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Determination of all non-quadratic imaginary cyclic number fields of 2-power degree with relative class number ≤ 20

by

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1. Introduction. Recently, Louboutin [L1] has determined all imaginary cyclic number fields of 2-power degree with relative class number 1 and 2. (More precisely, he reduced the determination of all non-quadratic imaginary cyclic fields of 2-power degree with cyclic ideal class groups of 2-power orders to the determination of all the non-quadratic imaginary cyclic fields of 2-power degree with relative class number one or two.) In [L1] Louboutin has obtained good lower bounds for the relative class number of non-quadratic imaginary cyclic number fields of 2-power degree. Using these lower bounds we prove the following:

Theorem 1. *There are* 204 *non-quadratic imaginary cyclic fields of* 2-*power degree with relative class number* $h_N^- \leq 20$. *They all have degrees* ≤ 16 *and conductors* ≤ 2355. *Moreover*, *there are* 169 *non-quadratic imaginary cyclic fields of* 2-*power degree with class number* $h_N \leq 20$. *They all have degrees* ≤ 16 *and conductors* ≤ 1789.

In Section 2, we give lower bounds on the relative class numbers of non-quadratic imaginary cyclic fields of 2-power degree. These bounds enable us to get reasonable upper bounds on the conductors of those fields which have relative class number $h_N^- \leq 20$. In Section 3, we explain how we construct any imaginary cyclic quartic or octic field. In Section 4, we explain how we compute the relative class number of any non-quadratic imaginary cyclic field of 2-power degree. Using Sections 2 and 3 we will be in a position to determine in Section 4 all the non-quadratic imaginary cyclic fields of 2-power degree with relative class number $h_N^- \leq 20$. Finally, we will explain how we computed the class numbers of the real subfields N_+ of those 204 non-quadratic imaginary cyclic fields of 2-power degree with relative class

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number $h_N^- \leq 20$. All non-quadratic imaginary cyclic number fields of 2-power degree with relative class number ≤ 20 are given in Tables 1, 2 and 3.

2. Lower bounds for the relative class number. Let N be a CM-field of degree $2n$, N_+ its maximal totally real subfield, h_N the class number of N and h_N^- the relative class number of N. In order to determine all CM-fields of a given degree and given class number, we begin with a reasonable lower bound for h_N^-, which leads us to a feasible computation. For this purpose we apply the following theorem, due to Louboutin [L1].

THEOREM 2. *Let N be an imaginary cyclic number field of 2-power degree $2n = 2^m \geq 4$, conductor f_N and discriminant d_N. Then*

$$h_N^- \geq \frac{2\varepsilon_N}{e(2n-1)}\left(\frac{\sqrt{f_N}}{\pi(\log f_N + 0.05)}\right)^n$$

where

$$\varepsilon_N = 1 - \frac{2\pi n e^{1/n}}{d_N^{1/(2n)}} \quad \textit{or} \quad \frac{2}{5}\exp\left(-\frac{2n\pi}{d_N^{1/(2n)}}\right).$$

In particular,

$$\begin{array}{llll}
\textit{if} & n = 2 \textit{ and } f_N \geq 118000 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 4 \textit{ and } f_N \geq 14800 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 8 \textit{ and } f_N \geq 4900 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 16 \textit{ and } f_N \geq 2000 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 32 \textit{ and } f_N \geq 1300 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 64 \textit{ and } f_N \geq 1000 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 128 \textit{ and } f_N \geq 900 & \textit{then} & h_N^- > 20; \\
\textit{if} & n = 256 \textit{ and } f_N \geq 800 & \textit{then} & h_N^- > 20.
\end{array}$$

If $h_N^- \leq 20$, then $n \leq 256$.

Proof. See Theorem 4 of [L1]. For the last statement, it suffices to notice that $f_N \geq 2n + 1$. ■

3. Conductors of cyclic number fields. Let N be a cyclic number field of degree $2n = 2^m$, f_N the conductor of N, d_N the discriminant of N and h_N the class number of N. Let χ_N be a primitive character of order $2n$ such that N is associated with the cyclic group generated by χ_N, $\{\chi_N^i : 0 \leq i \leq 2n-1\}$. For any positive integer n and prime q, let $v_q(n)$ denote the exponent of q in the prime factorization of n. The following properties are very useful in determining all possible conductors smaller than a fixed constant.

