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On the sequence of numbers of the form $\varepsilon_0 + \varepsilon_1 q + \ldots + \varepsilon_n q^n$, $\varepsilon_i \in \{0, 1\}$

by

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1. Introduction. Fix a real number $1 < q < 2$. For every nonnegative integer k let

$$k = \varepsilon_0 + 2\varepsilon_1 + \ldots + 2^n \varepsilon_n, \quad \varepsilon_i \in \{0, 1\} \tag{1}$$

be its dyadic expansion and set

$$x_k = \varepsilon_0 + \varepsilon_1 q + \ldots + \varepsilon_n q^n. \tag{2}$$

Denote by $y_0 < y_1 < \ldots$ the increasing rearrangement of the sequence (x_k), without repetitions. It is clear that

$$y_0 = 0, \quad y_1 = 1, \quad y_2 = q$$

and that

$$y_k \to \infty \quad \text{if } k \to \infty.$$

We are interested here in the behavior of the difference sequence $y_{k+1} - y_k$. Let us introduce for brevity the following notations:

$$l(q) = \inf(y_{k+1} - y_k), \quad L(q) = \limsup(y_{k+1} - y_k).$$

Note that $l(q) = \liminf(y_{k+1} - y_k)$. Indeed, fix $\varepsilon > 0$ arbitrarily. It is sufficient to show that there exist arbitrarily large integers $m < l$ such that $y_l - y_m < l(q) + \varepsilon$. By the definition of $l(q)$ there exists an integer k such that $y_{k+1} - y_k < l(q) + \varepsilon$. Then for every sufficiently large integer n (such that $q^n > y_{k+1}$) the numbers $q^n + y_k$ and $q^n + y_{k+1}$ are in the sequence (y_i). Denoting them by y_m and y_l we have $y_l - y_m = y_{k+1} - y_k < l(q) + \varepsilon$ and $l, m \to \infty$ as $n \to \infty$. Hence the claim follows.

We recall the following results; the first three of them were proved in [3], while the last one was obtained in [2].

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(a) $0 \le l(q) \le L(q) \le 1$ for all $1 < q < 2$.

(b) $L(q) = 1$ for all $A \le q < 2$ where $A = (1 + \sqrt{5})/2$.

(c) $L(q) > 0$ for all Pisot numbers.

(d) $l(q) = 1/q > 0$ for all Pisot numbers $1 < q < 2$ satisfying the equation $q^{r+1} = 1 + q + \ldots + q^r$ for some integer $r \ge 1$.

In the proof of (b) it was assumed that $q > A$, but the proof remains valid for $q = A$. (One can also give a different proof by adapting that of Proposition 3 below; see the remark following that proposition.)

In this paper we obtain several new estimates of $l(q)$ and of $L(q)$ for some special classes of numbers $1 < q < 2$. In particular, we obtain the following two results:

(e) $l(q) > 0$ for all Pisot numbers.

(f) $L(q) = 0$ (i.e. $y_{k+1} - y_k \to 0$) for all transcendental numbers $1 < q < \sqrt{2}$.

The property (e) was also obtained independently in another way by Y. Bugeaud [1]. He also proved a partial converse of this statement.

At the end of the paper we correct a small error in our previous paper [3] and we formulate some open problems.

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2. Pisot numbers. For any real number x let us denote by $\|x\|$ its distance from the closest integer. Our main result is the following:

THEOREM 1. *We have $l(q) > 0$ for all Pisot numbers. More precisely*,

$$L(q) \ge q^{-N}\Big(1 - \sum_{k=N}^{\infty} \|q^k\|\Big) \tag{3}$$

for all nonnegative integers N, and

$$l(q) \ge q^{-N}\Big(1 - \sum_{k=N}^{\infty} \|q^k\|\Big) \tag{4}$$

for all nonnegative integers N satisfying

$$\sum_{k=N}^{\infty} \|q^k\| < \frac{1}{q+1}. \tag{5}$$

Proof. Since $\|q^k\| \to 0$ exponentially if q is a Pisot number, the inequality (5) is satisfied if N is sufficiently large. Then it follows from (4) and (5) that

$$l(q) \ge q^{1-N}/(q+1) > 0.$$

It remains to prove the estimates (3) and (4).

