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Distinct zeros of L-functions

by

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1. Introduction. Let $L_1(s)$ and $L_2(s)$ be two "independent" L-functions, where the meaning of "independent" will be clarified later on. Since the L-functions are determined by their zeros, we may expect that $L_1(s)$ and $L_2(s)$ have few common zeros. This problem appears to be very difficult at present, therefore we may ask the easier question of getting a fair quantity of distinct zeros of such functions. In this paper we show that, under suitable conditions, $L_1(s)$ and $L_2(s)$ have a positive proportion of distinct zeros.

We state our results in the moderately general setting of Bombieri–Hejhal's paper [1], which also provides the basic ingredients of the present paper. Moreover, we will work out our main tool, Theorem 2 below, in the case of several L-functions. Hence, for a given integer $N \geq 2$, we consider N functions $L_1(s), \ldots, L_N(s)$ satisfying the following basic hypothesis.

Hypothesis B. (I) *Each function $L_j(s)$ has an Euler product of the form*

$$L_j(s) = \prod_p \prod_{i=1}^{d} (1 - \alpha_{ip} p^{-s})^{-1}$$

with $|\alpha_{ip}| \leq p^{\theta}$ for some fixed $0 \leq \theta < 1/2$ and $i = 1, \ldots, d$.

(II) *For every $\varepsilon > 0$ we have*

$$\sum_{p \leq x} \sum_{i=1}^{d} |\alpha_{ip}|^2 \ll x^{1+\varepsilon}.$$

(III) *The functions $L_j(s)$ have an analytic continuation to $\mathbb{C}$ as meromorphic functions of finite order with a finite number of poles, all on the line $\sigma = 1$, and satisfy a functional equation of the form*

$$\Phi(s) = \varepsilon \overline{\Phi}(1-s),$$

where $\Phi(s) = Q^s \prod_{i=1}^{m} \Gamma(\lambda_i s + \mu_i)$, $Q > 0$, $\lambda_i > 0$, $\operatorname{Re} \mu_i \geq 0$ and $|\varepsilon| = 1$.

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(IV) *The coefficients* $a_j(p)$ *of the Dirichlet series*

$$L_j(s)=\sum_{n=1}^{\infty} a_j(n)n^{-s}$$

satisfy

$$\sum_{p\le x}\frac{a_j(p)\overline{a_k(p)}}{p}=\delta_{jk}n_j\log\log x+c_{jk}+O\left(\frac{1}{\log x}\right)$$

for certain constants $n_j>0$.

We explicitly remark that all the data involved in Hypothesis B concerning a function $L_j(s)$ may depend on j. We also remark that the conditions of Hypothesis B may be somewhat relaxed (see Selberg [10]) in order to deduce our results below.

We refer to Section 3 of [1] for a thorough discussion of Hypothesis B, of its standard consequences and of several examples of functions satisfying it. Here we point out only that B(II) implies that both the Dirichlet series and the Euler product of $L_j(s)$ converge absolutely for $\sigma>1$, B(I) ensures that $L_j(s)\neq 0$ for $\sigma>1$ and B(III) gives rise to the familiar notions of critical strip, critical line and trivial and non-trivial zeros. Moreover, writing

$$\Lambda_j=\sum_{i=1}^{m}\lambda_i,$$

$$N_j(t)=|\{\varrho: L_j(\varrho)=0,\ 0\le \operatorname{Re}\varrho\le 1 \text{ and } 0\le \operatorname{Im}\varrho\le t\}|$$

and

$$S_j(t)=\frac{1}{\pi}\arg L_j(1/2+it),$$

for sufficiently large t we have

$$N_j(t)=\frac{\Lambda_j}{\pi}t\log t+c_jt+c_j'+S_j(t)+O(1/t) \tag{1}$$

with certain constants c_j and c_j'.

Condition B(IV), introduced by Selberg [10], plays a special role, since it provides a form of "near-orthogonality" of the functions $L_j(s)$; the "independence" alluded to at the beginning of the section comes from this "near-orthogonality". For instance, B(IV) implies that $L_1(s),\dots,L_N(s)$ are linearly independent over $\mathbb{C}$; see Bombieri–Hejhal [1] and Kaczorowski–Perelli [7] for further results in this direction.

