ACTA ARITHMETICA
LXXXIII.4 (1998)

l-adic L-functions and rational function measures

by

JANGHEON OH (Seoul)

1. Introduction. In [2], Sinnott used a measure-theoretic method to give a new proof of a theorem of Washington [3]. We follow his approach to prove that $L_l(1,*) \bmod l$, where l is an odd prime, is the Γ-transform of a rational function measure. As a result, we show that $\mathrm{ord}_l(L_l(1,\chi\psi)) = 0$ for almost all ψ's (Theorem 3), where χ is an even Dirichlet character of the Galois group of an abelian extension over $\mathbb{Q}$ and ψ is a character of the Galois group of the basic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ over $\mathbb{Q}$. Theorem 3 could also be proved using a result of Sinnott [2]. The aim of this paper is to give a direct proof of Theorem 3 by using our Theorem 2. For an algebraic interpretation of Theorem 3, see Theorem 5 of this paper.

Fix two distinct primes l and p. Let $\mathbb{Z}_p$ denote the ring of p-adic integers, $\mathbb{F}_l$ the prime field with l elements, and $\overline{\mathbb{F}}_l$ its algebraic closure. Recall that the group $\mathbb{Z}_p^\times$ of units in $\mathbb{Z}_p$ is the direct product of its torsion subgroup V and the subgroup $U = 1 + 2p\mathbb{Z}_p$. By a *measure* on $\mathbb{Z}_p$ with values in $\overline{\mathbb{F}}_l$ we mean a finitely additive $\overline{\mathbb{F}}_l$-valued set function on the collection of compact open subsets of $\mathbb{Z}_p$. If α is a measure, and $\phi : \mathbb{Z}_p \to \overline{\mathbb{F}}_l$ is a locally constant function, say constant on the cosets of $p^n\mathbb{Z}_p$ in $\mathbb{Z}_p$, then we define the integral

$$\int_{\mathbb{Z}_p} \phi(x)\, d\alpha(x) = \sum_{a \bmod p^n} \phi(a)\alpha(a + p^n\mathbb{Z}_p).$$

Let Φ denote the group of continuous characters $U \to \overline{\mathbb{F}}_l^\times$, viewed always as characters of $\mathbb{Z}_p^\times$ trivial on V. Let α be a measure. The *Γ-transform* $\Gamma_\alpha : \Phi \to \overline{\mathbb{F}}_l$ of α is defined by

$$\Gamma_\alpha(\psi) = \int_{\mathbb{Z}_p^\times} \psi(x)\, d\alpha(x).$$

1991 *Mathematics Subject Classification*: 11R23, 11R18, 11R42.

This paper is part of the author's Ph.D. thesis. I would like to thank my adviser, W. Sinnott, for introducing me to this problem and for many valuable comments.

Let μ_{p^∞} and μ_{p^n} be the set of all p-power roots of unity and the set of all p^nth roots of unity respectively. The *Fourier transform* $\widehat{\alpha} : \mu_{p^\infty} \to \overline{\mathbb{F}}_l$ of α is defined by

$$\widehat{\alpha}(\zeta) = \int\limits_{\mathbb{Z}_p} \zeta^x \, d\alpha(x).$$

We have a relation between the two transforms. Let $\psi \in \Phi$ and let $1 + p^n\mathbb{Z}_p$ be the kernel of ψ in U. Then

$$\Gamma_\alpha(\psi) = \sum_{\zeta \in \mu_{p^n}} \tau(\psi, \zeta)\widehat{\alpha}(\zeta), \tag{1}$$

where

$$\tau(\psi, \zeta) = \frac{1}{p^n} \sum_{x \bmod p^n,\, x \not\equiv 0 \bmod p} \psi(x)\zeta^{-x}.$$

We call a measure α a *rational function measure* if there is a rational function $R(Z) \in \overline{\mathbb{F}}_l(Z)$ such that

$$\widehat{\alpha}(\zeta) = R(\zeta) \qquad \text{for almost all } \zeta \in \mu_{p^\infty}.$$