For every real x there is a unique integer m satisfying $-1/2 < x - m \le 1/2$. Set $d(x) = x - m$. Then $-1/2 < d(x) \le 1/2$ and $|d(x)| = \|x\|$.

There is nothing to prove if $\sum_{k=N}^{\infty} \|q^k\| \ge 1$. Fix a nonnegative integer N such that

$$\sum_{k=N}^{\infty} \|q^k\| < 1$$

and set

$$\alpha = -\sum_{k=N}^{\infty} \min\{d(q^k), 0\}, \qquad \beta = \sum_{k=N}^{\infty} \max\{d(q^k), 0\}. \tag{6}$$

Then

$$\alpha \ge 0, \quad \beta \ge 0 \quad \text{and} \quad \alpha + \beta = \sum_{k=N}^{\infty} \|q^k\| < 1. \tag{7}$$

Consider the increasing sequence $y_0^N < y_1^N < \ldots$ of the numbers of the form (2) with $\varepsilon_0 = \ldots = \varepsilon_{N-1} = 0$. Since clearly $y_k^N = q^N y_k$ for all $k \ge 0$, it is sufficient to prove that

$$y_{k+1}^N - y_k^N \ge 1 - \sum_{j=N}^{\infty} \|q^j\| \tag{8}$$

for infinitely many $k \ge 0$, and that

$$y_{k+1}^N - y_k^N \ge 1 - \sum_{j=N}^{\infty} \|q^j\| \tag{9}$$

for all $k \ge 0$ if the condition (5) is satisfied.

It follows from (6) and (7) that for every $k \ge 0$ there is a unique integer $m = m(k)$ satisfying $m - \alpha \le y_k^N \le m + \beta$. Since $y_k^N \to \infty$, there are infinitely many k's for which $m(k) < m(k+1)$. For these k's we have (writing $m = m(k)$)

$$y_{k+1}^N - y_k^N \ge (m + 1 - \alpha) - (m + \beta) = 1 - (\alpha + \beta) = 1 - \sum_{j=N}^{\infty} \|q^j\| \tag{10}$$

and (8) follows.

Now assume (5). It follows from (6) and (7) that for any $l > k \ge 0$ we have either $0 < y_l^N - y_k^N \le \alpha + \beta$ or $y_l^N - y_k^N \ge 1 - (\alpha + \beta)$. It remains to prove that the first case never occurs.

Assume on the contrary that $0 < y_l^N - y_k^N \le \alpha + \beta$ for some $l > k \ge 0$. Choose an integer $m \ge 1$ such that $\alpha + \beta < q^m (y_l^N - y_k^N) \le q(\alpha + \beta)$ and consider the numbers $y_{l'}^N = q^m y_l^N$ and $y_{k'}^N = q^m y_k^N$. Then $\alpha + \beta < y_{l'}^N - y_{k'}^N \le q(\alpha + \beta)$. However, this is impossible because $q(\alpha + \beta) < 1 - (\alpha + \beta)$ by the assumption (5). ∎

Examples. 1. Let $q \approx 1.32472$ be the first Pisot number (the real root of $q^3 - q - 1 = 0$). Denoting its conjugates by q_2 and q_3, we have the (crude) estimates $\|q^k\| \le |q_2|^k + |q_3|^k$ for all $k \ge 0$. Applying the theorem with $N = 22$ resp. $N = 26$ and using these estimates we easily obtain $L(q) > 0.0006$ and $l(q) > 0.0004$.

2. Let $q \approx 1.46557$ be the fourth Pisot number (the real root of $q^3 - q^2 - 1 = 0$). Applying the theorem with $N = 15$ and with $N = 18$ we easily obtain $L(q) > 0.0011$ and $l(q) > 0.0006$. Simple numerical tests seem to indicate that $L(q) = q-1 \approx 0.46557$ and $l(q) = q^5 - q^4 - q^3 + q^2 - 1 \approx 0.1479$.

3. Let $q = A = (1+\sqrt{5})/2$. Applying the theorem with $N = 4$ we find that $l(q) > 0.09$. We recall from [2] (see (d) in the introduction) that $l(A) = 1/A \approx 0.618$. We also recall that $L(A) = 1$.