PROPOSITION 1. *Let N be a quartic cyclic number field and k the quadratic subfield of N.*

(i) *The conductor f_N can be written as*

$$f_N = \prod_{i=1}^{s} p_i \prod_{j=1}^{t} q_j, \qquad f_k = 2^{\varepsilon} \prod_{j=1}^{t} q_j, \qquad s \geq 0 \text{ and } t \geq 1.$$

Here, p_i's and q_j's are all distinct, p_i is $2^2, 2^3$ or an odd prime, q_j is 2^4 or odd prime equal to 1 modulo 4, and $\varepsilon = -1$ or 0 according as 16 divides $\prod q_j$ or not. In addition, $d_N = f_N^2 f_k$.

(ii) *For a given conductor f_N with $f_N \equiv 0 \bmod 8$, there are 2^{t-1} imaginary cyclic quartic fields and 2^{t-1} real fields.*

(iii) *For a given conductor f_N with $v_2(f_N) = 2$, we assume $p_1 = 2^2$. Then there are 2^{t-1} cyclic quartic fields of conductor f_N; all of them are real if*

$$1 + \sum_{i \geq 2}^{s} \frac{p_i - 1}{2} + \sum_{j=1}^{t} \frac{q_j - 1}{4} \equiv 0 \pmod 2$$

and all are imaginary otherwise.

(iv) *For a given odd conductor f_N, there are 2^{t-1} cyclic quartic fields of conductor f_N; all of them are real if*

$$\sum_{i=1}^{s} \frac{p_i - 1}{2} + \sum_{j=1}^{t} \frac{q_j - 1}{4} \equiv 0 \pmod 2$$

and all are imaginary otherwise.

Proof. (i) Let χ_N be a primitive Dirichlet character modulo f_N of order 4 such that the cyclic group $\langle \chi_N \rangle$ is associated with the field N. Let $f_N = \prod p^a$. Corresponding to the decomposition

$$(\mathbb{Z}/f_N\mathbb{Z})^* = \prod (\mathbb{Z}/p^a\mathbb{Z})^*,$$

we may write χ_N as $\chi_N = \prod \chi_p$ where χ_p is a character defined modulo p^a. As χ_N has order 4, every χ_p has order 2 or 4 and at least one of the χ_p has order 4. If χ_p has order 2, then $p^a = 2^2, 2^3$ or an odd prime, χ_p is the Legendre symbol when p_i is odd, and χ_p is one of two primitive non-conjugate quadratic characters modulo 8 when $p = 2^3$. If χ_p has order 4, then $p^a = 2^4$ or an odd prime equal to 1 modulo 4. Moreover, in that case χ_p is one of two conjugate primitive quartic characters modulo p when p is odd, and χ_p is one of two non-conjugate characters modulo 16 where $p^a = 16$. Denote by q_j the divisor of f_N such that the corresponding character χ_{q_j} is of order 4, and by p_i the divisor of f_N such that χ_{p_i} is of order 2. We rewrite f_N as

$$f_N = \prod_{i=1}^{s} p_i \prod_{j=1}^{t} q_j \quad \text{and} \quad \chi_N = \prod_{i=1}^{s} \chi_{p_i} \prod_{j=1}^{t} \chi_{q_j},$$

with the convention $q_1 = 2^4$ if $v_2(f_N) = 4$, $p_1 = 2^3$ if $v_2(f_N) = 3$ and $p_1 = 2^2$ if $v_2(f_N) = 2$ (we allow $s = 0$). Then $\chi_N^2 = \prod \chi_{q_j}^2$ is associated with the quadratic subfield k. If q_j is an odd prime then the conductor of $\chi_{q_j}^2$ is also q_j and if $q_1 = 2^4$ then the conductor of $\chi_{q_1}^2$ is 2^3. So we have

$$f_k = 2^{\varepsilon} \prod_{j=1}^{t} q_j \quad \text{and} \quad d_N = 2^{\varepsilon} \prod p_i^2 \prod q_j^3$$

where $\varepsilon = -1$ or 0 according as 16 divides $\prod q_j$ or not.