If α is a measure and $X \subset \mathbb{Z}_p$ is compact and open, we denote by $\alpha|_X$ the measure obtained by restricting α to X and extending by 0. If α is a rational function measure, then so is $\alpha|_X$ for any compact open subset $X \subset \mathbb{Z}_p$. In particular, if $X = \mathbb{Z}_p^\times$ and we put $\alpha^* = \alpha|_{\mathbb{Z}_p^\times}$, then

$$\widehat{\alpha}^*(\zeta) = \widehat{\alpha}(\zeta) - \frac{1}{p} \sum_{\varepsilon^p = 1} \widehat{\alpha}(\varepsilon\zeta).$$

We say a measure α is *supported* on $\mathbb{Z}_p^\times$ if $\alpha = \alpha^*$.

THEOREM 1 (Sinnott [2]). *Let α be a rational function measure on $\mathbb{Z}_p$ with values in $\overline{\mathbb{F}}_l$, and let $R(Z) \in \overline{\mathbb{F}}_l(Z)$ be the associated rational function. Assume that α is supported on $\mathbb{Z}_p^\times$. If $\Gamma_\alpha(\psi) = 0$ for infinitely many $\psi \in \Phi$, then*

$$R(Z) + R(Z^{-1}) = 0.$$

Let $\mathbb{C}_l^\times$ be the nonzero elements of $\mathbb{C}_l$, which is the completion of the algebraic closure of $\mathbb{Q}_l$.

LEMMA 1. *We have*

$$\mathbb{C}_l^\times = l^{\mathbb{Q}} \times W \times U_1,$$

where W is the group of all roots of unity of order prime to l, and $U_1 = \{x \in \mathbb{C}_l \mid |x - 1| < 1\}$.

Proof. See Washington [4, p. 50]. ∎

We now define

$$\log_l(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}X^n}{n}. \tag{2}$$

Now, by the above lemma, let $y = l^r\omega x \in \mathbb{C}_l^\times$. Define $\log_l y = \log_l x$, where for $x \in U_1$, $\log_l x$ is defined by the power series (2).

2. Statement of the main theorem. Let F be a totally real abelian number field, and χ be a Dirichlet character of $\mathrm{Gal}(F/\mathbb{Q})$ whose conductor f is relatively prime to lp. Let ψ be a character of the basic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ with values in $\mathbb{C}_l$; we view ψ as a character of $\mathbb{Z}_p^\times$ trivial on V.

THEOREM 2. *Let l be an odd prime. Then the function given by*

$$\psi \to (L_l(1,\chi\psi) \bmod l)$$

is the Gamma transform of a rational function measure.

3. Proof of theorems

LEMMA 2. *Assume that f is relatively prime to p. Then $\{1,2,\ldots,fp^n\} = \bigcup_{j=1}^f A_j$, where $A_j = \{j, j+f, \ldots, j+(p^n-1)f\}$, and A_j is a representative set of $\mathbb{Z}/p^n\mathbb{Z}$ for any $j = 1,\ldots,f$.*

Proof. The number of elements in A_j is p^n, and if $j + mf \equiv j + kf \bmod p^n$, then

$$(m-k)f \equiv 0 \bmod p^n.$$

Since f is relatively prime to p, $m \equiv k \bmod p^n$. ∎

We use the same notations as in the previous section. The value of the l-adic L-function at 1 for an even nontrivial character was evaluated by Leopoldt (see Washington [4, p. 63]):

$$L_l(1,\chi\psi) = -(1-\chi\psi(l)/l)\frac{\tau(\chi\psi)}{fp^n}\sum_{a=1}^{fp^n}\overline{\chi\psi}(a)\log_l(1-\zeta_{fp^n}^a),$$

where $\tau(\chi\psi) = \sum_{a=1}^{fp^n}\chi\psi(a)\zeta_{fp^n}^a$, and ζ_{fp^n} is a primitive fp^nth root of unity in $\mathbb{Q}_l$.

PROPOSITION 1. *Let $1+p^n\mathbb{Z}_p$ be the kernel of ψ in U. Then*

$$L_l(1,\chi\psi) = \sum_\zeta \tau(\psi,\zeta)\Big(-F(\zeta) + \frac{\chi(l)}{l}F(\zeta^l)\Big),$$

where $F(T) = (1/f)\sum_{i=1}^f \alpha_i \log_l(1-\zeta_f^i T)$ as a function on μ_{p^∞}, $\alpha_i = \sum_{j=1}^f \chi(j)\zeta_f^{ij}$ and the above sum runs over all p^nth roots of unity.