We expect that the functions $L_j(s)$ satisfy the Generalized Riemann Hypothesis. As a substitute of it in our arguments, we will instead assume the following density estimate. Let

$$N_j(\sigma,T)=|\{\varrho: L_j(\varrho)=0,\ \operatorname{Re}\varrho\ge\sigma \text{ and } |\operatorname{Im}\varrho|\le T\}|.$$

Hypothesis D. *There exists* $0 < a < 1$ *such that*

$$N_j(\sigma, T) \ll T^{1-a(\sigma-1/2)} \log T$$

uniformly for $\sigma \geq 1/2$ *and* $j = 1, \ldots, N$.

The main point in introducing Hypothesis D is that, unlike the Generalized Riemann Hypothesis, it can be verified in many interesting cases. In fact, it has been proved by Selberg [9] for the Riemann zeta function, by Fujii [5] for Dirichlet L-functions, and by Luo [8] in the more difficult case of L-functions attached to certain modular forms.

In order to state our main result, we define the counting function $D(T, L_1, L_2)$ of the distinct non-trivial zeros, counted with multiplicity, of two functions $L_1(s)$ and $L_2(s)$ as

$$D(T, L_1, L_2) = \sum_{\substack{0 \leq \mathrm{Re}\, \varrho \leq 1 \\ 0 \leq \mathrm{Im}\, \varrho \leq T}} \max(m_1(\varrho) - m_2(\varrho), 0),$$

where ϱ runs over the zeros of $L_1(s)L_2(s)$ and is counted without multiplicity. We also define

$$D(T) = D(T, L_1, L_2) + D(T, L_2, L_1) = \sum_{\substack{0 \leq \mathrm{Re}\, \varrho \leq 1 \\ 0 \leq \mathrm{Im}\, \varrho \leq T}} |m_1(\varrho) - m_2(\varrho)|,$$

with the same convention about ϱ.

Our main result is

Theorem 1. *Let* $L_1(s)$ *and* $L_2(s)$ *satisfy Hypotheses* B *and* D *and suppose that* $\Lambda_1 = \Lambda_2$. *Then*

$$D(T, L_1, L_2) \gg T \log T.$$

Clearly, the same lower bound holds for $D(T, L_2, L_1)$ and $D(T)$ too.

The first result of this type has been obtained by Fujii [6] in the case of two primitive Dirichlet L-functions, by means of Selberg's moments method. The problem of counting *strongly distinct* zeros, i.e., zeros placed at different points, appears to be more difficult, and the best result is due to Conrey–Ghosh–Gonek [3], [4]. They deal with this problem, in the case of two primitive Dirichlet L-functions, by considering the more difficult question of getting simple zeros of $L(s, \chi_1)L(s, \chi_2)$, and show that there are $\gg T^{6/11}$ such zeros up to T. Moreover, if the Riemann Hypothesis is assumed for one of the two functions, then a positive proportion of such zeros is obtained. However, the techniques in [3] and [4] do not extend to cover the case of more general L-functions, such as GL_2 L-functions.

Let us call *coprime* two functions in Selberg's class $\mathcal{S}$ (see [10]) each having a factorization into primitive functions (in the sense of Selberg [10]) such that there are no common factors of such factorizations. Assuming Selberg's Conjectures 1.1 and 1.2 in [10], we see that B(IV) holds for coprime functions. Hence, assuming Hypothesis D for every function in $\mathcal{S}$, we may regard the lower bound in Theorem 1, in the case of coprime functions, as a consequence of Selberg's conjectures. Another consequence of Selberg's conjectures is that $\mathcal{S}$ has unique factorization (see Conrey–Ghosh [2]). We remark here that the latter consequence of Selberg's conjectures is easily implied by a very weak form of the former. Precisely, assuming that two coprime functions in $\mathcal{S}$ have $D(T) \geq 1$ for sufficiently large T, we get the unique factorization in $\mathcal{S}$. In fact, the assumption implies that two coprime functions are necessarily distinct, and this clearly implies the unique factorization.

