r_{u+1}-r_i}l_i+1\bigg)$$

exceeds

$$\frac{1}{6\log M}\log x.$$

Thus by induction we have determined all $l_u$ $(1 \le u \le K)$. Let

$$D = \prod_{1 \le v < u \le K} \left( \frac{M!}{r_u - r_v} l_u - \frac{M!}{r_u - r_v} l_v + 1 \right).$$

Then each prime factor of $D$ exceeds

$$\frac{1}{6 \log M} \log x$$

and by (1),

$$\begin{aligned} |D| &\le \prod_{1 \le v < u \le K} |M! l_u + r_u - M! l_v - r_v| \\ &\le (2 b_l \sqrt{x})^{K(K-1)/2} \le (2 b_l)^{M(M-1)} x^{M(M-1)}. \end{aligned}$$

By Lemma 2, the number of $a$ such that $(a, D) > 1$ and $a \in (0, b_l \sqrt{x}]$ is $O(\sqrt{x}/\log\log x)$, where $O$ depends only on $\mathcal{I}$. Let

$$B = \left\{ a : a \in \bigcup_{i=1}^{l} (I_i \sqrt{x} \cap \mathbb{Z}),\ (a, D) = 1 \right\}.$$

If $a \in (0, \delta\sqrt{x}/2]$ and

$$M! l_u + r_u + 2a \in B, \qquad M! l_v + r_v + 2a \in B,$$

then for $u \ne v$ we have

$$\begin{aligned} (2) \qquad & (M! l_u + r_u + 2a, M! l_v + r_v + 2a) \\ &= (M! l_u + r_u + 2a, M!(l_v - l_u) + r_v - r_u) \\ &= (M! l_u + r_u + 2a, r_v - r_u) \le g(r_u, r_v)^{-1} |r_u - r_v|. \end{aligned}$$

Thus for $a \in (0, \delta\sqrt{x}/2]$ with

$$\begin{aligned} & M! l_{ij} + r_{ij} + 2a \in B, \\ & M! l_{uv} + r_{uv} + 2a \in B, \qquad (i-u)^2 + (j-v)^2 \ne 0, \end{aligned}$$

by (1), (2) and the conditions of the lemma we have

$$\begin{aligned} \text{l.c.m.}\{M! l_{ij} + r_{ij} + 2a, M! l_{uv} + r_{uv} + 2a\} &= \frac{(M! l_{ij} + r_{ij} + 2a)(M! l_{uv} + r_{uv} + 2a)}{(M! l_{ij} + r_{ij} + 2a, M! l_{uv} + r_{uv} + 2a)} \\ &> \frac{(a_i + t_{ij}\delta)(a_u + t_{uv}\delta)x}{|r_{ij} - r_{uv}|} g(r_{ij}, r_{uv}) \\ &\ge \frac{(a_i + t_{ij}\delta)(a_u + t_{uv}\delta)x}{a_i a_u} \ge x. \end{aligned}$$

So $|\Delta(a) \cap B \cap A_x| \le 1$. Since (see (1))

$$\begin{aligned} I_i(t_{ij})\sqrt{x} \cap (2\mathbb{Z} + r_{ij}) &\subseteq ((M!l_{ij} + r_{ij}, M!l_{ij} + r_{ij} + \delta\sqrt{x}] \\ &\quad \cup ((a_i + t_{ij}\delta)\sqrt{x}, (a_i + t_{ij}\delta)(\sqrt{x} + x^{1/4})]) \cap (2\mathbb{Z} + r_{ij}) \\ &\subseteq \Big( \bigcup_{0<a\le\delta\sqrt{x}/2} \{M!l_{ij} + r_{ij} + 2a\} \Big) \\ &\quad \cup (((a_i + t_{ij}\delta)\sqrt{x}, (a_i + t_{ij}\delta)(\sqrt{x} + x^{1/4})] \cap \mathbb{Z}), \end{aligned}$$

we have

$$\begin{aligned} \bigcup_{i,j} (I_i(t_{ij})\sqrt{x} \cap (2\mathbb{Z} + r_{ij})) &\subseteq \Big( \bigcup_{0<a\le\delta\sqrt{x}/2} \bigcup_{i,j} \{M!l_{ij} + r_{ij} + 2a\} \Big) \\ &\quad \cup \Big( \bigcup_{i,j} (((a_i + t_{ij}\delta)\sqrt{x}, (a_i + t_{ij}\delta)(\sqrt{x} + x^{1/4})] \cap \mathbb{Z}) \Big) \\ &\subseteq \Big( \bigcup_{0<a\le\delta\sqrt{x}/2} \Delta(a) \Big) \\ &\quad \cup \Big( \bigcup_{i,j} (((a_i + t_{ij}\delta)\sqrt{x}, (a_i + t_{ij}\delta)(\sqrt{x} + x^{1/4})] \cap \mathbb{Z}) \Big). \end{aligned}$$

Hence

$$\begin{aligned} &\Big| \bigcup_{i,j} (I_i(t_{ij})\sqrt{x} \cap A_x \cap B \cap (2\mathbb{Z} + r_{ij})) \Big| \\ &\quad \le \frac{1}{2}\delta\sqrt{x} + \sum_{i,j} ((a_i + t_{ij}\delta)x^{1/4} + 1) \le \frac{1}{2}\delta\sqrt{x} + \sum_{i,j} ((a_i + |I_i|)x^{1/4} + 1) \\ &\quad \le \frac{1}{2}\delta\sqrt{x} + \sum_{i,j} (b_i x^{1/4} + 1) \le \frac{1}{2}\delta\sqrt{x} + K \max_i b_i x^{1/4} + K \\ &\quad \le \frac{1}{2}\delta\sqrt{x} + O(x^{1/4}), \end{aligned}$$

where $O$ depends only on $\mathcal{I}$ (note that $K \le M$), whence

$$\begin{aligned} &\Big| \bigcup_{i,j} (I_i(t_{ij})\sqrt{x} \cap A_x \cap (2\mathbb{Z} + r_{ij})) \Big| \\ &\qquad \le \frac{1}{2}\delta\sqrt{x} + O(x^{1/4}) + O\bigg( \frac{\sqrt{x}}{\log\log x} \bigg) \le \frac{1}{2}\delta\sqrt{x} + O\bigg( \frac{\sqrt{x}}{\log\log x} \bigg), \end{aligned}$$

where $O$ depends only on $\mathcal{I}$. Since $I_1, \ldots, I_l$ are pairwise disjoint, we have

$$\Big|\bigcup_{i\le l-1}\bigcup_j (I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big| + \Big|\bigcup_{\substack{j=1\\ 2|r_{lj}}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|$$

$$+\Big|\bigcup_{\substack{j=1\\ 2\nmid r_{lj}}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$

$$\le \frac{1}{2}\delta\sqrt{x}+O\Big(\frac{\sqrt{x}}{\log\log x}\Big).$$

Hence, if $k_l^{(1)}\ge 1$ and $u\ge 0$, then

$$\Big|\bigcup_{i\le l-1}\bigcup_j (I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big| + \Big|\bigcup_{\substack{j=1\\ 2|r_{lj}}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|$$

$$+\Big|\bigcup_{\substack{r=0\\ k_l^{(1)}u+r\le|I_l|/\delta-1}}^{k_l^{(1)}-1}(I_l(k_l^{(1)}u+r)\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$

$$\le \frac{1}{2}\delta\sqrt{x}+O\Big(\frac{\sqrt{x}}{\log\log x}\Big).$$

Thus

$$\Big(\Big[\frac{|I_l|}{k_l^{(1)}\delta}\Big]+1\Big)\Big|\bigcup_{i\le l-1}\bigcup_j (I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big|$$

$$+\Big(\Big[\frac{|I_l|}{k_l^{(1)}\delta}\Big]+1\Big)\Big|\bigcup_{\substack{j=1\\ 2|r_{lj}}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|$$

$$+\sum_{0\le u\le[|I_l|/(k_l^{(1)}\delta)]}\Big|\bigcup_{\substack{r=0\\ k_l^{(1)}u+r\le|I_l|/\delta-1}}^{k_l^{(1)}-1}(I_l(k_l^{(1)}u+r)\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$

$$\le \frac{1}{2}\delta\sqrt{x}\Big(\Big[\frac{|I_l|}{k_l^{(1)}\delta}\Big]+1\Big)+O\Big(\Big(\Big[\frac{|I_l|}{k_l^{(1)}\delta}\Big]+1\Big)\frac{\sqrt{x}}{\log\log x}\Big).$$