Proof. First compute $\tau(\chi\psi)\overline{\chi\psi}(a)$:

$$\tau(\chi\psi)\overline{\chi\psi}(a) = \sum_{x=1}^{fp^n} \chi\psi(x)\overline{\chi\psi}(a)\zeta_{fp^n}^{x} = \sum_{x=1}^{fp^n} \chi\psi(x)\zeta_{fp^n}^{ax}, \tag{3}$$

so

$$L_l(1,\chi\psi) = -(1-\chi\psi(l)/l)\frac{1}{fp^n}\sum_{a=1}^{fp^n}\Big(\sum_{x=1}^{fp^n}\chi\psi(x)(\zeta_f\zeta_{p^n})^{ax}\Big)\log_l(1-\zeta_f^a\zeta_{p^n}^a).$$

Let us calculate

$$\frac{1}{fp^n}\sum_{a=1}^{fp^n}\Big(\sum_{x=1}^{fp^n}\chi\psi(x)(\zeta_f\zeta_{p^n})^{ax}\Big)\log_l(1-\zeta_f^a\zeta_{p^n}^a). \tag{4}$$

Define $\langle x\rangle$ and $\{x\}$ by $x = \langle x\rangle + dp^n$, $1 \le \langle x\rangle \le p^n$ and $x = \{x\} + ef$, $i \le \{x\} \le f$. Then, by the above lemma, we have

$$\begin{aligned}\sum_{x=1}^{fp^n}\chi\psi(x)(\zeta_f\psi_{p^n})^{ax} &= \sum_{x=1}^{fp^n}\chi(\{x\})\psi(\langle x\rangle)\zeta_f^{a\{x\}}\zeta_{p^n}^{a\langle x\rangle} \\ &= \sum_{j=1}^{f}\sum_{x\in A_j}\chi(\{x\})\psi(\langle x\rangle)\zeta_f^{a\{x\}}\zeta_{p^n}^{a\langle x\rangle} \\ &= \Big(\sum_{j=1}^{f}\chi(j)\zeta_f^{aj}\Big)\Big(\sum_{c=1}^{p^n}\psi(c)\zeta_{p^n}^{ac}\Big).\end{aligned} \tag{5}$$

Let $\alpha_a = \sum_{j=1}^{f}\chi(j)\zeta_f^{aj}$. Then $\alpha_a = \alpha_i$ for any $a \in A_i$, and $\zeta_f^b = \zeta_f^i$ for any $b \in A_i$. Therefore,

$$\begin{aligned}(4) &= \frac{1}{fp^n}\sum_{a=1}^{fp^n}\Big(\sum_{x=1}^{fp^n}\chi\psi(x)(\zeta_f\zeta_{p^n})^{ax}\Big)\log_l(1-\zeta_f^a\zeta_{p^n}^a) \\ &= \frac{1}{fp^n}\sum_{a=1}^{fp^n}\Big(\sum_{j=1}^{f}\chi(j)\zeta_f^{aj}\Big)\Big(\sum_{c=1}^{p^n}\psi(c)\zeta_{p^n}^{ac}\Big)\log_l(1-\zeta_f^a\zeta_{p^n}^a) \\ &= \frac{1}{fp^n}\sum_{a=1}^{fp^n}\Big(\alpha_a\sum_{c=1}^{p^n}\psi(c)\zeta_{p^n}^{ac}\Big)\log_l(1-\zeta_f^a\zeta_{p^n}^a) \\ &= \frac{1}{fp^n}\sum_{i=1}^{f}\Big[\Big(\sum_{b\in A_i}\alpha_b\Big(\sum_{c=1}^{p^n}\psi(c)\zeta_{p^n}^{bc}\Big)\Big)\log_l(1-\zeta_f^b\zeta_{p^n}^b)\Big] \\ &= \frac{1}{fp^n}\sum_{i=1}^{f}\Big[\sum_{b\in A_i}\alpha_i\Big(\sum_{c=1}^{p^n}\psi(c)\zeta_{p^n}^{bc}\Big)\log_l(1-\zeta_f^i\zeta_{p^n}^b)\Big]\end{aligned} \tag{6}$$