Theorem 1 appears to be the limit of our method, although much more is expected to hold. For instance, if $L_1(s)$ and $L_2(s)$ are distinct primitive functions, we expect that almost all zeros of $L_1(s)$ and $L_2(s)$ are distinct, i.e.,

$$D(T) \sim \frac{\Lambda_1 + \Lambda_2}{\pi} T \log T,$$

in which case almost all zeros are actually strongly distinct, or even that

$$D(T) = N_1(T) + N_2(T) + O(1),$$

i.e., $L_1(s)$ and $L_2(s)$ have $O(1)$ common non-trivial zeros.

The proof of Theorem 1 is based on Bombieri–Hejhal's [1] variant of Selberg's [9] moments method, which leads in a more direct way to the distribution function for the $\log L_j(1/2+it)$ (see Theorem B of [1]). Although we could follow a variant more in the spirit of Selberg [9] and Fujii [6], we will prove Theorem 1 by means of a short intervals analog of the above mentioned Theorem B, which we believe to be of interest in itself.

Let $M \geq 10$, write $h = M/\log T$ and

$$V_j(t) = \frac{\log L_j(1/2+i(t+h)) - \log L_j(1/2+it)}{(2\pi n_j \log M)^{1/2}},$$

and let μ_T denote the associated probability measure on $\mathbb{C}^N$, defined by

$$\mu_T(\Omega) = \frac{1}{T}\,|\{t \in [T,2T] : (V_1(t),\dots,V_N(t)) \in \Omega\}| \tag{2}$$

for every open set $\Omega \subset \mathbb{C}^N$. Moreover, let $e^{-\pi\|\mathbf{z}\|^2}$ denote the gaussian measure on $\mathbb{C}^N$ and let $d\omega$ be the euclidean density on $\mathbb{C}^N$.

THEOREM 2. *Let $L_1(s),\dots,L_N(s)$ satisfy Hypotheses* B *and* D *and let $M = M(T) \to \infty$ with $M \leq (\log T)/\log\log T$ as $T \to \infty$. Then, as $T \to \infty$, μ_T tends to the gaussian measure with associated density $e^{-\pi\|\mathbf{z}\|^2}\,d\omega$.*

We remark that we can easily get a slight variant of Theorem 2, where $h = M/\log t$ and $M = M(t) \to \infty$ with $M \le \log^{1-\varepsilon} t$ as $t \to \infty$. Therefore, if we separate the $V_j(t)$ into their real and imaginary parts, Theorem 2 can be expressed by saying that the functions

$$\frac{\log\left|L_j\left(\frac{1}{2} + i\left(t + \frac{M}{\log t}\right)\right)\right| - \log\left|L_j\left(\frac{1}{2} + it\right)\right|}{(2\pi n_j \log M)^{1/2}}, \qquad j = 1, \ldots, N,$$

and

$$\frac{\arg L_j\left(\frac{1}{2} + i\left(t + \frac{M}{\log t}\right)\right) - \arg L_j\left(\frac{1}{2} + it\right)}{(2\pi n_j \log M)^{1/2}}, \qquad j = 1, \ldots, N,$$

become distributed, in the limit of large t, like independent random variables, each having gaussian density $\exp(-\pi u^2)du$, provided $M \to \infty$ with $M \le \log^{1-\varepsilon} t$ as $t \to \infty$.

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2. Basic lemmas. In this section we follow the arguments in Section 5 of Bombieri–Hejhal [1]. For $\sigma > 1$ and $j = 1, \ldots, N$ we write

$$\log L_j(s) = \sum_{n=1}^{\infty} c_j(n)\Lambda_1(n)n^{-s}, \qquad \Lambda_1(n) = \begin{cases} 0, & n = 1, \\ \Lambda(n)/\log n, & n \ge 2, \end{cases}$$

and denote by $u(x)$ a real positive C^∞ function with compact support in $[1, e]$ and by $\widetilde{u}(s)$ its Mellin transform. We also write

$$v(x) = \int_x^\infty u(t)\,dt$$

and assume that u is normalized so that $v(0) = 1$. We refer to Lemma 1 of [1] and the remark following it for relevant properties of $\widetilde{u}(s)$.

