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Asymptotic behavior of Pascal's triangle modulo a prime

by

Brad Wilson (Brockport, N.Y.)

1. Definitions and notation. Let p be a prime and let $F_p(n)$ denote the number of entries in the first n rows of Pascal's triangle not divisible by p. In 1947 Fine [1] showed

Theorem (Fine, 1947).

$$\lim_{n\to\infty} F_p(n)\Big/\binom{n+1}{2} = 0.$$

Fine made use of the well-known result of Kummer [3]:

Theorem (Kummer, 1852). *The highest power of a prime p that divides $\binom{n}{m}$ is equal to the number of carries that occur when adding m and $n-m$ in base p.*

The following is a well-known corollary of Kummer's theorem which we will need.

Corollary (of Kummer's Theorem). *The number of entries in the nth row of Pascal's triangle not divisible by p is*

$$2^{r_1}3^{r_2}\dots p^{r_{p-1}}$$

where $r_i = |\{j : n_j = i\}|$ for $n = (n_k n_{k-1}\dots n_1 n_0)_p$ is the number of i's in the base p expansion of n.

Let $\theta = \ln 3/\ln 2$ and

$$\alpha = \limsup_{n\to\infty} F_2(n)/n^\theta, \qquad \beta = \liminf_{n\to\infty} F_2(n)/n^\theta.$$

In 1977 Harborth [2] showed

Theorem (Harborth, 1977). $\alpha = 1$ *and* $\beta = .812556\dots$

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The value of β was determined to six decimal places and Harborth conjectured that $\beta = \lim_{r\to\infty} q_r$ for $q_r = F(n_r)/n_r^\theta$ where $n_0 = 1, n_i = n_{i-1} \pm 1$ with sign chosen to minimize $F(n_i)/n_i^\theta$.

More generally, let $\theta_p = \ln(p(p+1)/2)/\ln p$ and

$$\alpha_p = \limsup_{n\to\infty} F_p(n)/n^{\theta_p}, \qquad \beta_p = \liminf_{n\to\infty} F_p(n)/n^{\theta_p}.$$

In 1989 Stein [4] proved

THEOREM (Stein, 1989). *$\alpha_p = 1$.*

In this paper we calculate $\beta_3, \beta_5, \ldots, \beta_{19}$ to six decimal places, get upper and lower bounds on β_p which allow us to show $\lim_{p\to\infty} \beta_p = .5$, and expand on Harborth's conjecture.

2. Lemmas on the behavior of $F_p(n)$. We first note some elementary facts about $F_p(n)$.

LEMMA 1. *For $1 \le a \le p-1$ and $0 \le b \le p^r$,*

$$F_p(ap^r + b) = \frac{a(a+1)}{2} F_p(p^r) + (a+1)F_p(b).$$

Proof. This is a corollary of [4], Lemma 3. ∎

Lemma 1 allows us to prove that $F_p(n) = n^{\theta_p}$ for infinitely many well-chosen n:

LEMMA 2. *$F_p(p^r) = (p(p+1)/2)^r = (p^r)^{\theta_p}$ for all $r \in \mathbb{N}$.*

Proof. This was shown in [1]. ∎

LEMMA 3. *For p a prime and $n, k \in \mathbb{N}$,*

$$F_p(p^k n) = \left(\frac{p(p+1)}{2}\right)^k F_p(n),$$

and

$$\frac{F_p(p^k n)}{(p^k n)^{\theta_p}} = \frac{F_p(n)}{n^{\theta_p}}.$$

Proof. This is a corollary of [4], Lemma 3. ∎

LEMMA 4. *For $n = a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0$, $a_k \ne 0$, $0 \le a_i < p$ we have*

$$F_p(p^k + n) = \begin{cases} \dfrac{2+a_k}{2}\left(\dfrac{p(p+1)}{2}\right)^k + \dfrac{2+a_k}{1+a_k} F_p(n) & \text{if } a_k \ne p-1, \\ \left(\dfrac{p(p+1)}{2}\right)^{k+1} - (p-1)\left(\dfrac{p(p+1)}{2}\right)^k + \dfrac{2}{p} F_p(n) & \text{if } a_k = p-1. \end{cases}$$

Proof. If $a_k < p-1$ then $1+a_k < p$ so

$$F_p(p^k+n) = F_p((1+a_k)p^k + a_{k-1}p^{k-1} + \ldots + a_1p + a_0).$$

By Lemma 1 this becomes

$$\begin{aligned}
&\frac{(2+a_k)(1+a_k)}{2}F_p(p^k) + (2+a_k)F_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\\
&= \frac{2+a_k}{2}F_p(p^k) + \frac{a_k(2+a_k)}{2}F_p(p^k)\\
&\quad + (2+a_k)F_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\\
&= \frac{2+a_k}{2}F_p(p^k)\\
&\quad + \frac{2+a_k}{1+a_k}\left(\frac{a_k(1+a_k)}{2}F_p(p^k) + (1+a_k)F_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\right).
\end{aligned}$$