$$= \frac{1}{fp^n} \sum_{i=1}^{f} \Big[\sum_{b=1}^{p^n} \alpha_i \Big(\sum_{c=1}^{p^n} \psi(c)\zeta_{p^n}^{bc} \Big) \log_l (1 - \zeta_f^i \zeta_{p^n}^b) \Big]$$
$$= \frac{1}{f} \sum_{i=1}^{f} \Big(\sum_{b=1}^{p^n} \alpha_i \Big(\frac{1}{p^n} \sum_{c=1}^{p^n} \psi(c)\zeta_{p^n}^{bc} \Big) \log_l (1 - \zeta_f^i \zeta_{p^n}^b) \Big).$$

Since ψ is an even character, $\psi(-1) = 1$. Upon replacing c by $-c$, the last expression becomes

$$\frac{1}{f} \sum_{b=1}^{p^n} \tau(\psi, \zeta_{p^n}^b) \Big(\sum_{i=1}^{f} \alpha_i \log_l (1 - \zeta_f^i \zeta_{p^n}^b) \Big). \tag{7}$$

Let

$$F(T) = \frac{1}{f} \sum_{i=1}^{f} \alpha_i \log_l (1 - \zeta_f^i T)$$

(as a function on μ_{p^∞}). Then we proved

$$(4) = \sum_{b=1}^{p^n} \tau(\psi, \zeta_{p^n}^b) F(\zeta_{p^n}^b) = \sum_{\zeta} \tau(\psi, \zeta) F(\zeta). \tag{8}$$

Now consider

$$\frac{\chi(l)}{l} F(T^l).$$

Since l and p are different primes, we have

$$\begin{aligned} \tau(\psi, \zeta_{p^n}^{bl}) &= \frac{1}{p^n} \sum_{c=1}^{p^n} \psi(c) \zeta_{p^n}^{-blc} \\ &= \frac{1}{p^n} \psi^{-1}(l) \sum_{t=1}^{p^n} \psi(t) \zeta_{p^n}^{-bt} \\ &= \psi^{-1}(l) \tau(\psi, \zeta_{p^n}^b), \end{aligned} \tag{9}$$

so that

$$\begin{aligned} \sum_{\zeta} \frac{\chi(l)}{l} \tau(\psi, \zeta) F(\zeta^l) &= \frac{\chi(l)}{l} \sum_{\zeta} \psi(l) \tau(\psi, \zeta^l) F(\zeta^l) \\ &= \frac{\chi\psi(l)}{l} \sum_{\zeta} \tau(\psi, \zeta) F(\zeta). \end{aligned} \tag{10}$$

Let

$$G_\chi(T) = -F(T) + \frac{\chi(l)}{l} F(T^l).$$

By (9), we have

$$\sum_{\zeta}\tau(\psi,\zeta)G_{\chi}(\zeta) = -\sum_{\zeta}\tau(\psi,\zeta)F(\zeta) + \frac{\chi(l)}{l}\sum_{\zeta}\tau(\psi,\zeta)F(\zeta^{l}) \tag{11}$$
$$= -\sum_{\zeta}\tau(\psi,\zeta)F(\zeta) + \frac{\chi\psi(l)}{l}\sum_{\zeta}\tau(\psi,\zeta)F(\zeta)$$
$$= -\left(1-\frac{\chi\psi(l)}{l}\right)\sum_{\zeta}\tau(\psi,\zeta)F(\zeta)$$
$$= L_l(1,\chi\psi).$$

This completes the proof. ■

Since l is prime to p, $rl + sf = 1$ for some $r, s \in \mathbb{Z}$.