By (5.4) of [1] we have the approximate formula

$$\log L_j(1/2 + it) = \sum_{n=1}^{\infty} \frac{c_j(n)\Lambda_1(n)}{n^{1/2+it}}\, v(e^{(\log n)/\log X}) + \sum_{\varrho} \int_{1/2}^{\infty} \frac{1}{\varrho - s}\, \widetilde{u}(1 + (\varrho - s)\log X)\, d\sigma + O(1), \tag{3}$$

where $|t|$ is sufficiently large and not the ordinate of a zero of $L_j(s)$, where $2 \le X \le t^2$ and where ϱ runs over zeros of $L_j(s)$ with $0 \le \operatorname{Re} \varrho \le 1$. We write (3) as

$$\log L_j(1/2 + it) = D_j(1/2 + it, X) + R_j(1/2 + it, X)$$

where $D_j(1/2 + it, X)$ is the Dirichlet series on the right hand side of (3).

From Lemma 3 of [1] we immediately get our first basic lemma.

LEMMA 1. *Assume Hypotheses* B *and* D, *and let* $2 \le X \le T^{a/2}$ *and* T *sufficiently large. Then for* $j = 1, \ldots, N$ *we have*

$$\int_T^{2T} |R_j(1/2+it, X)|\, dt \ll T\, \frac{\log T}{\log X}.$$

Our second basic lemma is a short intervals analog of Lemma 6 of [1], i.e., the mixed moments of the differences of the $D_j(1/2+it, X)$. Since the proof of Lemma 2 below follows that of Lemma 6 of [1], we will only sketch it. For sufficiently large M, write $h = M/\log T$ and

$$\Sigma_j(t) = D_j(1/2+i(t+h), X) - D_j(1/2+it, X).$$

Moreover, let $k_j \ge 0$ and $l_j \ge 0$, $j = 1, \ldots, N$, be integers and let us abbreviate $\mathbf{k} = (k_1, \ldots, k_N)$, $K_j = k_1 + \ldots + k_j$, $K = K_N$ and similarly for $\mathbf{l}, L_j$ and L. We also write $\mathbf{k}! = \prod_{j=1}^N k_j!$.

We state here the basic estimate we will repeatedly use in the proof of Lemma 2. For $X \ge 3$ we have

$$\sum_p \frac{a_j(p)\overline{a_k(p)}}{p}\, v(e^{(\log p)/\log X})^2 |e^{-ih\log p} - 1|^2 = \delta_{jk}\, 2n_j \log^+\left(\frac{h}{2}\log X\right) + O(1) \tag{4}$$

uniformly for $h \le 1/\log\log X$, where $\log^+ x = \max(\log x, 0)$. In fact, $|e^{-ih\log p} - 1| = 4\sin^2((h/2)\log p)$ and hence (4) follows from B(IV) by partial summation (see also (3.8) of [1]).

LEMMA 2. *Assume Hypothesis* B *and let* $X \le T^{1/(K+L+1)}$ *and* $M \le (\log T)/\log\log X$. *Write*

$$\Sigma_j(t) = \sum_{n=1}^\infty \frac{b_j(n)}{n^{1/2+it}}, \qquad b_j(n) = c_j(n)\Lambda_1(n) v(e^{(\log n)/\log X})(e^{-ih\log n} - 1).$$

Then

$$\int_T^{2T} \prod_{j=1}^N (\Sigma_j(t))^{k_j} (\overline{\Sigma_j(t)})^{l_j}\, dt = \delta_{\mathbf{k},\mathbf{l}}\, \mathbf{k}!\, T \prod_{j=1}^N \left(2n_j \log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\right)^{k_j} + O\left(T\left(\log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\right)^{(K+L-1)/2}\right).$$