Using Lemmas 1 and 2 this is equal to

$$\begin{aligned}
&\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}F_p(a_kp^k + a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\\
&\qquad = \frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}F_p(n).
\end{aligned}$$

If $a_k = p-1$ then $1+a_k = p$ so

$$F_p(p^k+n) = F_p(p^{k+1} + a_{k-1}p^{k-1} + \ldots + a_1p + a_0),$$

which by Lemma 1 is

$$F_p(p^{k+1}) + 2F_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0).$$

With some algebraic manipulation we get

$$\begin{aligned}
&F_p(p^{k+1}) - (p-1)F_p(p^k) + (p-1)F_p(p^k) + \frac{2}{p}\cdot pF_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\\
&= F_p(p^{k+1}) - (p-1)F_p(p^k)\\
&\quad + \frac{2}{p}\left(\frac{p(p-1)}{2}F_p(p^k) + pF_p(a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\right)\\
&= F_p(p^{k+1}) - (p-1)F_p(p^k) + \frac{2}{p}F_p((p-1)p^k + a_{k-1}p^{k-1} + \ldots + a_1p + a_0)\\
&= \left(\frac{p(p+1)}{2}\right)^{k+1} - (p-1)\left(\frac{p(p+1)}{2}\right)^k + \frac{2}{p}F_p(n). \ \blacksquare
\end{aligned}$$

The next lemma says that if we have a lower bound on $F_p(n)/n^{\theta_p}$ for n with lead coefficient a_k then we get a lower bound on the quotient where the lead coefficient of n is incremented by one. The new bound depends only on the value of the lead coefficient and not on k (i.e. the power of p it is a coefficient of).

LEMMA 5. *For* $1 \le a_k < p-1$ *fixed, if there is a constant* $\gamma_p(a_k)$ *so that*

$$\frac{F_p(n)}{n^{\theta_p}} > \gamma_p(a_k)$$

for all $n = a_k p^k + a_{k-1}p^{k-1} + \ldots + a_1 p + a_0$, *then*

$$\frac{F_p(p^k+n)}{(p^k+n)^{\theta_p}} > \frac{2+a_k}{2}\left(1+\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\right)^{1-\theta_p},$$

i.e., we may take $\gamma_p(1+a_k)$ *as the right hand side of the inequality.*

Proof. By Lemma 4 we have

$$\frac{F_p(p^k+n)}{(p^k+n)^{\theta_p}} = \frac{\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}F_p(n)}{(p^k+n)^{\theta_p}}.$$

Using our assumption $F_p(n) > n^{\theta_p}\gamma_p(a_k)$ we get

$$\frac{F_p(p^k+n)}{(p^k+n)^{\theta_p}} > \frac{\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}n^{\theta_p}\gamma_p(a_k)}{(p^k+n)^{\theta_p}}.$$

Define

$$f(n) = \frac{\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}n^{\theta_p}\gamma_p(a_k)}{(p^k+n)^{\theta_p}}$$

as a function of the continuous variable n (we are interested in showing the inequality of Lemma 5 only for integral values of n but to do this we will think of the bounding function f as a function of a continuous variable so we can use the Calculus). Differentiating gives

$$\frac{df}{dn} = \frac{\frac{2+a_k}{1+a_k}\gamma_p(a_k)\theta_p n^{\theta_p-1}p^k - \theta_p\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k}{(p^k+n)^{\theta_p+1}}.$$

Setting the numerator equal to zero and cancelling common factors we get

$$\frac{\gamma_p(a_k)}{1+a_k}n^{\theta_p-1} = \frac{1}{2}\left(\frac{p+1}{2}\right)^k$$

so

$$n = \left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\left(\left(\frac{p+1}{2}\right)^{1/(\theta_p-1)}\right)^k = \left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}p^k.$$

As there is only one critical point for $n \geq 0$ and it is a relative minimum as is easily checked by, for example, the Second Derivative Test, it is an absolute minimum for $f(n),\ n \geq 0$.