(ii) (iii) and (iv). It suffices to notice that N is real or imaginary according as $\chi_N(-1) = 1$ or $\chi_N(-1) = -1$ and that

$$\begin{cases} \chi_{p_i}(-1) = (-1)^{(p_i-1)/2} & \text{if } p_i \text{ is an odd prime,} \\ \chi_{q_j}(-1) = (-1)^{(q_j-1)/4} & \text{if } q_j \text{ is an odd prime equal to 1 mod 4.} \end{cases} \blacksquare$$

REMARK 1. If N is an imaginary cyclic quartic field, then $2^s 4^{t-1}$ divides h_N. In fact, let G_N be the genus field of N. Then $[G_N : \mathbb{Q}] = 2^s 4^t$ and $[G_N : N] \mid h_N$.

We can prove the similar properties for octic cyclic number fields:

PROPOSITION 2. *Let N be an octic cyclic number field, K the quartic subfield of N, and k the quadratic subfield of N.*

(i) *The conductor f_N can be written as*

$$f_N = \prod_{i=1}^{s} p_i \prod_{j=1}^{t} q_j \prod_{k=1}^{u} r_k, \qquad s \geq 0,\ t \geq 0 \text{ and } u \geq 1,$$

with

$$f_K = 2^{\varepsilon_1} \prod_{j=1}^{t} q_j \prod_{k=1}^{u} r_k, \qquad f_k = 2^{2\varepsilon_2} \prod_{k=1}^{u} r_k.$$

Moreover,

$$d_N = f_N^4 f_K^2 f_k = 2^{2(\varepsilon_1+\varepsilon_2)} \prod_{i=1}^{s} p_i^4 \prod_{j=1}^{t} q_j^6 \prod_{k=1}^{u} r_k^7.$$

Here, p_i's, q_j's and r_k's are all distinct, p_i is $2^2, 2^3$ or an odd prime, q_j is 2^4 or an odd prime equal to 1 modulo 4, r_k is 2^5 or an odd prime equal to 1 modulo 8, $\varepsilon_1 = -1$ or 0 according as 2 divides $\prod q_j \prod r_k$ or not, and $\varepsilon_2 = -1$ or 0 according as 2 divides $\prod r_k$ or not.

(ii) *For a given conductor* f_N *with* $f_N \equiv 0 \bmod 8$, *there are* $2^t 4^{u-1}$ *real fields and* $2^t 4^{u-1}$ *imaginary fields.*

(iii) *For a given conductor* f_N *with* $v_2(f_N) = 2$, *we assume* $p_1 = 2^2$. *Then there are* $2^t 4^{u-1}$ *cyclic octic fields of conductor* f_N; *all of them are real if*

$$1 + \sum_{i\ge 2}^{s} \frac{p_i - 1}{2} + \sum_{j=1}^{t} \frac{q_j - 1}{4} + \sum_{k=1}^{u} \frac{r_k - 1}{8} \equiv 0 \pmod 2$$

and all are imaginary otherwise.

(iv) *For a given odd conductor* f_N, *there are* $2^t 4^{u-1}$ *cyclic octic fields of conductor* f_N; *all of them are real if*

$$\sum_{i=1}^{s} \frac{p_i - 1}{2} + \sum_{j=1}^{t} \frac{q_j - 1}{4} + \sum_{k=1}^{u} \frac{r_k - 1}{8} \equiv 0 \pmod 2$$

and all are imaginary otherwise.

Corollary 1. (i) *If* N *is an imaginary cyclic octic field*, *then* $2^{s+2t+3u-3}$ *divides* h_N.

(ii) *Let* N *be a non-quadratic imaginary cyclic number field of degree* $2n = 2^m \ge 4$. *Then* N *has odd class number if and only if* f_N *is* 2^{m+2} *or an odd prime equal to* $2n + 1 \bmod 4n$.