Proof. We may clearly assume that $K+L \ge 1$. For notational simplicity, we abbreviate $\Sigma_j = \Sigma_j(t)$. Since Σ_j is supported at prime powers only, we split it as

$$\Sigma_j = \Sigma_j' + \Sigma_j''$$

where Σ'_j ranges over primes p and Σ''_j over prime powers p^r, $r \geq 2$. Then, accordingly, we get

$$\prod_{j=1}^{N} (\Sigma_j)^{k_j} (\overline{\Sigma}_j)^{l_j} = \prod_{j=1}^{N} (\Sigma'_j)^{k_j} (\overline{\Sigma}'_j)^{l_j} + R(t), \tag{5}$$

where, as in the proof of Lemma 6 of [1],

$$\int_T^{2T} |R(t)|\, dt \ll \int_T^{2T} |\Sigma''_{j_1}||\Sigma'_{j_2}|^{K+L-1}\, dt + \int_T^{2T} |\Sigma''_{j_3}|^{K+L}\, dt \tag{6}$$

for a suitable choice of j_1, j_2 and j_3.

Since $e^{-ih\log n} - 1 \ll 1$, by (5.14) of [1] we have

$$\int_T^{2T} |\Sigma''_j|^{2(K+L)}\, dt \ll T \tag{7}$$

for $j = 1, \ldots, N$, provided $X \leq T^{1/(K+L+1)}$.

By Montgomery–Vaughan's mean-value theorem for Dirichlet polynomials (see, e.g., Lemma 4 of [1]) we have

$$\int_T^{2T} |\Sigma'_j|^{2(K+L)}\, dt = T \sum \frac{|B'(n)|^2}{n} + O\Big(\sum |B'(n)|^2\Big),$$

where

$$B'(n) = \sum_{p_1 \ldots p_{K+L} = n} b_j(p_1) \ldots b_j(p_{K+L}).$$

Since $c_j(p) = a_j(p)$ and $\Lambda_1(p) = 1$, from (5.16) of [1] and (4) we get

$$\sum \frac{|B'(n)|^2}{n} \leq (K+L)! \Big(\sum_p \frac{|b_j(p)|^2}{p}\Big)^{K+L} \ll \left(\log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\right)^{K+L}.$$

Moreover, from (5.17) of [1] we obtain

$$\sum |B'(n)|^2 \ll X^{(1+\varepsilon)(K+L)},$$

and hence

$$\int_T^{2T} |\Sigma'_j|^{2(K+L)}\, dt \ll T\left(\log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\right)^{K+L} \tag{8}$$

provided $X \leq T^{1/(K+L+1)}$.

From (6)–(8) and Hölder's inequality we get

$$\int_T^{2T} |R(t)|\, dt \ll T\left(\log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\right)^{(K+L-1)/2} \tag{9}$$

provided $X \leq T^{1/(K+L+1)}$.

In order to treat the main product on the right hand side of (5) we use again Lemma 4 of [1]. We abbreviate $\mathbf{n} = (n_1, \ldots, n_K)$,

$$b(\mathbf{n},\mathbf{k}) = \prod_{j=1}^{N} \prod_{r=K_{j-1}+1}^{K_j} b_j(n_r) \quad \text{and} \quad B(n,\mathbf{k}) = \sum_{n_1 \ldots n_K = n} b(\mathbf{n},\mathbf{k}),$$

and as in (5.18) of [1] we have

$$(10) \quad \int_T^{2T} \prod_{j=1}^{N} (\Sigma_j')^{k_j} (\overline{\Sigma}_j')^{l_j} \, ,dt = T \sum \frac{B(n,\mathbf{k})\overline{B(n,\mathbf{l})}}{n} + O\Big(\Big(\sum |B(n,\mathbf{k})|^2\Big)^{1/2} \Big(\sum |B(n,\mathbf{l})|^2\Big)^{1/2}\Big),$$

where the sums are restricted to n of type $n = p_1 \ldots p_K$ for $\mathbf{k}$ and $n = q_1 \ldots q_L$ for $\mathbf{l}$; here p and q denote prime numbers. By a variant of the argument leading to (8) we see that

$$(11) \quad \Big(\sum |B(n,\mathbf{k})|^2\Big)^{1/2} \Big(\sum |B(n,\mathbf{l})|^2\Big)^{1/2} \ll T\Big(\log^+\Big(\frac{M}{2}\frac{\log T}{\log X}\Big)\Big)^{(K+L-1)/2}$$

provided $X \leq T^{1/(K+L+1)}$.