Hence

$$\frac{|I_l|}{k_l^{(1)}\delta}\Big|\bigcup_{i\le l-1}\bigcup_j (I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big|$$

$$+\frac{|I_l|}{k_l^{(1)}\delta}\Big|\bigcup_{\substack{j=1\\ 2|r_{lj}}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|$$

$$+\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$
$$\le \frac12\delta\sqrt{x}\bigg(\frac{|I_l|}{k_l^{(1)}\delta}+1\bigg)+O\bigg(\bigg(\frac{|I_l|}{k_l^{(1)}\delta}+1\bigg)\frac{\sqrt{x}}{\log\log x}\bigg).$$

So

$$\Big|\bigcup_{i\le l-1}\bigcup_j(I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big|$$
$$+\Big|\bigcup_{j=1,2|r_{lj}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|$$
$$+\frac{k_l^{(1)}\delta}{|I_l|}\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$
$$\le \frac12\delta\sqrt{x}+\frac12\cdot\frac{k_l^{(1)}\delta^2}{|I_l|}\sqrt{x}+O\bigg(\bigg(1+\frac{k_l^{(1)}\delta}{|I_l|}\bigg)\frac{\sqrt{x}}{\log\log x}\bigg)$$
$$\le \frac12\delta\sqrt{x}+\frac12\cdot\frac{K\delta^2}{|I_l|}\sqrt{x}+O\bigg(\bigg(1+\frac{K\delta}{|I_l|}\bigg)\frac{\sqrt{x}}{\log\log x}\bigg)$$
$$\le \frac12\delta\sqrt{x}+O\bigg(\delta^2\sqrt{x}+\frac{\sqrt{x}}{\log\log x}\bigg).$$

Noting that

$$\Big|\bigcup_{0\le t\le |I_l|/\delta-1}(I_l(t)\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1))\Big|$$
$$=|I_l\sqrt{x}\cap A_x\cap(2\mathbb{Z}+1)|-\theta_l^{(1)}\delta\sqrt{x}\quad(0\le\theta_l^{(1)}\le 1)$$
$$=\alpha_l^{(1)}|I_l|\sqrt{x}-\theta_l^{(1)}\delta\sqrt{x},$$

we have

$$\Big|\bigcup_{i\le l-1}\bigcup_j(I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big| \tag{3}$$
$$+\Big|\bigcup_{j=1,2|r_{lj}}^{k_l}(I_l(t_{lj})\sqrt{x}\cap A_x\cap(2\mathbb{Z}))\Big|+k_l^{(1)}\alpha_l^{(1)}\delta\sqrt{x}$$
$$\le \frac12\delta\sqrt{x}+\theta_l^{(1)}\frac{k_l^{(1)}\delta^2}{|I_l|}\sqrt{x}+O\bigg(\delta^2\sqrt{x}+\frac{\sqrt{x}}{\log\log x}\bigg)$$
$$\le \frac12\delta\sqrt{x}+O\bigg(\delta^2\sqrt{x}+\frac{\sqrt{x}}{\log\log x}\bigg).$$

It is clear that if $k_l^{(1)}=0$, (3) also holds. Similarly, we have

$$\Big|\bigcup_{i\le l-1}\bigcup_j (I_i(t_{ij})\sqrt{x}\cap A_x\cap(2\mathbb{Z}+r_{ij}))\Big| + k_l^{(0)}\alpha_l^{(0)}\delta\sqrt{x}+k_l^{(1)}\alpha_l^{(1)}\delta\sqrt{x} \le \frac{1}{2}\delta\sqrt{x}+O\bigg(\delta^2\sqrt{x}+\frac{\sqrt{x}}{\log\log x}\bigg).$$

Continuing this procedure we have

$$\sum_{i=1}^{l}(k_i^{(0)}\alpha_i^{(0)}+k_i^{(1)}\alpha_i^{(1)})\delta\sqrt{x}\le\frac{1}{2}\delta\sqrt{x}+O\bigg(\delta^2\sqrt{x}+\frac{\sqrt{x}}{\log\log x}\bigg),$$

where $O$ depends only on $\mathcal{I}$. Taking $\delta=(\log\log x)^{-1/2}$, we have

$$\sum_{i=1}^{l}(k_i^{(0)}\alpha_i^{(0)}+k_i^{(1)}\alpha_i^{(1)})\le\frac{1}{2}+O((\log\log x)^{-1/2}).$$

This completes the proof of Lemma 3.

Corollary. *Let the conditions be as in Lemma* 3 *and* $m_1^{(0)},\dots,m_l^{(0)}$, $m_1^{(1)},\dots,m_l^{(1)}$ *be nonnegative integers with*

$$\sum_{i=1}^{t}m_i^{(v)}\le\sum_{i=1}^{t}k_i^{(v)},\qquad t=1,\dots,l;\ v=0,1. \tag{4}$$

*Then*

$$\sum_{i=1}^{l}(m_i^{(0)}\alpha_i^{(0)}+m_i^{(1)}\alpha_i^{(1)})\le\frac{1}{2}+O((\log\log x)^{-1/2}),$$

*where $O$ depends only on $\mathcal{I}$.*

Proof. Let $m_i=m_i^{(0)}+m_i^{(1)}$. By (4) we may rearrange $\{r_{ij}\}$ as

$$\{w_{ij}:i=1,\dots,l;\ j=1,\dots,m_i\}\cup A$$

such that $w_{ij}=r_{uv}$ implies that $u\le i$, and

$$m_i^{(0)}=|\{w_{ij}:2\mid w_{ij},\ j=1,\dots,m_i\}|.$$

Thus

$$|w_{ij}-w_{i'j'}|=|r_{uv}-r_{u'v'}|\le g(r_{uv},r_{u'v'})a_ua_{u'}\le g(w_{ij},w_{i'j'})a_ia_{i'}.$$

Then the Corollary follows from Lemma 3.

Lemma 4. *Let $m$, $n_1,\dots,n_r$ be nonnegative integers with $m\le n_1+\dots+n_r$. Then there exist nonnegative integers $m_1,\dots,m_r$ such that*

$$m=m_1+\dots+m_r\quad\textit{and}\quad m_i\le n_i,\quad i=1,\dots,r.$$

The proof is clear.

LEMMA 5. *Let the conditions be as in Lemma* 3. *Let* $\beta_1^{(v)},\ldots,\beta_l^{(v)}$ $(v=0,1)$ *be nonnegative real numbers with*

$$\sum_{i=1}^{t}\beta_i^{(v)}\le\sum_{i=1}^{t}k_i^{(v)},\qquad t=1,\ldots,l;\ v=0,1. \tag{5}$$

*Then*

$$\sum_{i=1}^{l}(\beta_i^{(0)}\alpha_i^{(0)}+\beta_i^{(1)}\alpha_i^{(1)})\le\frac{1}{2}+O((\log\log x)^{-1/2}),$$

*where* $O$ *depends only on* $\mathcal{I}$.