Proof. This follows from genus field theory and Theorem 10.4(b) of [W]. ■

Set

$$\zeta_N = \exp\left(\frac{2i\pi}{f_N}\right) \quad \text{and} \quad \zeta_{N_+} = \exp\left(\frac{2i\pi}{f_{N_+}}\right).$$

From χ_N and χ_{N_+} we can compute numerically two polynomials defining the number fields N and N_+, respectively, for

$$\theta_N = \sum_{\substack{g=1 \\ \chi_N(g)=1}}^{f_N-1} \zeta_N^g \quad \text{and} \quad \theta_{N_+} = \sum_{\substack{g=1 \\ \chi_{N_+}(g)=1}}^{f_{N_+}-1} \zeta_{N_+}^g$$

are primitive elements of N and N_+, respectively. However, if N or N_+ is quartic we use [HHRW1] to get a more convenient primitive element for N or N_+.

4. Main results. We can evaluate precisely the relative class number by the following formula:

$$h_N^- = Q w_N \prod_{\chi \text{ odd}} \left(-\tfrac{1}{2} B_{1,\chi}\right)$$

where Q is the Hasse unit index of N, w_N is the number of roots of unity in N, f_χ is the conductor of χ and $B_{1,\chi} = (1/f_\chi)\sum_{a=1}^{f_\chi-1}\chi(a)a$. The $B_{1,\chi}$ are called the *generalized Bernoulli numbers*. (See [W], Chapter 4, Theorem 4.) Now, according to [H] or [Lm] imaginary cyclic fields have the Hasse unit indices equal to 1, and according to Lemma (b) of [L2] for N an imaginary cyclic field of degree $2n = 2^m \geq 4$ we have $w_N = 2$ except if $2n+1 = 2^m+1 = p$ is prime and $N = \mathbb{Q}(\zeta_p)$. Therefore, when N is an imaginary cyclic field of degree $2n = 2^m \geq 4$, setting $\alpha_N = \sum_{a=1}^{f_N-1}\chi_N(a)a \in \mathbb{Z}[\zeta_{2^m}]$ we get

$$h_N^- = \frac{w_N}{(2f_N)^n}\prod_{\substack{i=1\\ i\,\mathrm{odd}}}^{2^m-1}\Big(\sum_{a=1}^{f_N-1}\chi_N^i(a)a\Big) = \frac{w_N}{(2f_N)^n}N_{\mathbb{Q}(\zeta_{2^m})/\mathbb{Q}}(\alpha_N).$$

From this relative class number formula we get the following proposition which explains why our computation did not yield any field with some relative class numbers:

PROPOSITION 3 (Louboutin). *Let N be an imaginary cyclic number field of degree $2n = 2^m \geq 4$ and let q be an odd prime. If q divides h_N^- then $v_q(h_N^-)$, the exponent of q in the factorization of h_N^-, is divisible by f_q, the order of q in the multiplicative group $(\mathbb{Z}/2^m\mathbb{Z})^*$. Therefore*,

$h_N^- \leq 20$ *and* $2n = 4$ *imply* $h_N^- \in \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20\}$,

$h_N^- \leq 20$ *and* $2n = 8$ *imply* $h_N^- \in \{1, 2, 4, 8, 9, 16, 17, 18\}$,

$h_N^- \leq 20$ *and* $2n = 16$ *imply* $h_N^- \in \{1, 2, 4, 8, 16, 17\}$.

Proof. Use Theorem 2.13 of [W] and the prime ideal factorization of the principal ideal $(\alpha_N) = (\sum_{a=1}^{f_N-1} a\chi_N(a))$ of $\mathbb{Q}(\zeta_{2^m})$. ■

To determine all the non-quadratic imaginary cyclic fields of degree $2n = 2^m \geq 4$ with relative class number $h_N^- \leq 20$ we proceeded as follows.

First, according to Theorem 2 and using Propositions 1 and 2 we found all the imaginary cyclic quartic fields with conductor $f_N \leq 118000$ (there are 64078 of them) and all the imaginary cyclic octic fields with $f_N \leq 14800$ (there are 3599 of them).

Second, we computed the relative class numbers of all those 67677 imaginary cyclic fields. We found that there are 188 imaginary cyclic quartic fields with $h_N^- \leq 20$ and 13 imaginary cyclic octic fields with $h_N^- \leq 20$.

Third, for all those 201 quartic and octic fields we computed the class numbers of their real subfields N_+. If N_+ is real quadratic, then this computation was easy. If N_+ is cyclic quartic, then we used the table of [M.N.G]. We found that 166 out of those 201 fields have class number $h_N \leq 20$.