In view of (5), (9), (10) and (11), to complete the proof of Lemma 2 it suffices to show that

$$(12) \quad \sum \frac{B(n,\mathbf{k})\overline{B(n,\mathbf{l})}}{n} = \delta_{\mathbf{k},\mathbf{l}}\, \mathbf{k}! \prod_{j=1}^{N} \Big(2n_j \log^+\Big(\frac{M}{2}\frac{\log T}{\log X}\Big)\Big)^{k_j} + O\Big(\Big(\log^+\Big(\frac{M}{2}\frac{\log T}{\log X}\Big)\Big)^{(K+L-1)/2}\Big).$$

If $K \neq L$ there is nothing to prove, since $B(n,\mathbf{k})B(n,\mathbf{l}) = 0$ for every n; we can therefore assume $K = L \geq 1$ and proceed by induction as in Lemma 6 of [1].

If $K = 1$, (12) follows immediately from (4). Suppose now that $K \geq 2$. Arguing again as in Lemma 6 of [1] and using (3.8) of [1], we see that the contribution to the left hand side of (12) coming from n's which are not square-free is

$$\ll \Big(\log^+\Big(\frac{M}{2}\frac{\log T}{\log X}\Big)\Big)^{(K+L-1)/2}.$$

In order to deal with the remaining part of the sum on the left hand side of (12) we proceed as on pp. 847–849 of [1], with some obvious changes to take into account the factor $e^{-ih\log n}-1$ in our definition of the $b_j(n)$. In this way we see that (12) holds for any $K\geq 1$, and Lemma 2 is proved.

We remark that we can easily obtain a version of Lemma 2 with h replaced by $M/\log t$, provided an additional error term

$$O\left(T(\log\log X)^{K+L}\frac{M\log X}{\log^2 T}\right)$$

is added in the statement of Lemma 2. We leave its verification to the reader.

3. Proof of theorems. The proof of Theorem 2 follows closely that of Theorem B of [1]. Let $M\to\infty$ as $T\to\infty$ and choose

$$\log X=\frac{\log T}{(\log M)^{1/4}},$$

so that

$$\log^+\left(\frac{M}{2}\frac{\log T}{\log X}\right)\sim\log M,\qquad \frac{\log X}{\log T}=(\log M)^{-1/4},\qquad X=T^{o(1)}.$$

Moreover, let

$$U_j(t)=(2\pi n_j\log M)^{-1/2}\Sigma_j(t)$$

and $\widetilde{\mu}_T$ be the associated probability measure on $\mathbb{C}^N$, defined as in (2).

Then, assuming that $M\leq(\log T)/\log\log T$ and arguing exactly as in the proof of Theorem B of [1], from Lemma 2 we see that $\widetilde{\mu}_T$ converges, as $T\to\infty$, to the gaussian measure $e^{-\pi\|\mathbf{z}\|^2}$. Also, from Lemma 1 we easily deduce that

$$\frac{1}{T}\int_T^{2T}|V_j(t)-U_j(t)|\,dt\ll(\log M)^{-1/4},$$

and hence μ_T converges to the same gaussian measure, completing the proof.

The proof of Theorem 1 is by contradiction. Let T_ν be a sequence along which

$$D_\nu:=D(2T_\nu,L_1,L_2)-D(T_\nu,L_1,L_2)=o(T_\nu\log T_\nu).$$

We set

$$M_\nu=\min\left(\frac{\log T_\nu}{\log\log T_\nu},\sqrt{\frac{T_\nu\log T_\nu}{1+D_\nu}}\right). \tag{13}$$

Then $M_\nu\to\infty$ and $M_\nu\ll(\log T_\nu)/\log\log T_\nu$, so that Theorem 2 is applicable to L_1, L_2 and the sequence T_ν, M_ν.