Proof. Let $n$ and $n_i^{(v)}$ $(1\le i\le l;\ v=0,1)$ be nonnegative integers with

$$\frac{n_i^{(v)}}{n}\le\beta_i^{(v)}<\frac{n_i^{(v)}+1}{n},\qquad i=1,\ldots,l;\ v=0,1.$$

Then (5) implies that

$$\sum_{i=1}^{t}n_i^{(v)}\le n\sum_{i=1}^{t}k_i^{(v)},\qquad t=1,\ldots,l;\ v=0,1. \tag{6}$$

Now we use induction on $t$ to prove the following proposition $P(t)$: There exist nonnegative integers $n_{ij}^{(v)}$ $(1\le i\le t;\ 1\le j\le n;\ v=0,1)$ such that

$$n_i^{(v)}=\sum_{j=1}^{n}n_{ij}^{(v)},\qquad v=0,1;\ i=1,\ldots,t,$$

and

$$\sum_{i=1}^{s}n_{ij}^{(v)}\le\sum_{i=1}^{s}k_i^{(v)},\qquad s=1,\ldots,t;\ j=1,\ldots,n;\ v=0,1.$$

By (6) and Lemma 4, $P(1)$ is true. Suppose that $P(t)$ $(1\le t<l)$ is true. Now by (6) and the induction hypothesis we have

$$n_{t+1}^{(v)}\le n\sum_{i=1}^{t+1}k_i^{(v)}-\sum_{i=1}^{t}n_i^{(v)}\le\sum_{j=1}^{n}\Big(\sum_{i=1}^{t+1}k_i^{(v)}-\sum_{i=1}^{t}n_{ij}^{(v)}\Big)$$

and

$$\sum_{i=1}^{t+1}k_i^{(v)}-\sum_{i=1}^{t}n_{ij}^{(v)}\ge 0.$$

By Lemma 4 there exist nonnegative integers $n_{(t+1)j}^{(v)}$ $(1\le j\le n;\ v=0,1)$ such that

$$n_{(t+1)j}^{(v)}\le\sum_{i=1}^{t+1}k_i^{(v)}-\sum_{i=1}^{t}n_{ij}^{(v)},\qquad j=1,\ldots,n,$$

and

$$n_{t+1}^{(v)} = \sum_{j=1}^{n} n_{(t+1)j}^{(v)}, \qquad v = 0, 1.$$

So $P(t+1)$ is true. Hence $P(t)$ is true for all $t$, $1 \le t \le l$. In particular, $P(l)$ is true, that is, there exist nonnegative integers $n_{ij}^{(v)}$ ($1 \le i \le l$; $1 \le j \le n$; $v = 0, 1$) such that

$$n_i^{(v)} = \sum_{j=1}^{n} n_{ij}^{(v)}, \qquad v = 0, 1;\ i = 1, \ldots, l,$$

and

$$\sum_{i=1}^{t} n_{ij}^{(v)} \le \sum_{i=1}^{t} k_i^{(v)}, \qquad t = 1, \ldots, l;\ j = 1, \ldots, n;\ v = 0, 1.$$

By the Corollary of Lemma 3 we have

$$\sum_{i=1}^{l} (n_{ij}^{(0)} \alpha_i^{(0)} + n_{ij}^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}), \qquad j = 1, \ldots, n.$$

Hence

$$\sum_{i=1}^{l} (n_i^{(0)} \alpha_i^{(0)} + n_i^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} n + O(n(\log\log x)^{-1/2}),$$

that is,

$$\sum_{i=1}^{l} \left( \frac{n_i^{(0)}}{n} \alpha_i^{(0)} + \frac{n_i^{(1)}}{n} \alpha_i^{(1)} \right) \le \frac{1}{2} + O((\log\log x)^{-1/2}),$$

where $O$ depends only on $\mathcal{I}$. Letting $n \to \infty$ we obtain the statement of Lemma 5.

LEMMA 6. *Let*

$$\sum_{i=1}^{l} (k_{ij}^{(0)} \alpha_i^{(0)} + k_{ij}^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2})$$

($j = 1, \ldots, r$) *be* $r$ *relations obtained by using Lemma* 3 (*not necessarily from the same* $\{r_{ij}\}$). *Let* $\beta_1^{(v)}, \ldots, \beta_l^{(v)}$, $\delta_1, \ldots, \delta_r$ ($v = 0, 1$) *be nonnegative real numbers with*

$$\sum_{i=1}^{t} \beta_i^{(v)} \le \sum_{i=1}^{t} \sum_{j=1}^{r} \delta_j k_{ij}^{(v)}, \qquad t = 1, \ldots, l;\ v = 0, 1. \tag{7}$$

*Then*

$$\sum_{i=1}^{l} (\beta_i^{(0)} \alpha_i^{(0)} + \beta_i^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} \sum_{j=1}^{r} \delta_j + O\Big( \sum_{j=1}^{r} \delta_j (\log\log x)^{-1/2} \Big),$$

*where* $O$ *depends only on* $\mathcal{I}$.

Proof. As in Lemma 4, if $u$ and $v_j$ $(1 \le j \le r)$ are nonnegative real numbers with $u \le \sum_{j=1}^{r} \delta_j v_j$, then there exist nonnegative real numbers $u_1, \ldots, u_r$ such that

$$u = \sum_{j=1}^{r} \delta_j u_j, \quad u_j \le v_j, \quad j = 1, \ldots, r.$$

Using this fact and (7) we infer, as in the proof of Lemma 5, that there exist nonnegative real numbers $\beta_{ij}^{(v)}$ $(i = 1, \ldots, l;\ j = 1, \ldots, r;\ v = 0, 1)$ such that

$$\beta_i^{(v)} = \sum_{j=1}^{r} \delta_j \beta_{ij}^{(v)}, \quad i = 1, \ldots, l;\ v = 0, 1,$$

and

$$\sum_{i=1}^{t} \beta_{ij}^{(v)} \le \sum_{i=1}^{t} k_{ij}^{(v)}, \quad t = 1, \ldots, l;\ j = 1, \ldots, r;\ v = 0, 1.$$

By Lemma 5 we have

$$\sum_{i=1}^{l} (\beta_{ij}^{(0)} \alpha_i^{(0)} + \beta_{ij}^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}).$$

Hence

$$\sum_{i=1}^{l} \Big( \sum_{j=1}^{r} \delta_j \beta_{ij}^{(0)} \alpha_i^{(0)} + \sum_{j=1}^{r} \delta_j \beta_{ij}^{(1)} \alpha_i^{(1)} \Big) \le \frac{1}{2} \sum_{j=1}^{r} \delta_j + O\Big( \sum_{j=1}^{r} \delta_j (\log\log x)^{-1/2} \Big).$$

That is,

$$\sum_{i=1}^{l} (\beta_i^{(0)} \alpha_i^{(0)} + \beta_i^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} \sum_{j=1}^{r} \delta_j + O\Big( \sum_{j=1}^{r} \delta_j (\log\log x)^{-1/2} \Big),$$

where $O$ depends only on $\mathcal{I}$. This completes the proof of Lemma 6.