LEMMA 3. *Let l be an odd prime. Then*

$$\log_l(1-\zeta_f^i\zeta_{p^n}^l) \equiv \frac{-(1-\zeta_f^{ri}\zeta_{p^n})^l + (1-(\zeta_f^{ri}\zeta_{p^n})^l)}{1-\zeta_f^i\zeta_{p^n}^l} \bmod l^2.$$

Proof. Write $1-\zeta_f^i\zeta_{p^n}^l = \omega(1-\alpha)$, where ω is a root of unity whose order w is relatively prime to l. Since $(l, fp^n w) = 1$, there exists an integer f_n such that

$$\zeta_f^{l^{f_n}} = \zeta_f, \qquad \zeta_{p^n}^{l^{f_n}} = \zeta_{p^n}, \qquad \omega^{l^{f_n}} = \omega.$$

The number α is divisible by l, since l is unramified in $\mathbb{Q}_l(\omega, \zeta_f, \zeta_{p^n})$. Let

$$(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}-1} = (1-\alpha)^{l^{f_n}-1} = 1+\beta. \tag{12}$$

Then

$$\log_l(1-\zeta_f^i\zeta_{p^n}^l) = \frac{1}{l^{f_n}-1}\log_l(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}-1} \tag{13}$$
$$= \frac{1}{l^{f_n}-1}\log_l(1+\beta) \equiv \frac{1}{l^{f_n}-1}\beta \bmod l^2$$
$$\equiv -\beta = 1-(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}-1}$$
$$= \frac{-(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}} + (1-\zeta_f^i\zeta_{p^n}^l)}{1-\zeta_f^i\zeta_{p^n}^l}.$$

Now we simplify the expression $(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}}$. Write

$$(1-\zeta_f^iT)^l = 1-(\zeta_f^iT)^l + lf(T). \tag{14}$$

Then

$$(1-\zeta_f^iT)^{l^2} \equiv (1-(\zeta_f^iT)^l)^l \bmod l^2 \tag{15}$$
$$= 1-\zeta_f^{l^2 i}T^{l^2} + lf(\zeta_f^{(l-1)i}T^l).$$

Since $\zeta_f^{l^{f_n}} = \zeta_f$, we know that $l^{f_n} = 1 + kf$ for some integer k. Hence $rl^{f_n} = r + k'f$ and $rl^{f_n} = rll^{f_n-1} = (1-sf)l^{f_n-1}$, so we have $l^{f_n-1} = r + k''f$ for some integer k''. Continuing the above process, we have

$$(1-\zeta_f^i T)^{l^{f_n}} \equiv 1 - \zeta_f^{l^{f_n} i} T^{l^{f_n}} + lf(\zeta_f^{(l^{f_n-1}-1)i} T^{l^{f_n-1}}) \bmod l^2. \tag{16}$$

Substituting $T = \zeta_{p^n}^l$ and using the equation $l^{f_n-1} = r + k''f$, we have

$$(1-\zeta_f^i \zeta_{p^n}^l)^{l^{f_n}} \equiv 1 - \zeta_f^i \zeta_{p^n}^l + lf(\zeta_f^{(r-1)i} \zeta_{p^n}) \bmod l^2. \tag{17}$$

Finally, combining the above gives

$$\begin{aligned} \log_l(1-\zeta_f^i\zeta_{p^n}^l) &\equiv \frac{-(1-\zeta_f^i\zeta_{p^n}^l)^{l^{f_n}} + (1-\zeta_f^i\zeta_{p^n}^l)}{1-\zeta_f^i\zeta_{p^n}^l} \bmod l^2 \\ &= -\frac{lf(\zeta_f^{(r-1)i}\zeta_{p^n})}{1-\zeta_f^i\zeta_{p^n}^l} \\ &= \frac{-(1-\zeta_f^{ri}\zeta_{p^n})^l + (1-(\zeta_f^{ri}\zeta_{p^n})^l)}{1-\zeta_f^i\zeta_{p^n}^l}. \end{aligned} \tag{18}$$

This completes the proof. ■

PROPOSITION 2. *Let l be an odd prime and $1+p^n\mathbb{Z}_p$ be the kernel of ψ in U. Then*

$$L_l(1,\chi\psi) \equiv \sum_{\zeta_{p^n}} \tau(\psi,\zeta_{p^n}) \left[\frac{\chi(l)}{f} \sum_{i=1}^{f} \frac{-(1-\zeta_f^{ri}\zeta_{p^n})^l + (1-(\zeta_f^{ri}\zeta_{p^n})^l)}{l(1-\zeta_f^i\zeta_{p^n}^l)} \alpha_i \right] \bmod l,$$

where the sum runs over all p^nth roots of unity.