Write

$$h_\nu=M_\nu/\log T_\nu,$$
$$\Delta_N(t,h_\nu)=(N_1(t+h_\nu)-N_1(t))-(N_2(t+h_\nu)-N_2(t)),$$
$$\Delta_S(t,h_\nu)=(S_1(t+h_\nu)-S_1(t))-(S_2(t+h_\nu)-S_2(t))$$

and observe that (1) and $\Lambda_1 = \Lambda_2$ imply

$$\Delta_N(t,h_\nu) = \Delta_S(t,h_\nu) + O\left(\frac{M_\nu}{\log T_\nu}\right) \tag{14}$$

uniformly for $t \in [T_\nu, 2T_\nu]$.

For $j = 1,2$ and $t \in [T_\nu, 2T_\nu]$ we have

$$\operatorname{Im} V_j(t) = \frac{\pi}{(2\pi n_j \log M_\nu)^{1/2}}\,(S_j(t+h_\nu) - S_j(t)). \tag{15}$$

Thus from (14) and (15) we see that if $t \in [T_\nu, 2T_\nu]$ is such that

$$\operatorname{Im} V_2(t) < 0 \quad \text{and} \quad \operatorname{Im} V_1(t) > 1, \tag{16}$$

then

$$\begin{aligned} \Delta_N(t,h_\nu) &= \frac{1}{\pi}(2\pi n_1 \log M_\nu)^{1/2} \operatorname{Im} V_1(t) \\ &\quad - \frac{1}{\pi}(2\pi n_2 \log M_\nu)^{1/2} \operatorname{Im} V_2(t) + O(M_\nu/\log T_\nu) \\ &\ge \frac{1}{\pi}(2\pi n_1 \log M_\nu)^{1/2} + O(M_\nu/\log T_\nu). \end{aligned}$$

Denote by E_ν the set of $t \in [T_\nu, 2T_\nu]$ for which (16) holds.

In order to get a lower bound for $|E_\nu|$, we consider the set

$$\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : \operatorname{Im} z_1 > 1 \text{ and } \operatorname{Im} z_2 < 0\},$$

so that

$$|E_\nu| = T_\nu\, \mu_{T_\nu}(\Omega). \tag{17}$$

From Theorem 2 we obtain

$$\lim_{\nu\to\infty} \mu_{T_\nu}(\Omega) = \int_\Omega e^{-\pi\|\mathbf{z}\|^2}\, d\omega \gg 1. \tag{18}$$

From (15), (17) and (18) we see that $|E_\nu| \gg T_\nu$, and hence we deduce the existence of $\gg T_\nu/h_\nu$ values $t_r \in [T_\nu, 2T_\nu]$, with $|t_r - t_s| \ge h_\nu$ if $r \ne s$, such that

$$\Delta_N(t_r,h_\nu) \ge \frac{1}{\pi}(2\pi n_1 \log M_\nu)^{1/2} + O(M_\nu/\log T_\nu).$$

Therefore

$$D_\nu \ge \sum_r \Delta_N(t_r,h_\nu) \gg \frac{\sqrt{\log M_\nu}}{M_\nu}\, T_\nu \log T_\nu. \tag{19}$$

Now recall that

$$M_\nu = \min\left(\frac{\log T_\nu}{\log\log T_\nu}, \sqrt{\frac{T_\nu \log T_\nu}{1+D_\nu}}\right). \tag{13}$$

If in (13) we have $M_\nu = (\log T_\nu)/\log\log T_\nu$ we must also have

$$D_\nu \le T_\nu \frac{(\log\log T_\nu)^2}{\log T_\nu},$$

while (19) gives $D_\nu \gg T_\nu(\log\log T_\nu)^{3/2}$, a contradiction. The other alternative in (13) gives $M_\nu = \sqrt{(T_\nu \log T_\nu)/(1+D_\nu)}$, which substituted in (19) shows that $D_\nu \gg T_\nu \log T_\nu$; this contradicts our assumption $D_\nu = o(T_\nu \log T_\nu)$.

The proof of Theorem 1 is complete.

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