LEMMA 7. *Let $r_{ij}$ $(j = 1, \ldots, k_i;\ i = 1, \ldots, l)$ be distinct integers with*

$$|r_{ij} - r_{uv}| \le a_i a_u.$$

*Then*

$$\sum_{i=1}^{l} k_i \alpha_i \le 1 + O((\log\log x)^{-1/2}),$$

*where $O$ depends only on $\mathcal{I}$.*

Proof. By Lemma 3 we have

$$\sum_{i=1}^{l} (k_i^{(0)} \alpha_i^{(0)} + k_i^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}). \tag{8}$$

Let

$$w_{ij} = r_{ij} + 1,$$
$$n_i = k_i, \quad j = 1, \ldots, n_i;\ i = 1, \ldots, l.$$

Then

$$|w_{ij} - w_{uv}| \le a_i a_u, \quad n_i^{(0)} = k_i^{(1)}, \quad n_i^{(1)} = k_i^{(0)}.$$

By Lemma 3 we have

$$\sum_{i=1}^{l} (n_i^{(0)} \alpha_i^{(0)} + n_i^{(1)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}).$$

That is,

$$\sum_{i=1}^{l} (k_i^{(1)} \alpha_i^{(0)} + k_i^{(0)} \alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}). \tag{9}$$

By (8), (9), $k_i^{(0)} + k_i^{(1)} = k_i$ and $\alpha_i^{(0)} + \alpha_i^{(1)} = \alpha_i$, we have

$$\sum_{i=1}^{l} k_i \alpha_i \le 1 + O((\log\log x)^{-1/2}).$$

This completes the proof of Lemma 7.

Lemma 8. *Let*

$$\sum_{i=1}^{l} k_{ij} \alpha_i \le 1 + O((\log\log x)^{-1/2}), \quad j = 1, \ldots, r,$$

*be $r$ relations obtained by using Lemma 7 (not necessarily from the same $\{r_{ij}\}$). Let $\beta_1, \ldots, \beta_l$, $\delta_1, \ldots, \delta_r$ be nonnegative real numbers with*

$$\sum_{i=1}^{t} \beta_i \le \sum_{i=1}^{t} \sum_{j=1}^{r} \delta_j k_{ij}, \quad t = 1, \ldots, l. \tag{10}$$

*Then*

$$\sum_{i=1}^{l} \beta_i \alpha_i \le \sum_{j=1}^{r} \delta_j + O\Big(\sum_{j=1}^{r} \delta_j (\log\log x)^{-1/2}\Big),$$

*where $O$ depends only on $\mathcal{I}$.*

Proof. By (10) and $k_{ij} = k_{ij}^{(0)} + k_{ij}^{(1)}$ we have

$$\sum_{i=1}^{t} \beta_i \le \sum_{i=1}^{t} \sum_{j=1}^{r} \delta_j k_{ij}^{(0)} + \sum_{i=1}^{t} \sum_{j=1}^{r} \delta_j k_{ij}^{(1)}, \quad t = 1, \ldots, l.$$

By the argument used in the proof of Lemma 5 there exist nonnegative real numbers $\beta_i^{(v)}$ $(1 \le i \le l;\ v = 0,1)$ such that

$$\sum_{i=1}^{t} \beta_i^{(v)} \le \sum_{i=1}^{t}\sum_{j=1}^{r} \delta_j k_{ij}^{(v)}, \qquad t = 1,\ldots,l;\ v = 0,1.$$

By the argument used in the proof of Lemma 7 we have for $j = 1,\ldots,r$,

$$\sum_{i=1}^{l} (k_i^{(0)}\alpha_i^{(0)} + k_i^{(1)}\alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}),$$

$$\sum_{i=1}^{l} (k_i^{(1)}\alpha_i^{(0)} + k_i^{(0)}\alpha_i^{(1)}) \le \frac{1}{2} + O((\log\log x)^{-1/2}).$$

By Lemma 6 we have

$$\sum_{i=1}^{l} (\beta_i^{(0)}\alpha_i^{(0)} + \beta_i^{(1)}\alpha_i^{(1)}) \le \frac{1}{2}\sum_{j=1}^{r}\delta_j + O\Big(\sum_{j=1}^{r}\delta_j(\log\log x)^{-1/2}\Big),$$

$$\sum_{i=1}^{l} (\beta_i^{(1)}\alpha_i^{(0)} + \beta_i^{(0)}\alpha_i^{(1)}) \le \frac{1}{2}\sum_{j=1}^{r}\delta_j + O\Big(\sum_{j=1}^{r}\delta_j(\log\log x)^{-1/2}\Big).$$

Since $\alpha_i = \alpha_i^{(0)} + \alpha_i^{(1)}$ and $\beta_i = \beta_i^{(0)} + \beta_i^{(1)}$, we have

$$\sum_{i=1}^{l} \beta_i\alpha_i \le \sum_{j=1}^{r}\delta_j + O\Big(\sum_{j=1}^{r}\delta_j(\log\log x)^{-1/2}\Big).$$

This completes the proof of Lemma 8.

**3. The asymptotic formula for $|A_x|$.** Let $L$ and $S$ be suitable large integers and

$$q = 2^{1/(2L)}, \qquad I_i = (q^i,\ q^{i+1}], \qquad T = 2LS - 1.$$

For positive real numbers $\alpha$, $\beta$, let

$$\begin{aligned} B(\alpha,\beta) &= \{a : a \in \mathbb{Z},\ 1 \le a \le \alpha\beta\} \\ &\quad \cup \{a : a \in \mathbb{Z},\ -\min\{\alpha\beta, \alpha^{-1}\beta - 1\} \le a \le 0\}, \end{aligned}$$

$$A_{ij} = \begin{cases} B(q^j, q^i) & \text{if } i \ge j, \\ \emptyset & \text{if } i < j. \end{cases}$$

In the following we make the *convention* that $\sum_{a\in\emptyset} h(a) = 0$ for any function $h(t)$.

LEMMA 9. *Let* $0 < \alpha \le \min\{\beta,\gamma\}$. *If* $a \in B(\alpha,\beta)$ *and* $b \in B(\alpha,\gamma)$, *then* $|a-b| \le \beta\gamma$.

Proof. If $ab \ge 0$, then $|a-b| \le \max\{|a|,|b|\} \le \max\{\alpha\beta,\alpha\gamma\} \le \beta\gamma$.

Now we assume that $ab < 0$. Without loss of generality, we may assume that $a > 0$ and $b < 0$. In this case we have $\alpha\beta \geq 1$ and $\alpha\gamma \geq 1$. Thus

$$\begin{aligned}|a-b| &= a-b \leq \alpha\beta + \min\{\alpha\gamma, \alpha^{-1}\gamma - 1\}\\ &\leq \alpha\beta + \alpha^{-1}\gamma - 1 = \beta\gamma + (\beta - \alpha^{-1})(\alpha-\gamma) \leq \beta\gamma.\end{aligned}$$

This completes the proof of Lemma 9.