Fourth, for imaginary cyclic fields of degree $2n = 2^m \geq 16$ results similar to those of Propositions 1 and 2 enabled us to make a list of all the imaginary cyclic fields of degree $2n = 2^m \geq 16$ with $f_N \leq 5000$ (see Theorem 2). There are 996 such fields.

Fifth, we computed their relative class numbers and found that 3 out of them have $h_N^- \leq 20$. Finally, using PARI-GP and polynomials defining N_+ for those 3 fields (see Section 3), we found that all have $h_N \leq 20$.

We list all imaginary cyclic quartic fields with relative class number ≤ 20 in Table 1, all imaginary cyclic octic fields with relative class number ≤ 20 in Table 2, and all imaginary cyclic fields of degree $2n = 2^m \geq 16$ with relative class number ≤ 20 in Table 3. The results of our computation agree with those of [G], [H], [HHRW1], [HHRW2], [HHRWH], [L1], [L2], [MM], [M.N.G], [S], [Y], [YH1] and [YH2].

Table 1. The imaginary cyclic quartic fields $N = \mathbb{Q}(\sqrt{-\beta_N})$ with $h_N^- \leq 20$

$h_N^- = 1$					
f	h_{N_+}	β_N	f	h_{N_+}	β_N
5	1	$5+2\sqrt{5}$	37	1	$37+6\sqrt{37}$
13	1	$13+2\sqrt{13}$	53	1	$53+2\sqrt{53}$
16	1	$2+\sqrt{2}$	61	1	$61+6\sqrt{61}$
29	1	$29+2\sqrt{29}$			

$h_N^- = 2$					
40	1	$5+\sqrt{5}$	80	2	$10+3\sqrt{10}$
48	1	$3(2+\sqrt{2})$	85	1	$17(5+2\sqrt{5})$
65	1	$13(5+2\sqrt{5})$	85	2	$85+6\sqrt{85}$
65	1	$5(13+2\sqrt{13})$	104	1	$13+3\sqrt{13}$
80	1	$5(2+\sqrt{2})$	119	1	$7(17+4\sqrt{17})$

$h_N^- = 4$					
60	1	$3(5+2\sqrt{5})$	164	1	$41+4\sqrt{41}$
68	1	$17+4\sqrt{17}$	195	2	$3(65+8\sqrt{65})$
105	1	$21(5+2\sqrt{5})$	205	2	$205+6\sqrt{205}$
112	1	$7(2+\sqrt{2})$	219	1	$3(73+8\sqrt{73})$
120	1	$3(5+\sqrt{5})$	221	1	$17(13+2\sqrt{13})$
136	1	$17+\sqrt{17}$	221	2	$221+14\sqrt{221}$
140	1	$7(5+2\sqrt{5})$	255	1	$15(17+4\sqrt{17})$
145	1	$29(5+2\sqrt{5})$	272	2	$34+3\sqrt{34}$
145	1	$5(29+2\sqrt{29})$			

Table 1 (cont.)

$h_N^- = 5$					
f	h_{N_+}	β_N	f	h_{N_+}	β_N
101	1	$101+10\sqrt{101}$	197	1	$197+14\sqrt{197}$
157	1	$157+6\sqrt{157}$	349	1	$349+18\sqrt{349}$
173	1	$173+2\sqrt{173}$	373	1	$373+18\sqrt{373}$

$h_N^- = 8$					
156	1	$3(13+2\sqrt{13})$	285	1	$57(5+2\sqrt{5})$
165	1	$33(5+2\sqrt{5})$	305	1	$61(5+2\sqrt{5})$
205	1	$41(5+2\sqrt{5})$	356	1	$89+8\sqrt{89}$
220	1	$11(5+2\sqrt{5})$	377	1	$29(13+2\sqrt{13})$
240	2	$3(10+3\sqrt{10})$	435	4	$3(145+8\sqrt{145})$
260	2	$65+4\sqrt{65}$	455	2	$7(65+4\sqrt{65})$
272	1	$17(2+\sqrt{2})$	545	1	$5(109+10\sqrt{109})$
273	1	$21(13+2\sqrt{13})$			

