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A generalization of Maillet and Demyanenko determinants

by

Mikihito Hirabayashi (Ishikawa)

1. Introduction. Let K be an imaginary abelian number field with conductor m and of degree $[K:\mathbb{Q}]=2n$. Let Q_K be the unit index of K and w_K the number of roots of unity in K. Then the relative class number h_K^* of K is given by

$$h_K^* = Q_K w_K \prod_{\chi} \frac{1}{2f(\chi)} \sum_{a=1}^{f(\chi)} (-\chi(a)a) \tag{1}$$

where the product $\prod_\chi$ is taken over the odd primitive characters χ of K with conductor $f(\chi)$ (see Hasse [4]).

Several ways of representing the product in (1) by a determinant are known, some of them holding for certain types of fields only (see Carlitz and Olson [1], Hazama [5] and further references in Hirabayashi [6]). Here our main concern is a unified approach to the Maillet determinant on the one hand and to the Demyanenko determinant on the other hand. This approach relies on the "b-division vector" introduced by Girstmair [3]. We obtain a relative class number formula for an arbitrary imaginary abelian number field which generalizes formulae of Girstmair [2] and [6]. Tsumura [7] also generalized both type of determinant formulae, but his generalization for the Demyanenko determinant requires the oddness of the conductor.

2. Lemma obtained by the b-division vector. In this section we state one of the results of [3] which we need to describe our results.

For $m\in\mathbb{Z}$, $m\ge 3$, $m\not\equiv 2 \bmod 4$ let $G_m=(\mathbb{Z}/m\mathbb{Z})^\times$ be the prime residue group mod m and H a subgroup of G_m of relative index $q=[G_m:H]=|G_m/H|$. We assume that H does not contain the class $\overline{-1}$, letting $\overline{x}=x+m\mathbb{Z}$ for $x\in\mathbb{Z}$.

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For a class $C \in G_m/H$ and for an integer b, $b \geq 2$, $m \nmid b$, let

$$S_C^{(b)} = \sum_{\overline{k} \in C,\, 0<k<m} \left[\frac{bk}{m}\right],$$

where $[x]$ denotes the integral part of a real number x, and let

$$S^{(b)} = \frac{1}{q} \sum_{C \in G_m/H} S_C^{(b)} \quad \text{and} \quad T_C^{(b)} = S_C^{(b)} - S^{(b)}.$$

Then we have

$$T_{-C}^{(b)} = -T_C^{(b)}, \tag{2}$$

where $-C = \{\overline{-x} \in G_m : \overline{x} \in C\}$.

Since H does not contain $\overline{-1}$, letting $n = q/2$, we can take classes $C_1, C_2, \ldots, C_n$ of G_m/H which satisfy

$$G_m/H = \{C_1, -C_1, C_2, -C_2, \ldots, C_n, -C_n\}.$$

Then $R = \{C_1, \ldots, C_n\}$ can be regarded as a complete system of representatives for $G_m/(H\{\overline{\pm 1}\})$. We assume $R \ni H$.

We call the vector $T^{(b)} = (T_C^{(b)})_{C \in R}$ the *b-division vector* of G_m/H. (For an explanation of this name, see [3].)

When a Dirichlet character χ is defined mod m, we sometimes use χ_m instead of χ. We denote by χ_f the primitive character corresponding to χ.

For a Dirichlet character $\chi = \chi_m$ let

$$B_\chi = B_{\chi_m} = \frac{1}{m} \sum_{\substack{k=1 \\ (k,m)=1}}^{m} \chi(k)k.$$

Then we have

$$B_{\chi_m} = \prod_{p \mid m} (1 - \chi_f(p)) \cdot B_{\chi_f},$$

where the product $\prod_{p\mid m}$ is taken over the primes p dividing m.

For integers m and b we let $m_b = (m, b)$ be the greatest common divisor of m and b, and let $m' = m/m_b$, $b' = b/m_b$.

Lemma 1 ([3, (17)–(20)]). *Let the notation be as above. Then, for an odd character $\chi = \chi_m$, we have*

$$\frac{1}{2} c_\chi(b) B_{\overline{\chi}_f} = \sum_{C \in R} \overline{\chi}(C) T_C^{(b)}$$

where $\overline{\chi}$ is the complex conjugate character of χ, i.e., $\overline{\chi}(C) = \overline{\chi(C)}$ and

$c_\chi(b)$ is defined by

$$c_\chi(b) = \begin{cases} b\prod_{p|m}(1-\overline{\chi}_f(p)) & \text{if } f(\chi)\nmid m',\\ m_b\prod_{p|m'}(1-\overline{\chi}_f(p))\cdot c' & \text{if } f(\chi)\,|\,m',\end{cases}$$

with

$$c' = b' \prod_{p|m,\, p\nmid m'} (1-\overline{\chi}_f(p)) - \chi_{m'}(b') \prod_{p|m,\, p\nmid m'} \left(1-\frac{1}{p}\right),$$

$\chi_{m'}$ being the Dirichlet character mod m' belonging to χ.