Proof. Since $\log_l(1-\zeta_f^i\zeta_{p^n}) \equiv 0 \bmod l$, we have

$$\begin{aligned} L_l(1,\chi\psi) &\equiv \sum_{\zeta} \tau(\psi,\zeta)\left(\frac{\chi(l)}{l}F(\zeta^l)\right) \\ &\equiv \sum_{\zeta_{p^n}} \tau(\psi,\zeta_{p^n}) \\ &\quad \times \left[\frac{\chi(l)}{f} \sum_{i=1}^{f} \frac{-(1-\zeta_f^{ri}\zeta_{p^n})^l + (1-(\zeta_f^{ri}\zeta_{p^n})^l)}{l(1-\zeta_f^i\zeta_{p^n}^l)} \alpha_i \right] \bmod l. \end{aligned}$$ ■

Proof of Theorem 2. The proof comes directly from Proposition 2 and the equation (1). ■

Let

$$G_\chi(T) = \frac{\chi(l)}{f} \sum_{i=1}^{f} \frac{-(1-\zeta_f^{ri}T)^l + (1-(\zeta_f^{ri}T)^l)}{l(1-\zeta_f^i T^l)} \alpha_i.$$

Then

(19) $G_\chi(T)$

$$= -\sum_{i=1}^{f} \frac{\chi(l)(-\zeta_f^{ri}T + \ldots + \zeta_f^{r(l-1)i}T^{l-1})}{f(1-\zeta_f^i T^l)} \alpha_i$$

$$= -\frac{\chi(l)((-\sum_{i=1}^{f}\zeta_f^{ri}\alpha_i)T + \ldots + (\sum_{i=1}^{f}\alpha_i(-1)^{f-1}\zeta_f^{r(l-1)i}\prod_{k\neq i}\zeta_f^k)T^{fl-1})}{f\prod_{i=1}^{f}(1-\zeta_f^i T^l)}$$

$$= -\frac{\chi(l)((-f\chi(-r))T + \ldots + f\chi(r)T^{fl-1})}{f\prod_{i=1}^{f}(1-\zeta_f^i T^l)}.$$

Hence

$$G_\chi(T^{-1}) = -\frac{\chi(l)}{f} \cdot \frac{-f\chi(r)T^{lf-1} + \ldots + f\chi(r)T}{\prod_{i=1}^{f}(T^l - \zeta_f^i)}.$$

Let us compute

$$(20) \qquad G_\chi(T) + G_\chi(T^{-1}) = \frac{\chi(rl)T + \ldots}{\prod_{i=1}^{f}(1-\zeta_f^i T^l)} - \frac{\chi(rl)T + \ldots}{\prod_{i=1}^{f}(T^l - \zeta_f^i)}$$
$$= \frac{-2\chi(rl)T + \ldots}{\prod_{i=1}^{f}(1-\zeta_f^i T^l)\prod_{i=1}^{f}(T^l - \zeta_f^i)}.$$

Hence $G_\chi(T) + G_\chi(T^{-1})$ is not identically zero when we reduce the coefficients modulo l since the value $\chi(rl)$ is a unit. Let

$$G_\chi^*(T) = G_\chi(T) - \frac{1}{p}\sum_{\varepsilon^p=1} G_\chi(\varepsilon T).$$

Let $\widetilde{R}(T)$ be the power series in $\overline{\mathbb{F}}_l[[T]]$ obtained by reducing the coefficients of $R[[T]] \in \mathbb{Z}_l[[T]]$. Since $G_\chi(T)$ and $G_\chi^*(T)$ have the same coefficient of T, we have

$$(21) \qquad \widetilde{G}_\chi^*(T) + \widetilde{G}_\chi^*(T^{-1}) \neq 0.$$

By Theorem 2 and a result of Sinnott (Theorem 1), we can prove the following theorem. Let F be a totally real abelian number field, and χ be a Dirichlet character of $\mathrm{Gal}(F/\mathbb{Q})$ whose conductor f is relatively prime to lp. Let ψ be a character of $\mathbb{Q}_\infty/\mathbb{Q}$ as a character on $\mathbb{Z}_p^\times$ trivial on V.