$h_N^- = 9$					
149	1	$149+10\sqrt{149}$	661	1	$661+6\sqrt{661}$
293	1	$293+2\sqrt{293}$			

$h_N^- = 10$					
51	1	$3(17+4\sqrt{17})$	365	2	$365+14\sqrt{365}$
80	2	$10+\sqrt{10}$	391	1	$23(17+4\sqrt{17})$
85	2	$85+2\sqrt{85}$	464	2	$58+3\sqrt{58}$
176	1	$11(2+\sqrt{2})$	481	1	$13(37+6\sqrt{37})$
185	1	$37(5+2\sqrt{5})$	485	2	$485+14\sqrt{485}$
185	1	$5(37+6\sqrt{37})$	493	2	$493+18\sqrt{493}$
208	1	$13(2+\sqrt{2})$	527	1	$31(17+4\sqrt{17})$
208	2	$26+\sqrt{26}$	533	1	$41(13+2\sqrt{13})$
208	2	$26+5\sqrt{26}$	533	2	$533+22\sqrt{533}$
265	1	$53(5+2\sqrt{5})$	565	2	$565+6\sqrt{565}$
265	1	$5(53+2\sqrt{53})$	685	2	$685+18\sqrt{685}$
267	1	$3(89+8\sqrt{89})$	699	1	$3(233+8\sqrt{233})$
287	1	$7(41+4\sqrt{41})$	771	3	$3(257+16\sqrt{257})$
304	1	$19(2+\sqrt{2})$	803	1	$11(73+8\sqrt{73})$
339	1	$3(113+8\sqrt{113})$	1261	2	$1261+6\sqrt{1261}$
365	1	$73(5+2\sqrt{5})$			

Table 1 (cont.)

$h_N^- = 13$					
f	h_{N_+}	β_N	f	h_{N_+}	β_N
269	1	$269+10\sqrt{269}$	509	1	$509+22\sqrt{509}$
317	1	$317+14\sqrt{317}$	557	1	$557+14\sqrt{557}$
397	1	$397+6\sqrt{397}$	1789	1	$1789+42\sqrt{1789}$

$h_N^- = 16$					
240	2	$3(10+\sqrt{10})$	520	2	$65+7\sqrt{65}$
260	2	$65+8\sqrt{65}$	580	4	$145+12\sqrt{145}$
312	1	$3(13+3\sqrt{13})$	584	1	$73+3\sqrt{73}$
336	1	$21(2+\sqrt{2})$	609	1	$21(29+2\sqrt{29})$
340	1	$5(17+4\sqrt{17})$	615	1	$15(41+4\sqrt{41})$
380	1	$19(5+2\sqrt{5})$	663	1	$39(17+4\sqrt{17})$
385	1	$77(5+2\sqrt{5})$	689	1	$53(13+2\sqrt{13})$
408	1	$3(17+\sqrt{17})$	795	2	$3(265+16\sqrt{265})$
429	1	$33(13+2\sqrt{13})$	799	1	$47(17+4\sqrt{17})$
440	1	$11(5+\sqrt{5})$	905	1	$5(181+10\sqrt{181})$
444	1	$3(37+6\sqrt{37})$	979	1	$11(89+8\sqrt{89})$
445	4	$445+18\sqrt{445}$	1015	4	$7(145+12\sqrt{145})$
452	1	$113+8\sqrt{113}$	1271	1	$31(41+4\sqrt{41})$
465	1	$93(5+2\sqrt{5})$	1351	1	$7(193+12\sqrt{193})$
496	1	$31(2+\sqrt{2})$	1595	4	$11(145+8\sqrt{145})$
505	1	$101(5+2\sqrt{5})$			

$h_N^- = 17$					
109	1	$109+10\sqrt{109}$	821	1	$821+14\sqrt{821}$
229	3	$229+2\sqrt{229}$	853	1	$853+18\sqrt{853}$
277	1	$277+14\sqrt{277}$			

$h_N^- = 18$					
424	1	$53+7\sqrt{53}$	949	2	$949+18\sqrt{949}$
493	1	$17(29+2\sqrt{29})$	1059	1	$3(353+8\sqrt{353})$
592	2	$74+7\sqrt{74}$	1165	2	$1165+18\sqrt{1165}$
629	2	$629+2\sqrt{629}$	1207	1	$71(17+4\sqrt{17})$
848	2	$106+9\sqrt{106}$			

Table 1 (cont.)