As usual, we let $\prod_{p|m,\, p\nmid m'}$ be equal to 1 if there is no prime p such that $p\,|\,m$ and $p\nmid m'$.

3. Results. Let K be an imaginary abelian number field with conductor m. Let H be the subgroup of $G_m = (\mathbb{Z}/m\mathbb{Z})^\times$ corresponding to K. Let X^- be the set of odd characters defined mod m of K. Put

$$c_K^*(b) = \prod_{\chi\in X^-} c_\chi(b).$$

THEOREM. *Let K be an imaginary abelian number field with conductor m and of degree $2n$. Let $b\ge 2$ be an integer not divisible by m. Then*

$$c_K^*(b)h_K^* = (-1)^n Q_K w_K \det(T^{(b)}_{C_1C_2^{-1}})_{C_1,C_2\in R} \tag{3}$$

$$= \pm Q_K w_K \det(T^{(b)}_{C_1C_2})_{C_1,C_2\in R}. \tag{4}$$

When $b = m+1$ and $b = 2$, we have the following Corollary, where the determinants $\det(T^{(b)}_{C_1C_2^{-1}})_{C_1,C_2\in R}$ turn out to be the Maillet determinant and the Demyanenko determinant, respectively.

For $c\in\mathbb{Z}, (c,m) = 1$ we let c^{-1} be an integer such that $cc^{-1}\equiv 1 \bmod m$ and let

$$C(c) = |\overline{c}H \cap \{a \bmod m : 1\le a < m/2\}|.$$

Let

$$g_K^* = \prod_{\chi\in X^-}\prod_{p|m}(1-\chi_f(p)), \qquad F_K = \prod_{\chi\in X^-}(2-\chi_f(2))$$

and

$$e_K = \begin{cases} 1 & \text{if } m \text{ is odd, or if } m \text{ is even and } \chi((m-2)/2) = +1 \text{ for every } \chi\in X^-,\\ 0 & \text{otherwise.}\end{cases}$$

COROLLARY. *Let K be an imaginary abelian number field with conductor m and of degree $2n$. Then we have the following formulae of* [2] *and* [6], *respectively*:

$$(5)\quad g_K^* h_K^* = (-1)^n \frac{Q_K w_K}{m^n} \det\Bigg(\sum_{\overline{k} \in C_1 C_2^{-1},\, 0<k<m} \Big(k - \frac{m}{2} \Big) \Bigg)_{C_1, C_2 \in R},$$

$$(6)\qquad e_K g_K^* h_K^* = (-1)^n \frac{Q_K w_K}{F_K} \det\Bigg(\frac{\varphi(m)}{2q} - C(c_1 c_2^{-1}) \Bigg)_{C_1, C_2 \in R},$$

where φ is Euler's totient function and $C_i = \overline{c}_i H$ for $i = 1, 2$.

4. Proofs of Theorem and Corollary

Proof of Theorem. By the formula (1) and Lemma 1 we have

$$\begin{aligned} c_K^*(b) h_K^* &= Q_K w_K \prod_{\chi \in X^-} \Big(-\frac{1}{2} c_\chi(b) B_{\overline{\chi}_f} \Big) \\ &= (-1)^n Q_K w_K \prod_{\chi \in X^-} \sum_{C \in R} \overline{\chi}(C) T_C^{(b)}. \end{aligned}$$

Here we fix *one* odd character χ_1 of K. Then

$$\begin{aligned} \prod_{\chi \in X^-} \sum_{C \in R} \overline{\chi}(C) T_C^{(b)} &= \prod_{\chi \in X^-} \sum_{C \in R} \chi(C) T_C^{(b)} \\ &= \prod_{\chi_0 \in X^+} \sum_{C \in R} (\chi_0(C) \cdot \chi_1(C) T_C^{(b)}), \end{aligned}$$

where X^+ is the group of even characters of K.

Noting that R is regarded as a complete system of representatives of the group $G_m/(H\{\overline{\pm 1}\})$ and that X^+ is the group of characters of $G_m/(H\{\overline{\pm 1}\})$, by the group determinant formula (cf. [4, p. 23]) we have

$$\begin{aligned} \prod_{\chi \in X^-} \sum_{C \in R} \overline{\chi}(C) T_C^{(b)} &= \det(\chi_1(C_1 C_2^{-1}) T_{C_1 C_2^{-1}}^{(b)})_{C_1, C_2 \in R} \\ &= \det(T_{C_1 C_2^{-1}}^{(b)})_{C_1, C_2 \in R}. \end{aligned}$$

Since $\{C^{-1} \in G_m/H : C \in R\}$ is also a complete system of representatives of $G_m/(H\{\overline{\pm 1}\})$, by the formula (2) we have

$$\det(T_{C_1 C_2^{-1}}^{(b)})_{C_1, C_2 \in R} = \pm \det(T_{C_1 C_2}^{(b)})_{C_1, C_2 \in R}.$$

Thus we get the desired formulae (3) and (4). ∎

For the proof of the Corollary we need

LEMMA 2. *Suppose that the conductor m of an imaginary abelian number field K is even. Then, for an odd character χ of K, $\chi((m-2)/2) = +1$ if and only if the 2-part of the conductor $f(\chi)$ coincides with that of m.*

P r o o f. If $f(\chi)$ divides $m/2$, then $\chi((m-2)/2) = \chi(-1) = -1$.