Using the definition of $f(n)$ we get

$$\frac{F_p(p^k+n)}{(p^k+n)^{\theta_p}} > f(n) \geq f\left(\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)} p^k\right)$$
$$= \frac{\frac{2+a_k}{2}\left(\frac{p(p+1)}{2}\right)^k + \frac{2+a_k}{1+a_k}\gamma_p(a_k)(p^{\theta_p})^k\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{\theta_p/(\theta_p-1)}}{\left(p^k + p^k\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\right)^{\theta_p}}.$$

Since $p^{\theta_p} = p(p+1)/2$ we get

$$\frac{F_p(p^k+n)}{(p^k+n)^{\theta_p}} > \frac{2+a_k}{2}\cdot\frac{1+\frac{2\gamma_p(a_k)}{1+a_k}\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{\theta_p/(\theta_p-1)}}{\left(1+\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\right)^{\theta_p}}$$
$$= \frac{2+a_k}{2}\cdot\frac{1+\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}}{\left(1+\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\right)^{\theta_p}}$$
$$= \frac{2+a_k}{2}\left(1+\left(\frac{1+a_k}{2\gamma_p(a_k)}\right)^{1/(\theta_p-1)}\right)^{1-\theta_p}. \ \blacksquare$$

The next lemma will allow us to get a lower bound on $F_p(n)/n^{\theta_p}$ where n has $k+1$ digits base p given a lower bound on $F_p(m)/m^{\theta_p}$ for all m with at most k digits base p.

LEMMA 6. *For* $n = p^{k+r+1} + a_k p^{k+r} + \ldots + a_1 p^{r+1} + b$ *with* $0 \leq b = b_r p^r + \ldots + b_1 p + b_0 < p^{r+1}$ *and for*

$$A = \left(\frac{p(p+1)}{2}\right)^{k+1} + 2\sum_{i=1}^{k}\left(\left(\frac{p(p+1)}{2}\right)^i \frac{a_i(a_i+1)}{2}\prod_{j=i+1}^{k}(a_j+1)\right)$$

and

$$B = 2\gamma_p(b_r)\prod_{i=1}^{k}(a_i+1),$$

with γ_p *as in the previous lemma, we have*

$$\frac{F_p(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}} > \frac{Ap^r}{n-b}\biggl(\frac{n-b}{p^r}+\biggl(\frac{Ap^r}{(n-b)B}\biggr)^{1/(\theta_p-1)}\biggr)^{1-\theta_p}.$$

Proof. Using Lemmas 1 and 2 we have

$$\begin{aligned}
&\frac{F_p(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}}\\
&\quad= \frac{\Bigl(\frac{p(p+1)}{2}\Bigr)^{k+r+1}+2\sum_{i=1}^{k}\Bigl(\frac{a_i(1+a_i)}{2}\Bigl(\frac{p(p+1)}{2}\Bigr)^{r+i}\prod_{j>i}(1+a_j)\Bigr)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}}\\
&\qquad+\frac{2\Bigl(\prod_{i=1}^{k}(1+a_i)\Bigr)F_p(b)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}}\\
&\quad> \frac{\Bigl(\frac{p(p+1)}{2}\Bigr)^{k+r+1}+2\sum_{i=1}^{k}\Bigl(\frac{a_i(1+a_i)}{2}\Bigl(\frac{p(p+1)}{2}\Bigr)^{r+i}\prod_{j>i}(1+a_j)\Bigr)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}}\\
&\qquad+\frac{2\Bigl(\prod_{i=1}^{k}(1+a_i)\Bigr)\gamma_p(b_r)b^{\theta_p}}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}}.
\end{aligned}$$

Factoring out $p^{r\theta_p}$ from the denominator and noting that $p^{\theta_p}=p(p+1)/2$ we define

$$f(b)=\frac{A+B(b/p^r)^{\theta_p}}{(p^{k+1}+a_kp^k+\ldots+a_1p^1+b/p^r)^{\theta_p}}$$

so

$$\frac{F_p(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}} > f(b).$$

Treating $f(b)$ as a continuous function of b we find the only critical point for $f(b)$ is a relative minimum at

$$b=\biggl(\frac{Ap^{r\theta_p}}{B(n-b)}\biggr)^{1/(\theta_p-1)}.$$

Therefore

$$\frac{F_p(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)}{(p^{k+r+1}+a_kp^{k+r}+\ldots+a_1p^{r+1}+b)^{\theta_p}} > f(b)$$

$$\geq f\left(\left(\frac{Ap^{r\theta_p}}{B(n-b)}\right)^{1/(\theta_p-1)}\right)$$

$$= \frac{A+B\left(\frac{Ap^r}{B(n-b)}\right)^{\theta_p/(\theta_p-1)}}{\left(\frac{n-b}{p^r}+\left(\frac{Ap^r}{B(n-b)}\right)^{1/(\theta_p-1)}\right)^{\theta_p}}$$

$$= \frac{Ap^r}{n-b}\left(\frac{n-b}{p^r}+\left(\frac{Ap^r}{B(n-b)}\right)^{1/(\theta_p-1)}\right)^{1-\theta_p}.$$

Since $(n-b)/p^r = p^{k+1}+a_kp^k+\ldots+a_1p$ it is obvious that this lower bound does not depend on r. ■

3. Values of β_p for small p. Using Lemmas 3, 5 and 6 and a fair bit of machine computation it is possible to approximate the values of β_p to any number of desired decimal places. In the theorem below we give the values of β_p, $p < 20$, to six decimal places. In the next section we will get some bounds on β_p and use these to investigate the nature of β_p for large p.