$h_N^- = 20$					
f	h_{N_+}	β_N	f	h_{N_+}	β_N
205	2	$205+14\sqrt{205}$	728	1	$7(13+3\sqrt{13})$
221	2	$221+10\sqrt{221}$	745	1	$149(5+2\sqrt{5})$
240	1	$15(2+\sqrt{2})$	772	1	$193+12\sqrt{193}$
280	1	$7(5+\sqrt{5})$	776	1	$97+9\sqrt{97}$
305	1	$5(61+6\sqrt{61})$	805	1	$161(5+2\sqrt{5})$
328	1	$41+5\sqrt{41}$	880	2	$11(10+\sqrt{10})$
345	1	$69(5+2\sqrt{5})$	935	1	$55(17+4\sqrt{17})$
348	1	$3(29+2\sqrt{29})$	959	1	$7(137+4\sqrt{137})$
368	1	$23(2+\sqrt{2})$	1001	1	$77(13+2\sqrt{13})$
377	1	$13(29+2\sqrt{29})$	1011	1	$3(337+16\sqrt{337})$
445	1	$89(5+2\sqrt{5})$	1040	4	$130+9\sqrt{130}$
460	1	$23(5+2\sqrt{5})$	1145	3	$5(229+2\sqrt{229})$
520	1	$13(5+\sqrt{5})$	1168	2	$146+11\sqrt{146}$
528	1	$33(2+\sqrt{2})$	1235	2	$19(65+8\sqrt{65})$
545	1	$109(5+2\sqrt{5})$	1243	1	$11(113+8\sqrt{113})$
555	2	$3(185+8\sqrt{185})$	1252	1	$313+12\sqrt{313}$
560	2	$7(10+3\sqrt{10})$	1295	2	$7(185+4\sqrt{185})$
572	1	$11(13+2\sqrt{13})$	1313	1	$101(13+2\sqrt{13})$
624	2	$3(26+5\sqrt{26})$	1313	1	$13(101+10\sqrt{101})$
645	1	$129(5+2\sqrt{5})$	1405	2	$1405+6\sqrt{1405}$
656	1	$41(2+\sqrt{2})$	1495	2	$23(65+4\sqrt{65})$
680	2	$85+9\sqrt{85}$	1599	1	$39(41+4\sqrt{41})$
696	1	$3(29+5\sqrt{29})$	1855	2	$7(265+12\sqrt{265})$
715	2	$11(65+8\sqrt{65})$	2355	6	$3(785+16\sqrt{785})$

Table 2. The imaginary cyclic octic fields N with $h_N^- \leq 20$

h_N^-	f	h_{N_+}	f_+	quartic subfield N_+
polynomial defining N				
1	32	1	16	$\mathbb{Q}(\sqrt{2+\sqrt{2}})$
$x^8+8x^6+20x^4+16x^2+2 \quad (N=\mathbb{Q}(\sqrt{-(2+\sqrt{2+\sqrt{2}})}))$				
1	41	1	41	$\mathbb{Q}(\sqrt{41+4\sqrt{41}})$
$x^8+x^7+3x^6+11x^5+44x^4-53x^3+153x^2-160x+59$				

Table 2 (cont.)