To use Lemma 8, let

$$\begin{aligned}\alpha &= (10-7\sqrt{2})/32,\\ k_{ij} &= |A_{ij} \setminus A_{(i-1)j}|,\ -T \leq i \leq T, -T \leq j \leq L-1,\\ k_{iL} &= 0\ (-T \leq i \leq T,\ i \neq 0),\ k_{0L} = 1,\\ \beta_i &= q^i(q-1),\ -T \leq i \leq L-1,\\ \beta_i &= (1+\alpha)q^i(q-1),\ L \leq i \leq T,\\ \delta_j &= q^j(q-1),\ -T \leq j \leq -1,\\ \delta_j &= \tfrac{1}{2}(q-1)(q^j - q^{-j-1}),\ 0 \leq j \leq L-1,\\ \delta_L &= 1 - q^{-1}.\end{aligned}$$

LEMMA 10. *For $-T \leq j \leq L$, we have*

$$\sum_{-T\leq i\leq T} k_{ij}\alpha_i \leq 1 + O((\log\log x)^{-1/2}),$$

*where $O$ depends only on $L$ and $S$.*

Proof. The inequality

$$\sum_{i=-T}^{T} k_{iL}\alpha_i \leq 1 + O((\log\log x)^{-1/2})$$

can be deduced from Lemma 7 by taking

$$\{r_{ij} : j = 1, \ldots, k_{iL};\ i = -T, \ldots, T\} = \{r_{01} = 1\}.$$

Now we assume that $-T \leq j \leq L-1$. If $i < j$, then

$$|A_{ij} \setminus A_{(i-1)j}| = \emptyset.$$

If $j \leq i \leq i' \leq T$ and

$$a \in A_{ij} \setminus A_{(i-1)j}, \quad b \in A_{i'j} \setminus A_{(i'-1)j},$$

then by Lemma 9 we have $|a-b| \leq q^{i+i'}$. Then Lemma 10 follows from Lemma 7.

LEMMA 11. *There exists a $L_0$ such that if $L \geq L_0$, then*

$$\sum_{i=-T}^{t} \beta_i \leq \sum_{i=-T}^{t}\sum_{j=-T}^{L} \delta_j k_{ij}, \quad t = -T, -T+1, \ldots, T.$$

P r o o f. For convenience let

$$f(t)=\sum_{i=-T}^{t}\sum_{j=-T}^{L}\delta_j k_{ij}.$$

That is,

$$f(t)=\sum_{-T\le i\le t}\Big(\sum_{-T\le j\le -1}|A_{ij}\setminus A_{(i-1)j}|q^j(q-1)\\ +\frac{1}{2}\sum_{0\le j\le L-1}|A_{ij}\setminus A_{(i-1)j}|(q-1)(q^j-q^{-j-1})\Big)+\varepsilon_t(1-q^{-1}),$$

where $\varepsilon_t=0$ if $t\le -1$, and $\varepsilon_t=1$ if $t\ge 0$. Since

$$A_{ij}\supseteq A_{(i-1)j},\qquad A_{(-T-1)j}=\emptyset,\qquad j\ge -T,$$

we have

$$\sum_{i=-T}^{t}|A_{ij}\setminus A_{(i-1)j}|=|A_{tj}|,\qquad j\ge -T.$$

Hence

$$\begin{aligned}(11)\quad f(t)&=\sum_{-T\le j\le -1}\sum_{-T\le i\le t}|A_{ij}\setminus A_{(i-1)j}|q^j(q-1)\\&\quad+\frac{1}{2}\sum_{0\le j\le L-1}\sum_{-T\le i\le t}|A_{ij}\setminus A_{(i-1)j}|(q-1)(q^j-q^{-j-1})\\&\quad+\varepsilon_t(1-q^{-1})\\&=\sum_{-T\le j\le -1}|A_{tj}|q^j(q-1)\\&\quad+\frac{1}{2}\sum_{0\le j\le L-1}|A_{tj}|(q-1)(q^j-q^{-j-1})+\varepsilon_t(1-q^{-1}).\end{aligned}$$

Case 1: $-T\le t\le -1$. Then by (11) we have

$$f(t)\ge\sum_{-T\le j\le t}|A_{tj}|q^j(q-1)\ge\sum_{-T\le j\le t}q^j(q-1)\ge\sum_{-T\le i\le t}\beta_i.$$

Case 2: $0\le t\le L-1$. Then by (11) we have

$$\begin{aligned}f(t)&\ge\sum_{-T\le j\le -t-1}|A_{tj}|q^j(q-1)+\sum_{-t\le j\le -1}|A_{tj}|q^j(q-1)\\&\quad+\frac{1}{2}\sum_{0\le j\le t}|A_{tj}|(q-1)(q^j-q^{-j-1})+1-q^{-1}\\&\ge\sum_{-T\le j\le -t-1}q^j(q-1)+2\sum_{-t\le j\le -1}q^j(q-1)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0\le j\le t}(q-1)(q^j-q^{-j-1})+1-q^{-1}\\
&\ge q^{t+1}-q^{-T}+(1-q^{-1})(1-q^{-t})\\
&\ge q^{t+1}-q^{-T}\ge \sum_{-T\le i\le t}\beta_i.
\end{aligned}
$$

CASE 3: $L\le t\le 2L-1$. Then by (11) we have

$$
\begin{aligned}
f(t)\ge{}& \sum_{-T\le j\le -t-1}|A_{tj}|q^j(q-1)+\sum_{-t\le j\le t-2L}|A_{tj}|q^j(q-1)\\
&+\sum_{t-2L<j\le -1}|A_{tj}|q^j(q-1)+\frac12\sum_{0\le j<2L-t}|A_{tj}|(q-1)(q^j-q^{-j-1})\\
&+\frac12\sum_{2L-t\le j\le L-1}|A_{tj}|(q-1)(q^j-q^{-j-1})+1-q^{-1}\\
\ge{}& \sum_{-T\le j\le -t-1}q^j(q-1)+3\sum_{-t\le j\le t-2L}q^j(q-1)+2\sum_{t-2L<j\le -1}q^j(q-1)\\
&+\sum_{0\le j<2L-t}(q-1)(q^j-q^{-j-1})+\frac32\sum_{2L-t\le j\le L-1}(q-1)(q^j-q^{-j-1})\\
&+1-q^{-1}\\
\ge{}& q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)+\frac94\sqrt2+\sqrt2\alpha+1-q^{-1}\\
&+q^t\Big(-\frac12 q-\frac14-\alpha q\Big)-3q^{-t}\\
\ge{}& q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)+\frac94\sqrt2+\sqrt2\alpha+1-q^{-1}\\
&+\min\Big\{q^L\Big(-\frac12 q-\frac14-\alpha q\Big)-3q^{-L},\,q^{2L-1}\Big(-\frac12 q-\frac14-\alpha q\Big)\\
&\qquad\qquad -3q^{-2L+1}\Big\}\\
\ge{}& q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)\\
\ge{}& \sum_{-T\le i\le t}\beta_i.
\end{aligned}
$$

The last inequality but one holds for all sufficiently large $L$ (note that $q=2^{1/(2L)}$).

CASE 4: $T\ge t\ge 2L$. Then by (11) we have

$$
f(t)\ge \sum_{-T\le j\le -1}|A_{tj}|q^j(q-1)+\frac12\sum_{0\le j\le L-1}|A_{tj}|(q-1)(q^j-q^{-j-1}) \tag{12}
$$

$$= \sum_{-T\le j\le -1} q^j(q-1)\Big(\sum_{1\le a\le q^{j+t}} 1 + \sum_{-\min\{q^{j+t},q^{t-j}-1\}\le a\le 0} 1\Big)$$
$$+\frac{1}{2}\sum_{0\le j\le L-1}(q-1)(q^j-q^{-j-1})\Big(\sum_{1\le a\le q^{j+t}} 1 + \sum_{1-q^{t-j}\le a\le 0} 1\Big).$$