We can give lower bounds of $L(q)$ and $l(q)$ without using N.

Corollary 2. *Let q be a Pisot number. Denote by d the degree of its minimal polynomial, by $q_2, \ldots, q_d$ the conjugates of q and by Q the largest absolute value of these conjugates, so that $Q < 1$. Then*

$$L(q) \ge (2q)^{-1} q^{(\log(2d-2) - \log(1-Q))/\log Q}$$

and

$$l(q) \ge (1+q)^{-1} q^{(\log(d-1) + \log(1+q) - \log(1-Q))/\log Q}.$$

Proof. If we choose N such that

$$\sum_{k=N}^{\infty} \|q^k\| < 0.5,$$

then $L(q) > 2^{-1} q^{-N}$ by the preceding theorem. Since

$$\sum_{k=N}^{\infty} \|q^k\| \le \sum_{k=N}^{\infty} \sum_{j=2}^{d} |q_j|^k \le (d-1) \sum_{k=N}^{\infty} Q^k = (d-1)Q^N/(1-Q), \tag{11}$$

it is sufficient to choose N so that $(d-1)Q^N/(1-Q) < 0.5$, or equivalently,

$$N > \frac{\log(1-Q) - \log(d-1) - \log 2}{\log Q}.$$

Choosing the smallest integer N satisfying this inequality, we have

$$N - 1 \le \frac{\log(1-Q) - \log(d-1) - \log 2}{\log Q}$$

and therefore

$$L(q) > 2^{-1} q^{-N} = (2q)^{-1} q^{-(N-1)} \ge (2q)^{-1} q^{(\log(d-1) + \log 2 - \log(1-Q))/\log Q},$$

proving the first part of the corollary.

Next, if we choose N such that

$$\sum_{k=N}^{\infty} \|q^k\| < 1/(1+q),$$

then $l(q) > q^{1-N}(1+q)^{-1}$. By (11) it is sufficient to choose N so that

$$(d-1)Q^N/(1-Q) < 1/(1+q),$$

or equivalently,

$$N > \frac{\log(1-Q) - \log(d-1) - \log(1+q)}{\log Q}.$$

Choosing the smallest integer N satisfying this inequality, we have

$$N - 1 \le \frac{\log(1-Q) - \log(d-1) - \log(1+q)}{\log Q}$$

and therefore

$$l(q) > q^{1-N}(1+q)^{-1} \ge (1+q)^{-1} q^{(\log(d-1)+\log(1+q)-\log(1-Q))/\log Q},$$

proving the second part of the corollary. ∎

It is possible to obtain more accurate lower bounds of $L(q)$ by *ad hoc* arguments for special Pisot numbers. Let us give an example.

PROPOSITION 3. *If* $q \approx 1.46557$ *is the fourth Pisot number* (*i.e. the only real root of the equation* $q^3 = q^2 + 1$), *then none of the open intervals* $(q^n - (q-1), q^n)$ *contains any element* y_k. *Hence* $L(q) \ge q - 1$.

Proof. Assume that this is false and let $n \ge 0$ be the smallest integer such that there exists $y_k \in (q^n - (q-1), q^n)$. It follows easily from the relations

$$y_0 = 0, \quad y_1 = 1, \quad y_2 = q, \quad y_3 = q^2, \quad y_4 = q + 1$$

that $n \ge 4$. Furthermore, we have obviously

$$y_k = \varepsilon_0 + \varepsilon_1 q + \ldots + \varepsilon_{n-1} q^{n-1}.$$

Observe that $\varepsilon_{n-1} = 0$. Indeed, otherwise we would have

$$y_l := y_k - q^{n-1} \in (q^{n-3} - (q-1), q^{n-3}),$$

contradicting the minimality of n.

Similarly, we have $\varepsilon_{n-3} = 0$, for otherwise

$$y_l := y_k - q^{n-3} \in (q^{n-1} - (q-1), q^{n-1}),$$

again contradicting the minimality of n.