Conversely, we assume that the 2-part of the conductor $f(\chi)$ coincides with that of m. Let the character χ be decomposed into the characters ξ_1 and ξ_2 with 2-power conductor and odd conductor, respectively: $\chi = \xi_1\xi_2$.

If $4 \parallel f(\chi)$, then $\xi_1((m-2)/2) = \xi_2((m-2)/2) = +1$ and therefore $\chi((m-2)/2) = +1$.

If $8 \mid f(\chi)$, then $\xi_1((m-2)/2) = \xi_2((m-2)/2) = -1$ or $+1$ according as $\xi_1(-1) = +1$ or -1. Hence we have $\chi((m-2)/2) = +1$. ∎

Proof of Corollary. First we consider the case of $b = m+1$. As shown in [3], we have

$$S^{(b)} = \frac{m\varphi(m)}{2q} \quad \text{and} \quad S_C^{(m+1)} = \sum_{\overline{k}\in C,\, 0<k<m} k.$$

Therefore,

$$T_C^{(m+1)} = S_C^{(m+1)} - S^{(m+1)} = \sum_{\overline{k}\in C,\, 0<k<m} \left(k - \frac{m}{2}\right). \tag{7}$$

On the other hand, since $m' = m$ and $b' = b$, by Lemma 1 we have

$$c_\chi(b) = m_b \prod_{p\mid m'} (1 - \overline{\chi}_f(p)) \cdot c' = m \prod_{p\mid m} (1 - \overline{\chi}_f(p)).$$

Hence $c_K^*(b) = m^n g_K^*$.

Thus, by the formulae (3) and (7) we obtain

$$m^n g_K^* h_K^* = (-1)^n Q_K w_K \det\left(\sum_{\overline{k}\in C_1C_2^{-1},0<k<m} \left(k - \frac{m}{2}\right)\right)_{C_1,C_2\in R},$$

which is our desired formula (5).

Secondly we consider the case of $b = 2$. By [3], we have

$$T_C^{(2)} = S_C^{(2)} - \frac{\varphi(m)}{2q}$$

and

$$S_C^{(2)} = |\{\overline{k} \in C : m/2 < k < m\}| = \frac{\varphi(m)}{q} - C(c),$$

where $C = \overline{c}H$. Consequently, we have

$$T_C^{(2)} = \frac{\varphi(m)}{2q} - C(c) \quad \text{for } C = \overline{c}H. \tag{8}$$

Now we let m be odd. Since $m' = m$, $b' = 2$, it follows from Lemma 1 that

$$c_\chi(b) = \prod_{p\mid m} (1 - \overline{\chi}_f(p)) \cdot (2 - \chi(2))$$

and hence $c_K^*(b) = g_K^* F_K$.

Thus, by the formulae (3) and (8) we obtain

$$F_K g_K^* h_K^* = (-1)^n Q_K w_K \det\left(\frac{\varphi(m)}{2q} - C(c_1 c_2^{-1})\right)_{C_1, C_2 \in R},$$

where $C_1 = \overline{c}_1 H$ and $C_2 = \overline{c}_2 H$.

Because of the oddness of m we have $e_K = 1$. Thus we get the formula (6) under this condition.

Next we let m be even. Then we have $m' = m/2$, $b' = 1$.

If for some odd character χ of K the conductor $f(\chi)$ divides m', then $\chi((m-2)/2) = -1$ and so $e_K = 0$.

On the other hand, by Lemma 1 we have

$$c_\chi(b) = 2 \prod_{p \mid m'} (1 - \overline{\chi}_f(p)) \cdot c'$$

and $c' = 0$. Hence, $c_K^*(b) = 0$. Then we see by the Theorem that $\det(T^{(2)}_{C_1 C_2^{-1}})$ $= 0$. Thus we have proved the formula (6) under this condition.

Finally, we suppose that for every $\chi \in X^-$ the 2-part of the conductor $f(\chi)$ coincides with that of m. Then it follows from Lemma 1 that

$$c_\chi(b) = 2 \prod_{p \mid m} (1 - \overline{\chi}_f(p)) \qquad \text{for every } \chi \in X^-$$

and hence $c_K^*(b) = 2^n g_K^*$. Consequently, from (3) and (8) we obtain

$$2^n g_K^* h_K^* = (-1)^n Q_K w_K \det\left(\frac{\varphi(m)}{2q} - C(c_1 c_2^{-1})\right)_{C_1, C_2 \in R},$$

where $C_1 = \overline{c}_1 H$ and $C_2 = \overline{c}_2 H$.

By Lemma 2 we have $e_K = 1$. And under this condition we get $F_K = 2^n$. Thus we obtain the formula (6). ∎

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Kanazawa Institute of Technology
7-1, Ogigaoka, Nonoichi-machi
Ishikawa 921, Japan
E-mail: hira@neptune.kanazawa-it.ac.jp

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