Now we estimate each part in (12):

$$\sum_{-T\le j\le -1} q^j(q-1)\sum_{1\le a\le q^{j+t}} 1 = \sum_{1\le a\le q^{t-1}}\ \sum_{\substack{-T\le j\le -1\\ j\ge 2L\log_2 a-t}} q^j(q-1)$$
$$\ge \sum_{1\le a\le q^{t-1}}(1-q^{2L\log_2 a-t+1})$$
$$\ge \sum_{1\le a\le q^{t}}(1-q^{2L\log_2 a-t+1})$$
$$\ge \sum_{1\le a\le q^{t}}(1-aq^{-t+1})$$
$$\ge [q^t]-\frac{1}{2}q^{-t+1}[q^t]([q^t]+1);$$

$$\sum_{-T\le j\le -1} q^j(q-1)\sum_{-\min\{q^{t+j},q^{t-j}-1\}\le a\le 0} 1$$
$$= \sum_{-T\le j\le -1} q^j(q-1)\sum_{1\le a\le \min\{q^{t+j}+1,q^{t-j}\}} 1$$
$$\ge \sum_{1\le a\le 1/2+q^t}\ \sum_{\substack{-T\le j\le -1\\ \min\{q^{t+j}+1,q^{t-j}\}\ge a}} q^j(q-1)$$
$$=: \sum_{2\le a\le q^t}\ \sum_{\substack{-T\le j\le -1\\ \min\{q^{t+j}+1,q^{t-j}\}\ge a}} q^j(q-1)+\delta(t)+1-q^{-T}$$
$$= \sum_{2\le a\le q^t}\ \sum_{2L\log_2(a-1)-t\le j\le -1} q^j(q-1)+\delta(t)+1-q^{-T}$$
$$\ge \sum_{2\le a\le q^t}(1-q^{2L\log_2(a-1)-t+1})+\delta(t)+1-q^{-T}$$
$$\ge \sum_{2\le a\le q^t}(1-(a-1)q^{-t+1})+\delta(t)+1-q^{-T}$$
$$\ge [q^t]-\frac{1}{2}q^{-t+1}[q^t]([q^t]-1)-q^{-T}+\delta(t);$$

$$\sum_{0\le j\le L-1}(q-1)(q^j-q^{-j-1})\Big(\sum_{1\le a\le q^{j+t}}1+\sum_{1-q^{t-j}\le a\le 0}1\Big)$$
$$\ge \sum_{0\le j\le L-1}(q-1)(q^j-q^{-j-1})(q^{j+t}+q^{t-j}-2)$$
$$\ge \frac{q^t}{1+q}\left(1-\frac{1}{2}q\right)+L(q-1)q^t(1-q^{-1})+4-3\sqrt{2}$$
$$\ge \frac{q^t}{1+q}\left(1-\frac{1}{2}q\right)+4-3\sqrt{2}.$$

Thus by these estimates and (12) we have

$$\begin{aligned}(13)\qquad f(t)&\ge [q^t]-\frac{1}{2}q^{-t+1}[q^t]([q^t]+1)\\&\quad+[q^t]-\frac{1}{2}q^{-t+1}[q^t]([q^t]-1)-q^{-T}+\delta(t)\\&\quad+\frac{1}{2}\left(1-\frac{1}{2}q\right)\frac{q^t}{1+q}+2-\frac{3}{2}\sqrt{2}\\&\ge 2[q^t]-q^{-t+1}[q^t]^2-q^{-T}+\delta(t)\\&\quad+\frac{1}{2}\left(1-\frac{1}{2}q\right)\frac{q^t}{1+q}+2-\frac{3}{2}\sqrt{2}.\end{aligned}$$

If $q^t-1/2\le [q^t]\le q^t$ and $q^t\ge 2$ (i.e. $t\ge 2L$), then by (13) and $\delta(t)\ge 0$ we have

$$\begin{aligned}f(t)&\ge \min\left\{2q^t-q^{-t+1}q^{2t},2\left(q^t-\frac{1}{2}\right)-q^{-t+1}\left(q^t-\frac{1}{2}\right)^2\right\}\\&\quad-q^{-T}+\frac{1}{2}\left(1-\frac{1}{2}q\right)\frac{q^t}{1+q}+2-\frac{3}{2}\sqrt{2}\\&\ge q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)+q^t\left(2-2q+\frac{1}{2}\left(1-\frac{1}{2}q\right)\frac{1}{1+q}-\alpha q\right)\\&\quad+2-\frac{3}{2}\sqrt{2}+\sqrt{2}\alpha-\frac{1}{4q^t}q\\&\ge q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)\\&\quad+2\left(2-2q+\frac{1}{2}\left(1-\frac{1}{2}q\right)\frac{1}{1+q}-\alpha q\right)+2-\frac{3}{2}\sqrt{2}+\sqrt{2}\alpha-\frac{1}{8}q\\&\ge q^{t+1}-q^{-T}+\alpha(q^{t+1}-q^L)\\&\ge \sum_{-T\le i\le t}\beta_i.\end{aligned}$$

The last three inequalities hold for all sufficiently large $L$. If $q^t-1<[q^t]<$

$q^t - 1/2$ and $q^t \ge 2$ (i.e. $t \ge 2L$), then

$$\begin{aligned}\delta(t) &= \sum_{q^t < a \le 1/2 + q^t} \sum_{\substack{-T \le j \le -1 \\ \min\{q^{t+j}+1, q^{t-j}\} \ge a}} q^j(q-1) \\ &\ge \sum_{\min\{q^{t+j}+1, q^{t-j}\} \ge [q^t]+1} q^j(q-1) \\ &\ge \sum_{2L\log_2[q^t] - t \le j \le t - 2L\log_2([q^t]+1)} q^j(q-1) \\ &\ge q^{t-2L\log_2([q^t]+1)} - q^{2L\log_2[q^t]-t+1} \ge \frac{1}{[q^t]+1} q^t - [q^t]q^{-t+1}.\end{aligned}$$

In this subcase, by (13) we have

$$\begin{aligned}f(t) &\ge 2[q^t] - q^{-t+1}[q^t]^2 + \frac{1}{[q^t]+1}q^t - [q^t]q^{-t+1} \\ &\quad - q^{-T} + \frac{1}{2}\left(1 - \frac{1}{2}q\right)\frac{q^t}{1+q} + 2 - \frac{3}{2}\sqrt{2} \\ &\ge \min\left\{2(q^t-1) - q^{-t+1}(q^t-1)^2 + \frac{1}{q^t-1+1}q^t - (q^t-1)q^{-t+1},\right. \\ &\quad \left. 2\left(q^t - \frac{1}{2}\right) - q^{-t+1}\left(q^t - \frac{1}{2}\right)^2 + \frac{1}{q^t - 1/2 + 1}q^t - \left(q^t - \frac{1}{2}\right)q^{-t+1}\right\} \\ &\quad - q^{-T} + \frac{1}{2}\left(1 - \frac{1}{2}q\right)\frac{q^t}{1+q} + 2 - \frac{3}{2}\sqrt{2} \\ &\ge q^{t+1} - q^{-T} + \alpha(q^{t+1} - q^L) \\ &\quad + q^t\left(\frac{1}{2}\left(1 - \frac{1}{2}q\right)\frac{1}{1+q} + 2 - 2q - \alpha q\right) - \frac{1}{4q^{t-1}} + 2 - \frac{3}{2}\sqrt{2} + \alpha\sqrt{2} \\ &\ge q^{t+1} - q^{-T} + \alpha(q^{t+1} - q^L) \\ &\quad + 2\left(\frac{1}{2}\left(1 - \frac{1}{2}q\right)\frac{1}{1+q} + 2 - 2q - \alpha q\right) - \frac{1}{8}q + 2 - \frac{3}{2}\sqrt{2} + \alpha\sqrt{2} \\ &\ge q^{t+1} - q^{-T} + \alpha(q^{t+1} - q^L) \\ &\ge \sum_{-T \le i \le t} \beta_i.\end{aligned}$$

The last three inequalities hold for all sufficiently large $L$. This completes the proof of Lemma 11.