THEOREM 3. *Let l be an odd prime. Then* $\mathrm{ord}_l(L_l(1,\chi\psi)) = 0$ *for all but finitely many* ψ*'s.*

Proof. In Proposition 2, we proved

$$L_l(1,\chi\psi) = \sum_{\zeta_{p^n}} \tau(\psi,\zeta_{p^n})\Big[\frac{\chi(l)}{f}\sum_{i=1}^{f}\frac{-(1-\zeta_f^{ri}\zeta_{p^n})^l+(1-(\zeta_f^{ri}\zeta_{p^n})^l)}{l(1-\zeta_f^i\zeta_{p^n}^l)}\alpha_i\Big] \bmod l,$$

that is,

$$L_l(1,\chi\psi)\equiv\sum_{\zeta}\tau(\psi,\zeta)G_\chi(\zeta) \bmod l. \tag{22}$$

Let α and α^* be the corresponding measures of $\widetilde{G}_\chi(T)$ and $\widetilde{G}^*_\chi(T)$, respectively. If $\psi\in\Phi$, let ψ' be the character of $\mathbb{Q}_\infty/\mathbb{Q}$ which satisfies $\widetilde{\psi}'(n)=\psi(n)$ for integers n prime to p, where the tilde stands for reduction mod l; on the right we are viewing ψ as a character of $\mathbb{Z}_p^\times$ trivial on V. Then $\tau(\widetilde{\psi',\zeta})=\tau(\psi,\zeta)$. Hence, by (1) and Proposition 2, we have

$$\Gamma_{\alpha^*}(\psi)=\int_{\mathbb{Z}_p^\times}\psi\,d\alpha^*=\int_{\mathbb{Z}_p^\times}\psi\,d\alpha=L_l(\widetilde{1,\chi\psi'}). \tag{23}$$

Now $\widetilde{G}^*_\chi(T)+\widetilde{G}^*_\chi(T^{-1})\neq 0$ by (21); hence $\Gamma_{\alpha^*}(\psi)=0$ for only finitely many ψ, by Theorem 1. This completes the proof. ∎

We let ord_l denote the usual valuation on $\overline{\mathbb{Q}}_l$, normalized by $\mathrm{ord}_l(l)=1$. Let $R_l(K)$ be the l-adic regulator of a number field K, $h(K)$ be the class number of K, and $d(K)$ be the discriminant of K. Then we have the l-adic class number formula.

THEOREM 4. *Let K be a totally real abelian number field of degree n corresponding to a group X of Dirichlet characters. Then*

$$\frac{2^{n-1}h(K)R_l(K)}{\sqrt{d(K)}}=\prod_{\chi\in X,\,\chi\neq 1}\Big(1-\frac{\chi(l)}{l}\Big)^{-1}L_l(1,\chi). \tag{24}$$

Proof. See Washington [4, p. 71]. ∎

COROLLARY 1. *Let l be an odd prime. Let K be a totally real abelian number field whose conductor is relatively prime to lp, and K_n be the nth layer of the basic $\mathbb{Z}_p$-extension K_∞/K. Then*

$$\mathrm{ord}_l(R_l(K_n))=dp^n+C,\qquad \textit{for } n \textit{ sufficiently large}, \tag{25}$$

for some constant C independent of n.

Proof. Washington [3] proved that $\mathrm{ord}_l h(K_n)$ is constant if n is sufficiently large. By assumption, $\mathrm{ord}_l d(K_n)=0$. By Theorem 3, $\mathrm{ord}_l(L_l(1,\chi\psi))$ is nonzero for only finitely many ψ's. Note that $[K_n:\mathbb{Q}]=[K:\mathbb{Q}]p^n$, and

$\mathrm{ord}_l(1-\chi(l)/l) = -\mathrm{ord}_l(l)$ since $\chi(l)$ is a unit. Hence equation (25) follows from the l-adic class number formula since

$$\mathrm{ord}_l\Big(\prod_{\chi\in X,\,\chi\neq 1}\Big(1-\frac{\chi(l)}{l}\Big)\Big) = -dp^n+1,$$

where $d=[K:\mathbb{Q}]$. ∎

Let L be a number field, and n a positive integer. Let $w(L)$ be the number of roots of unity in L. Let $S_{F,l}$ be the set of primes of a totally real number field F above a rational prime l. Let M be the maximal abelian l-extension of F which is unramified outside $S_{F,l}$ and $F_\infty^{(l)}$ be the basic $\mathbb{Z}_l$-extension of F. Coates [1, p. 348] proved the following theorem.