h_N^-	f	h_{N_+}	f_+	quartic subfield N_+	polynomial defining N
2	51	1	17	$\mathbb{Q}(\sqrt{17+4\sqrt{17}})$	$x^8 - x^7 + 10x^6 - 11x^5 + 15x^4 - 61x^3 + 58x^2 - 47x + 103$
2	85	2	85	$\mathbb{Q}(\sqrt{5(17+4\sqrt{17})})$	$x^8 - x^7 + 10x^6 - 79x^5 + 134x^4 + 41x^3 + 245x^2 - 846x + 596$
4	68	1	17	$\mathbb{Q}(\sqrt{17+4\sqrt{17}})$	$x^8 + 17x^6 + 68x^4 + 85x^2 + 17$
8	221	2	221	$\mathbb{Q}(\sqrt{13(17+4\sqrt{17})})$	$x^8 - x^7 + 27x^6 - 96x^5 + 576x^4 - 3512x^3 - 1421x^2 - 20515x + 139129$
17	137	1	137	$\mathbb{Q}(\sqrt{137+4\sqrt{137}})$	$x^8 + x^7 + 9x^6 + 105x^5 + 954x^4 + 3767x^3 + 9149x^2 + 12828x + 7607$
17	281	1	281	$\mathbb{Q}(\sqrt{281+16\sqrt{281}})$	$x^8 + x^7 + 18x^6 + 145x^5 - 794x^4 - 4463x^3 + 23729x^2 - 26540x + 559952$
18	96	1	16	$\mathbb{Q}(\sqrt{2+\sqrt{2}})$	$x^8 + 24x^6 + 180x^4 + 432x^2 + 162 \quad (N = \mathbb{Q}(\sqrt{-3(2+\sqrt{2+\sqrt{2}})}))$
18	119	1	17	$\mathbb{Q}(\sqrt{17+4\sqrt{17}})$	$x^8 - x^7 + 27x^6 - 28x^5 + 151x^4 - 350x^3 + 500x^2 - 846x + 1157$
18	160	2	80	$\mathbb{Q}(\sqrt{5(2+\sqrt{2})})$	$x^8 + 40x^6 + 500x^4 + 2000x^2 + 50$
18	365	1	73	$\mathbb{Q}(\sqrt{73+8\sqrt{73}})$	$x^8 - x^7 + 78x^6 + 17x^5 + 1706x^4 + 3421x^3 + 14117x^2 + 45478x + 272444$
18	485	2	485	$\mathbb{Q}(\sqrt{5(97+4\sqrt{97})})$	$x^8 - x^7 + 55x^6 + 156x^5 + 7384x^4 + 27896x^3 + 179695x^2 + 549x + 85941$

Table 3. The imaginary cyclic fields N of degree 16 with $h_N^- \le 20$ which are the only ones of degree $2^m \ge 16$ with $h_N^- \le 20$

f	h_N^-	polynomial defining N
f_+	h_{N_+}	polynomial defining the real octic subfield N_+
f_L	h_L	the quartic subfield L
17	1	$\mathbb{Q}(\zeta_{17})$
17	1	$\mathbb{Q}(\cos(2\pi/17))$
17	1	$\mathbb{Q}(\sqrt{17+4\sqrt{17}})$
64	17	$\mathbb{Q}(\sqrt{-(2+\sqrt{2+\sqrt{2+\sqrt{2}}})})$
32	1	$\mathbb{Q}(\sqrt{2+\sqrt{2+\sqrt{2}}})$
16	1	$\mathbb{Q}(\sqrt{2+\sqrt{2}})$
113	17	$x^{16}+x^{15}+4x^{14}+20x^{13}+110x^{12}+525x^{11}+325x^{10}-425x^9$ $+12062x^8-21729x^7+64244x^6-119403x^5+154492x^4$ $-132177x^3+210865x^2-281708x+132937$
113	1	$x^8+x^7-49x^6+16x^5+511x^4-367x^3-1499x^2+798x+1372$
113	1	$\mathbb{Q}(\sqrt{113+8\sqrt{113}})$

Remark 2. Some of the fields which appear in Tables 2 and 3 could be given explicitly. In Table 2, the first field of conductor 32 is $N = \mathbb{Q}\big(\sqrt{-(2+\sqrt{2+\sqrt{2}})}\big)$ and the ninth field of conductor 96 is $N = \mathbb{Q}\big(\sqrt{-3(2+\sqrt{2+\sqrt{2}})}\big)$. In Table 3, the first field of conductor 17 is $N = \mathbb{Q}(\zeta_{17})$ and $N_+ = \mathbb{Q}(\cos(2\pi/17))$ and the second field of conductor 64 is $N = \mathbb{Q}\big(\sqrt{-(2+\sqrt{2+\sqrt{2+\sqrt{2}}})}\big)$ and $N_+ = \mathbb{Q}\big(\sqrt{2+\sqrt{2+\sqrt{2}}}\big)$ (see [L2]).

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