Next we claim that $\varepsilon_{n-2} = 1$. Indeed, otherwise y_k would be too small: we would have $y_k \le q^n - (q-1)$ by the following computation:

$$
\begin{aligned}
q^n-(q-1)-y_k &\ge q^n-(q-1)-(1+q+\ldots+q^{n-4})\\
&= q^n-q+1-\frac{q^{n-3}-1}{q-1}\\
&= \frac{q^{n+1}-q^n-q^2+2q-1-q^{n-3}+1}{q-1}\\
&= \frac{q^{n-2}-q^2+2q-q^{n-3}}{q-1}=\frac{q^{n-3}(q-1)+q(2-q)}{q-1}>0.
\end{aligned}
$$

Now it follows that $\varepsilon_{n-4}=0$. Indeed, otherwise

$$y_l := y_k-q^{n-2}-q^{n-4}=y_k-q^{n-1}\in(q^{n-3}-(q-1),q^{n-3}),$$

contradicting the minimality of n.

However, this is also impossible, because now we have $y_k<q^n-(q-1)$. Indeed, using also the relation $q^2(q-1)=1$ and the inequality $q>\sqrt{2}$, we obtain

$$
\begin{aligned}
q^n-y_k &\ge q^n-(1+q+\ldots+q^{n-5}+q^{n-2})\\
&= q^n-q^{n-2}-\frac{q^{n-4}-1}{q-1}=\frac{q^n(q-1)-q^{n-2}(q-1)-q^{n-4}+1}{q-1}\\
&= \frac{q^{n-2}-2q^{n-4}+1}{q-1}>\frac{1}{q-1}. \ \blacksquare
\end{aligned}
$$

Remark. One can prove by a similar but simpler argument that if $q=A$, then none of the open intervals (q^n,q^n+1) $(n=1,2,\ldots)$ contains any element of the sequence (y_k). Hence $L(q)=1$.

3. Numbers q close to 1. We do not know whether $L(q)=0$ for all q sufficiently close to 1. We have the following weaker result:

Theorem 4. *We have $L(q)\to 0$ as $q\to 1$. More precisely, $L(q)\le (q^2-1)e$ for all $1<q<2$.*

Proof. If $q\ge 6/5$, then $(q^2-1)e>1$ and the estimate follows from the inequality $L(q)\le 1$. Assume therefore that $1<q<1.2$; then there exists an *odd* integer $n\ge 5$ satisfying

$$1+\frac{1}{n+2}\le q<1+\frac{1}{n}.$$

Consider the numbers $q<q^3<\ldots<q^{n+2}$. First of all, we have

$$q^{n+2}\ge\left(\frac{n+3}{n+2}\right)^{n+2}>\left(\frac{4}{3}\right)^3>2+\frac{1}{3}>q+1$$

because $n\ge 3$. Furthermore,

$$q^3-q<q^5-q^3<\ldots<q^{n+2}-q^n$$

and

$$q^{n+2} - q^n = (q^2 - 1)q^n < (q^2 - 1)e =: \delta.$$

We claim that for every real number $\alpha > q$ there exists a y_k satisfying $\alpha - \delta < y_k < \alpha$. Indeed, since $1 < q^2 < 2$, we have $L(q^2) \le 1$ by (a) of the introduction. Hence there exists

$$\overline{y} = \varepsilon_0 + \varepsilon_2 q^2 + \ldots + \varepsilon_{2m} q^{2m}$$

such that $\alpha - q - 1 \le \overline{y} < \alpha - q$. Consider the numbers

$$\overline{y} + q < \overline{y} + q^3 < \ldots < \overline{y} + q^{n+2}.$$

The first of them is clearly less than α, while the last one is greater than α:

$$\overline{y} + q^{n+2} > \overline{y} + q + 1 \ge \alpha.$$

Furthermore, the distance of two consecutive numbers is always less than δ. It follows that if we denote by y_k the largest term of this sequence which is still less than α, then $\alpha - \delta < y_k < \alpha$.

The above claim implies that

$$\limsup (y_{k+1} - y_k) \le \delta,$$

and the proof is complete. ∎

Our next result shows that $y_{k+1} - y_k \to 0$ for almost all numbers q sufficiently close to 1.

THEOREM 5. *Let q be a real number satisfying $1 < q < \sqrt{2}$ and $l(q^2) = 0$. Then $L(q) = 0$, i.e. $y_{k+1} - y_k \to 0$. In particular, this is true when $1 < q < \sqrt{2}$ and q is transcendental.*

We need three lemmas.