LEMMA 4. *$G(M/F_\infty^{(l)})$ is finite if and only if $R_l(F)\neq 0$. If $R_l(F)\neq 0$, then the order of $G(M/F_\infty)$ is the inverse of the l-adic valuation of*

$$w(F(\zeta_l))h(F)R_l(F)\prod_{\mathfrak{l}\in S_{F,l}}(1-(N\mathfrak{l})^{-1})/\sqrt{d(F)}. \tag{26}$$

Fix an odd prime l relatively prime to p. Let M_n be the maximal abelian l-extension of K_n which is unramified outside $S_{K_n,l}$ and $K_{n,\infty}^{(l)}$ be the basic $\mathbb{Z}_l$-extension of K_n. Let Y_K be the maximal abelian l-extension of K_∞ unramified above l. Then $Y_K=\bigcup_n M_n$. By assumption, l is unramified in K_n. Hence

$$\mathrm{ord}_l\Big(\prod_{S_{K_n,l}}(1-(N\mathfrak{l})^{-1})\Big) = \mathrm{ord}_l\Big(\prod_{\chi\in X_n}\Big(1-\frac{\chi(l)}{l}\Big)\Big). \tag{27}$$

By assumption, the number of roots of unity in $K_n(\zeta_l) = K\mathbb{Q}_n(\zeta_l)$ is bounded independently of n. Therefore, by Lemma 4, Theorem 3 and Theorem 4, the order of $G(M_n/K_{n,\infty}^{(l)})$ is constant if n is large enough. Thus we proved:

THEOREM 5. *The order of* $\mathrm{Gal}(M_n/K_{n,\infty}^{(l)})$ *is constant if n is sufficiently large.*

REMARK 1. W. Sinnott pointed out to me that there was an alternative proof of Theorem 3. We include the proof: Suppose $\chi\psi$ is not of the second kind for l. Write $L_l(s,\chi\psi)=f(u^s-1)$, where $f(X)=a_0+a_1X+\dots$ with $a_i\in\mathbb{Z}_p$ [values of $\chi\psi$], $u=1+l$. Then

$$\begin{aligned} L_l(1,\chi\psi) &= a_0+a_1(u-1)+a_2(u-1)^2+\dots\\ &\equiv a_0 \bmod l\\ &\equiv L_l(0,\chi\psi) \bmod l\\ &\equiv (1-\chi\psi\omega_l^{-1}(l))L(0,\chi\psi\omega_l^{-1}) \bmod l. \end{aligned}$$

Since the conductor of χ is assumed to be prime to lp, $\chi\psi\omega_l^{-1}(l)$ is zero. It is known [2] that the map $\psi \to L(0, \chi\psi\omega_l^{-1})$ is the Γ-transform of a rational function measure for which $R(Z) + R(Z^{-1}) \neq 0 \bmod l$ and so is a unit for all but finitely many ψ.

References

[1] J. Coates, *p-adic L-functions and Iwasawa theory*, in: Algebraic Number Fields (Durham Symposium, 1975), A. Fröhlich (ed.), Academic Press, 1977, 269–353.

[2] W. Sinnott, *On a theorem of L. Washington*, Astérisque 147–148 (1987), 209–224.

[3] L. Washington, *The non-p-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension*, Invent. Math. 49 (1979), 87–97.

[4] —, *Introduction to Cyclotomic Fields*, Springer, Berlin, 1982.

KIAS
207-43 Cheongryangri-dong
Dongdaemun-gu
Seoul 130-012, Korea
E-mail: ohj@kias.kaist.ac.kr

Received on 21.1.1997
and in revised form on 21.8.1997 (3116)