LEMMA 12. *Let $D$ be a positive integer and $x \ge 2D$. Then*

$$|(\sqrt{Dx}, x] \cap A_x| = O\left(\frac{1}{\sqrt{D}}\sqrt{x}\right),$$

*where the $O$-constant is absolute.*

P r o o f. For any integer $k \geq 1$, since for $a \geq \sqrt{kx}$,

$$|\{a, a+1, \ldots, a+k-1\} \cap A_x| \leq 1,$$

we have

$$|[\sqrt{kx}, 2\sqrt{kx}] \cap A_x| \leq \frac{1}{k}(2\sqrt{k} - \sqrt{k})\sqrt{x} + 1 = \frac{1}{\sqrt{k}}\sqrt{x} + 1.$$

Thus

$$\begin{aligned}|(\sqrt{Dx}, x] \cap A_x| &= \sum_{1 \leq i \leq \frac{1}{2}\log_2(x/D)+1} |(2^{i-1}\sqrt{Dx}, 2^i\sqrt{Dx}] \cap A_x| \\ &\leq \sum_{1 \leq i \leq \frac{1}{2}\log_2(x/D)+1} \left(\frac{1}{2^{i-1}\sqrt{D}}\sqrt{x} + 1\right) \\ &= O\left(\frac{1}{\sqrt{D}}\sqrt{x}\right).\end{aligned}$$

This completes the proof of Lemma 12.

THEOREM 1.

$$|A_x| = \sqrt{\tfrac{9}{8}x} + o(\sqrt{x}), \qquad |(\sqrt{2x}, x] \cap A_x| = o(\sqrt{x}).$$

P r o o f. By Lemmas 10, 11 and 8 we have

$$\begin{aligned}&\sum_{-T \leq i \leq L-1} q^i(q-1)\alpha_i + (1+\alpha)\sum_{L \leq i \leq T} q^t(q-1)\alpha_i \\ &\qquad = \sum_{-T \leq i \leq T} \beta_i\alpha_i \leq \sum_{-T \leq j \leq L} \delta_j + O((\log\log x)^{-1/2}) \\ &\qquad \leq \sqrt{\tfrac{9}{8}} - q^{-T} + 1 - q^{-1} + O((\log\log x)^{-1/2}),\end{aligned}$$

where $\alpha = (10 - 7\sqrt{2})/32$. Hence

$$\begin{aligned}&|(q^{-T}\sqrt{x}, \sqrt{2x}] \cap A_x| + (1+\alpha)|(\sqrt{2x}, q^{T+1}\sqrt{x}] \cap A_x| \\ &\qquad \leq \sqrt{\tfrac{9}{8}x} - q^{-T}\sqrt{x} + (1 - q^{-1})\sqrt{x} + O(\sqrt{x}(\log\log x)^{-1/2}).\end{aligned}$$

So

$$\begin{aligned}&|[1, \sqrt{2x}] \cap A_x| + (1+\alpha)|(\sqrt{2x}, 2^S\sqrt{x}] \cap A_x| \\ &\qquad \leq \sqrt{\tfrac{9}{8}x} + (1 - q^{-1})\sqrt{x} + O_1(\sqrt{x}(\log\log x)^{-1/2}),\end{aligned}$$

where $O_1$ depends only on $L$ and $S$. By Lemma 12 we have

$$|(2^S\sqrt{x}, x] \cap A_x| = O_2\left(\frac{1}{2^S}\sqrt{x}\right),$$

where the $O_2$-constant is absolute. Therefore

$$|[1,\sqrt{2x}]\cap A_x|+(1+\alpha)|(\sqrt{2x},x]\cap A_x| \le \sqrt{\tfrac{9}{8}x}+(1-q^{-1})\sqrt{x}+O_1(\sqrt{x}(\log\log x)^{-1/2})+O_2\left(\frac{1}{2^S}\sqrt{x}\right),$$

where $O_1$ is independent of $x$, and $O_2$ is independent of $S$ and $x$. Thus

$$|[1,\sqrt{2x}]\cap A_x|+(1+\alpha)|(\sqrt{2x},x]\cap A_x|\le \sqrt{\tfrac{9}{8}x}+o(\sqrt{x}).$$

That is,

$$|A_x|+\alpha|(\sqrt{2x},x]\cap A_x|\le \sqrt{\tfrac{9}{8}x}+o(\sqrt{x}).$$

Since $|A_x|\ge |B_x|=\sqrt{\tfrac{9}{8}x}+O(1)$, we have

$$|A_x|=\sqrt{\tfrac{9}{8}x}+o(\sqrt{x}),\qquad |(\sqrt{2x},x]\cap A_x|=o(\sqrt{x}).$$

This completes the proof of Theorem 1.

**4. Proof of the Theorem.** First we prove

THEOREM 2.

$$\left|\left(\sqrt{\tfrac{1}{2}x},\sqrt{2x}\,\right]\cap A_x\cap(2\mathbb{Z}+1)\right|=o(\sqrt{x}).$$

Proof. Let $L$ be an integer and

$$q=2^{1/(2L)},\qquad I_i=(q^i,q^{i+1}],\qquad -L\le i\le L.$$

Let

$$\{r_{iu}:u=1,\ldots,k_i;\ i=-L,-L+1,\ldots,L\}_j = \{r_{j,1}=0,\ r_{-j,1}=-1,\ r_{-j,2}=1\},\qquad -L\le j\le -1.$$

Then by Lemma 3 we have

$$\alpha_j^{(0)}+2\alpha_{-j}^{(1)}\le \frac{1}{2}+O((\log\log x)^{-1/2}),\qquad -L\le j\le -1.$$

Let

$$\{r_{iu}:u=1,\ldots,k_i;\ i=-L,-L+1,\ldots,L\}_j = \{r_{-j,1}=-1,\ r_{j,1}=0,\ r_{j,2}=1\},\qquad 1\le j\le L.$$

Then by Lemma 3 we have

$$\alpha_{-j}^{(1)}+\alpha_j^{(0)}+\alpha_j^{(1)}\le \frac{1}{2}+O((\log\log x)^{-1/2}),\qquad 1\le j\le L.$$

Let

$$\{r_{iu}:u=1,\ldots,k_i;\ i=-L,-L+1,\ldots,L\} = \{r_{0,1}=-1,\ r_{0,2}=0,\ r_{0,3}=1\}.$$

Then we have

$$\alpha_0^{(0)}+2\alpha_0^{(1)}\le \frac{1}{2}+O((\log\log x)^{-1/2}).$$

To use Lemma 6, let

$$\begin{aligned} k_{ij}^{(0)} &= 0 \ (i \neq j), & k_{jj}^{(0)} &= 1, \\ k_{ij}^{(1)} &= 0 \ (i \neq j, -j), & k_{jj}^{(1)} &= 1 \ (j \geq 1), \\ k_{jj}^{(1)} &= 0 \ (j \leq -1), & k_{(-j)j}^{(1)} &= 2 \ (j \leq -1), \\ k_{(-j)j}^{(1)} &= 1 \ (j \geq 1), & k_{00}^{(1)} &= 2, \\ \delta_j &= q^j(q-1), & \beta_i^{(0)} &= q^i(q-1), \\ \alpha &= (10 - 7\sqrt{2})/32, & \beta_i^{(1)} &= (1+\alpha)q^i(q-1). \end{aligned}$$