LEMMA 6. *Let $1 < q < 2$ satisfy $l(q) = 0$ and fix $\delta > 0$. Then there exists a subsequence (z_k) of (y_k) satisfying the following two conditions*:

(a) *if $i \ne j$, then z_i and z_j have no common term q^n*;
(b) *$\delta < z_{2i} - z_{2i-1} < 2\delta$ for all $i = 1, 2, \ldots$*

Proof. Since $l(q) = 0$, there exist $l > k \ge 1$ such that $0 < y_l - y_k < \delta$. (We may even choose $l = k+1$.) By omitting the common terms q^n (if any), we may assume that y_k and y_l have no common terms. Choose a positive integer m such that $\delta < q^m(y_l - y_k) < 2\delta$ (possible because $1 < q < 2$), and set $z_1 = q^m y_k$, $z_2 = q^m y_l$.

Now we proceed by induction. Assume that $z_1 < \ldots < z_{2n}$ are already defined for some $n \ge 1$ and that they satisfy the conditions (a) and (b).

Fix a positive integer N such that $q^N > z_{2n}$. Then none of the numbers $z_1 < \ldots < z_{2n}$ contains any term q^i with $i \ge N$. Since $l(q) = 0$, there exist $l > k \ge 1$ such that $0 < y_l - y_k < q^{-N}\delta$. We may also assume that y_k and y_l have no common terms. Choose a positive integer m such that

$q^{-N}\delta < q^m(y_l - y_k) < 2q^{-N}\delta$ (possible because $1 < q < 2$), and set $z_{2n+1} = q^{N+m}y_k$, $z_{2n+2} = q^{N+m}y_l$. Then $\delta < z_{2n+2} - z_{2n+1} < 2\delta$. Furthermore, z_{2n+2}, z_{2n+1} have no common term, and no term q^i with $i \le N$. Hence the properties (a) and (b) continue to hold. ■

LEMMA 7. *Let $1 < q < 2$ satisfy $l(q) = 0$ and fix $\delta > 0$, $D > 0$. Then there exists a finite subsequence*

$$w_0 < w_1 < \ldots < w_m \tag{12}$$

of (y_k) such that

$$w_i - w_{i-1} < 2\delta, \quad i = 1, \ldots, m, \tag{13}$$

and

$$w_m - w_0 > D. \tag{14}$$

Proof. Consider the sequence (z_k) of the preceding lemma. Choose an integer $m > D/\delta$ and define

$$\begin{aligned} w_0 &= z_1 + z_3 + z_5 + \ldots + z_{2m-3} + z_{2m-1}, \\ w_1 &= z_2 + z_3 + z_5 + \ldots + z_{2m-3} + z_{2m-1}, \\ w_2 &= z_2 + z_4 + z_5 + \ldots + z_{2m-3} + z_{2m-1}, \\ &\vdots \\ w_{m-1} &= z_2 + z_4 + z_6 + \ldots + z_{2m-2} + z_{2m-1}, \\ w_m &= z_2 + z_4 + z_6 + \ldots + z_{2m-2} + z_{2m}. \end{aligned}$$

We clearly have (12) and it follows from property (a) of the preceding lemma that (w_i) is a subsequence of (y_k). It is also clear from (b) that (13) is satisfied. Finally, (14) also follows from (b):

$$w_m - w_0 = (z_2 - z_1) + \ldots + (z_{2m} - z_{2m-1}) > m\delta > D. \ ■$$

LEMMA 8. *If $1 < q < 2$ and q does not satisfy any algebraic equation with integer coefficients belonging to the set $\{-1, 0, 1\}$, then $l(q) = 0$.*

Proof. Fix $\delta > 0$. Choose a sufficiently large n with $(q^n - 1)/(q - 1) < (2^n - 1)\delta$ and consider the numbers x_i, $0 \le i < 2^n$, constructed in the introduction. It follows from our assumption on q that they are all different. Furthermore, all these 2^n numbers belong to the interval $[0, 1 + \ldots + q^{n-1}]$ whose length is less than $(2^n - 1)\delta$ by the choice of n. Therefore, by the box principle there are two x_i whose distance is less than δ. Hence $l(q) < \delta$. Letting $\delta \to 0$ we conclude that $l(q) = 0$. ■