Then

$$\sum_{-L \leq i \leq L} (k_{ij}^{(0)} \alpha_i^{(0)} + k_{ij}^{(1)} \alpha_i^{(1)}) = \begin{cases} \alpha_j^{(0)} + 2\alpha_{-j}^{(1)} & \text{if } -L \leq j \leq 0, \\ \alpha_{-j}^{(1)} + \alpha_j^{(0)} + \alpha_j^{(1)} & \text{if } 1 \leq j \leq L. \end{cases}$$

Hence

$$\sum_{-L \leq i \leq L} (k_{ij}^{(0)} \alpha_i^{(0)} + k_{ij}^{(1)} \alpha_i^{(1)}) \leq \frac{1}{2} + O((\log\log x)^{-1/2}). \tag{14}$$

Let

$$g^{(v)}(t) = \sum_{-L \leq i \leq t} \sum_{-L \leq j \leq L} \delta_j k_{ij}^{(v)}, \qquad v = 0, 1.$$

Then

$$g^{(0)}(t) = \sum_{i=-L}^{t} \delta_i k_{ii}^{(0)} = \sum_{i=-L}^{t} \delta_i = \sum_{i=-L}^{t} \beta_i^{(0)}, \qquad -L \leq t \leq L. \tag{15}$$

Now we show that

$$g^{(1)}(t) \geq \sum_{i=-L}^{t} \beta_i^{(1)}, \qquad -L \leq t \leq L.$$

For $-L \leq t \leq -1$ we have (note that $q^L = \sqrt{2}$)

$$\begin{aligned} g^{(1)}(t) &= \sum_{i=-L}^{t} \delta_{-i} k_{i(-i)}^{(1)} = \sum_{i=-L}^{t} \delta_{-i} = q^{L+1} - q^{-t} \\ &= (1+\alpha)(q^{t+1} - q^{-L}) + q^{L+1} + (1+\alpha)q^{-L} \\ &\quad - q^{-t} - (1+\alpha)q^{t+1} \\ &\geq (1+\alpha)(q^{t+1} - q^{-L}) + q^{L+1} + (1+\alpha)q^{-L} \\ &\quad - \max\{q^L + (1+\alpha)q^{-L+1}, q + 1 + \alpha\} \\ &\geq (1+\alpha)(q^{t+1} - q^{-L}) \geq \sum_{i=-L}^{t} \beta_i^{(1)}. \end{aligned} \tag{16}$$

For $0 \le t \le L$ we have

$$(17)\qquad \begin{aligned} g^{(1)}(t) &= \sum_{i=-L}^{0} \delta_{-i} k^{(1)}_{i(-i)} + \sum_{1\le i\le t} (\delta_i k^{(1)}_{ii} + \delta_{-i} k^{(1)}_{i(-i)}) \\ &= \sum_{i=-L}^{-1} \delta_{-i} + 2\delta_0 + \sum_{1\le i\le t} (\delta_i + 2\delta_{-i}) \\ &= \sum_{i=-L}^{t} \beta^{(1)}_i + q^{L+1} + (1+\alpha)q^{-L} - 2q^{-t} - \alpha q^{t+1} \\ &\ge \sum_{i=-L}^{t} \beta^{(1)}_i + q^{L+1} + (1+\alpha)q^{-L} \\ &\quad - \max\{2+\alpha q, 2q^{-L} + \alpha q^{L+1}\} \\ &\ge \sum_{i=-L}^{t} \beta^{(1)}_i. \end{aligned}$$

By (14)–(17) and Lemma 6 we have

$$\begin{aligned} &\sum_{i=-L}^{L} q^i(q-1)\alpha^{(0)}_i + \sum_{i=-L}^{L} q^i(q-1)\alpha^{(1)}_i + \alpha \sum_{i=-L}^{L} q^i(q-1)\alpha^{(1)}_i \\ &\quad = \sum_{i=-L}^{L} (\beta^{(0)}_i \alpha^{(0)}_i + \beta^{(1)}_i \alpha^{(1)}_i) \le \frac{1}{2} \sum_{j=-L}^{L} \delta_j + O((\log\log x)^{-1/2}) \\ &\quad \le \tfrac{1}{4}\sqrt{2} + \tfrac{1}{2}\sqrt{2}(q-1) + O((\log\log x)^{-1/2}). \end{aligned}$$

Hence

$$\begin{aligned} &|(q^{-L}, q^{L+1}]\sqrt{x} \cap A_x| + \alpha |(q^{-L}, q^{L+1}]\sqrt{x} \cap A_x \cap (2\mathbb{Z}+1)| \\ &\qquad\qquad \le \tfrac{1}{4}\sqrt{2x} + \tfrac{1}{2}\sqrt{2x}(q-1) + O(\sqrt{x}(\log\log x)^{-1/2}). \end{aligned}$$

It is clear that

$$|[1, q^{-L}]\sqrt{x} \cap A_x| \le \tfrac{1}{2}\sqrt{2x}.$$

So

$$\begin{aligned} &|[1, \sqrt{2x}] \cap A_x| + \alpha \big|\big(\sqrt{\tfrac{1}{2}x}, \sqrt{2x}\,\big] \cap A_x \cap (2\mathbb{Z}+1)\big| \\ &\qquad\qquad \le \sqrt{\tfrac{9}{8}x} + \tfrac{1}{2}\sqrt{2x}(q-1) + O(\sqrt{x}(\log\log x)^{-1/2}). \end{aligned}$$

Hence

$$(18)\quad |[1, \sqrt{2x}] \cap A_x| + \alpha \big|\big(\sqrt{\tfrac{1}{2}x}, \sqrt{2x}\,\big] \cap A_x \cap (2\mathbb{Z}+1)\big| \le \sqrt{\tfrac{9}{8}x} + o(\sqrt{x}).$$

By Theorem 1 we have

$$|[1, \sqrt{2x}] \cap A_x| = \sqrt{\tfrac{9}{8}x} + o(\sqrt{x}).$$

Thus by (18) we have

$$\Big|\Big(\sqrt{\tfrac{1}{2}x}, \sqrt{2x}\,\Big] \cap A_x \cap (2\mathbb{Z}+1)\Big| = o(\sqrt{x}).$$

This completes the proof of Theorem 2.

*Proof of the Theorem.* By Theorem 1 we have

$$|A_x| = |B_x| + o(\sqrt{x}) = \sqrt{\tfrac{9}{8}x} + o(\sqrt{x}).$$

By Theorems 1 and 2 we have

$$|A_x \setminus B_x| \le \Big|A_x \cap \Big[\sqrt{\tfrac{1}{2}x}, \sqrt{2x}\,\Big] \cap (2\mathbb{Z}+1)\Big| + |A_x \cap (\sqrt{2x}, x]| = o(\sqrt{x}).$$

This completes the proof of the Theorem.

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**References**

[1] S. L. G. Choi, *The largest subset in* $[1, n]$ *whose integers have pairwise l.c.m. not exceeding* $n$, Mathematika 19 (1972), 221–230.

[2] —, *The largest subset in* $[1, n]$ *whose integers have pairwise l.c.m. not exceeding* $n$, *II*, Acta Arith. 29 (1976), 105–111.

[3] P. Erdős, *Problem*, Mat. Lapok 2 (1951), 233.

[4] —, *Extremal problems in number theory*, in: Theory of Numbers, Proc. Sympos. Pure Math. 8, Amer. Math. Soc., 1965, 181–189.

[5] R. K. Guy, *Unsolved Problems in Number Theory*, 2nd ed., Springer, New York, 1994.

[6] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press, London, 1974.

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