Proof of Theorem 5. Fix $\delta > 0$ and apply Lemma 7 with q^2 instead of q. It follows that there exists a finite sequence $a_0 < a_1 < \ldots < a_m$ of numbers of the form

$$\varepsilon_0 + \varepsilon_2 q^2 + \varepsilon_4 q^4 + \ldots + \varepsilon_{2n} q^{2n}, \qquad \varepsilon_i \in \{0,1\},$$

satisfying

$$0 < a_i - a_{i-1} < 2\delta, \qquad i = 1, \ldots, m, \qquad a_m - a_0 > q.$$

On the other hand, since $L(q^2) \le 1$ (see (a) in the introduction), every open interval $I \subset (0, \infty)$ of length q contains at least one number of the form

$$\varepsilon_1 q + \varepsilon_3 q^3 + \varepsilon_5 q^5 + \ldots + \varepsilon_{2n+1} q^{2n+1}, \qquad \varepsilon_i \in \{0,1\}. \tag{15}$$

It follows that every interval $(x, x+2\delta)$, $x > a_0 + q$, contains at least one y_k. Indeed, choose b of the form (15) in $(x - a_0 - q, x - a_0)$ and consider the numbers

$$b + a_0 < b + a_1 < \ldots < b + a_m.$$

It is clear that they all are in the sequence (y_k). Since $b + a_0 < x$, $b + a_m > b + a_0 + q > x$ and since the difference of two consecutive elements is always less than 2δ, it follows that at least one of them lies in $(x, x+\delta)$.

We have thus proved that $L(q) \le 2\delta$. Since $\delta > 0$ was arbitrary, we conclude that $L(q) = 0$.

The last part of the theorem follows from Lemma 8. ∎

The following result completes Theorem 5:

PROPOSITION 9. *We have* $L(\sqrt{2}) = 0$.

Proof. Fix $\delta > 0$ and choose an integer $N > 1/\delta$. There exist two integers $0 \le k < l \le N$ such that the fractional part of $l\sqrt{2} - k\sqrt{2}$ is in $(0, 1/N)$. Taking integer multiples of $l\sqrt{2} - k\sqrt{2}$, it follows easily that there exists a finite sequence of integers $k_1 < \ldots < k_N$ such that every interval of length δ contains at least one number having the same fractional part as one of $k_i\sqrt{2}$, $1 \le i \le N$.

It follows that every interval $(x, x+\delta)$, $x > k_N\sqrt{2}$, contains at least one y_k. Indeed, let $x < x' < x + \delta$ and $1 \le i \le N$ be such that x' and $k_i\sqrt{2}$ have the same fractional part. Then $l := x' - k_i\sqrt{2}$ is a nonnegative integer and hence $x' = l + k_i\sqrt{2}$ is in the sequence (y_k). ∎

Correction. We have proven in [3] that if $1 < q < 2$ and $L(q) = 0$, then the number 1 has an infinite expansion containing arbitrarily long sequences of consecutive 0 digits (Theorem 4, part (c)). In the proof, at the bottom of page 388, the sentence "It is equal to y_n for some $n \ge 1$." should be changed to: "If m is sufficiently large, then this number belongs to the interval $(y_{n-1}, y_n]$ for some $n \ge 2$." Next, the first sentence on the top of page 389 should be changed to: "It follows from (19) that $y_{n-1} < q^{m-k-n_{i_k}}$ and therefore $n_{1+i_k} > k + n_{i_k} \ge n_{i_k}$." The rest of the proof is the same.

Let us note that the proof can easily be modified to prove, more generally, that under the same assumption $L(q) = 0$, *every* $x \in (0, 1/(q-1))$ has an infinite expansion containing arbitrarily long sequences of consecutive 0 digits.

4. Open problems

1. Is it true that $l(q) > 0$ if and only if q is a Pisot number?

2. It would be interesting to determine the exact values of $l(q)$ and $L(q)$ for the Pisot numbers. Is it possible to adapt the proof of Proposition 3 for all Pisot numbers?

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