THEOREM 1.

$$\beta_3 = .774281\ldots,\quad \beta_5 = .758226\ldots,\quad \beta_7 = .749117\ldots,\quad \beta_{11} = .736495\ldots,$$
$$\beta_{13} = .732663\ldots,\quad \beta_{17} = .727582\ldots,\quad \beta_{19} = .725754\ldots$$

Proof. We will prove this for $p = 5$. The other proofs are similar. First note that

$$\frac{F_5(2929687)}{2929687^{\theta_5}} = .758226\ldots$$

and by Lemma 3,

$$\frac{F_5(5^k\cdot 2929687)}{(5^k\cdot 2929687)^{\theta_5}} = \frac{F_5(2929687)}{2929687^{\theta_5}}.$$

This means

$$\beta_5 \leq \frac{F_5(2929687)}{2929687^{\theta_5}} = .758226\ldots$$

To complete the proof we must show $F_5(n)/n^{\theta_5} \geq .758226$ for all n. Note that if $F_5(n)/n^{\theta_5} \geq .758226$ for all $1 \leq n < 2\cdot 5^k$, then Lemma 5 says

$$\frac{F_5(n)}{n^{\theta_5}} \geq .802517\ldots \quad \text{for all } 2\cdot 5^k \leq n < 3\cdot 5^k,$$

$$\frac{F_5(n)}{n^{\theta_5}} \geq .850443\ldots \qquad \text{for all } 3\cdot 5^k \leq n < 4\cdot 5^k,$$
$$\frac{F_5(n)}{n^{\theta_5}} \geq .895474\ldots \qquad \text{for all } 4\cdot 5^k \leq n < 5^{k+1}.$$

Therefore if $F_5(n)/n^{\theta_5} \geq .758226$ for all $1 \leq n < 2\cdot 5^k$ then $F_5(n)/n^{\theta_5} \geq .758226$ for all $1 \leq n < 5^{k+1}$. We would like to show this is true for all n so we will extend the region where $F_5(n)/n^{\theta_5} \geq .758226$ to $1 \leq n < 2\cdot 5^{k+1}$ and induct on k.

By Lemma 6, if $n = 5^{k+1} + b$, $b = b_k 5^k + \ldots + b_1 5 + b_0$, then

$$\frac{F_5(n)}{n^{\theta_5}} > .784931\ldots > .758226 \qquad \text{if } b_k = 4,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .772610\ldots > .758226 \qquad \text{if } b_k = 3,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .758226\ldots > .758226 \qquad \text{if } b_k = 2,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .743618\ldots < .758226 \qquad \text{if } b_k = 0,1.$$

Using more information we get better bounds when $b_k = 0,1$. If $b_k = 1$ then $n = 5^{k+1} + 5^k + b$, $b = b_{k-1}5^{k-1} + \ldots + b_1 5 + b_0$. Using Lemma 6 we get

$$\frac{F_5(n)}{n^{\theta_5}} > .780155\ldots > .758226 \qquad \text{if } b_{k-1} = 4,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .778535\ldots > .758226 \qquad \text{if } b_{k-1} = 3,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .776573\ldots > .758226 \qquad \text{if } b_{k-1} = 2,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .774498\ldots > .758226 \qquad \text{if } b_{k-1} = 0,1.$$

If $b_k = 0$ then $n = 5^{k+1} + b$, $b = b_{k-1}5^{k-1} + \ldots + b_1 5 + b_0$. Using Lemma 6 we get

$$\frac{F_5(n)}{n^{\theta_5}} > .772683\ldots > .758226 \qquad \text{if } b_{k-1} = 4,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .768335\ldots > .758226 \qquad \text{if } b_{k-1} = 3,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .763119\ldots > .758226 \qquad \text{if } b_{k-1} = 2,$$
$$\frac{F_5(n)}{n^{\theta_5}} > .757661\ldots < .758226 \qquad \text{if } b_{k-1} = 0,1.$$

Our two exceptional cases are thus $n = 5^{k+1} + 5^{k-1} + b$, $n = 5^{k+1} + b$, $1 \leq b < 5^{k-1}$. We could continue to apply Lemma 6 but at each step we would still get two exceptional cases; however, these two values for n can be shown to satisfy the requisite inequality in another way:

$$\frac{F_5(5^{k+1}+5^{k-1}+b)}{(5^{k+1}+5^{k-1}+b)^{\theta_5}} > \frac{F_5(5^{k+1}+5^{k-1})}{(5^{k+1}+2\cdot 5^{k-1})^{\theta_5}}$$
$$= \frac{F_5(26)}{27^{\theta_5}} = .886348\ldots > .758226$$

and similarly,

$$\frac{F_5(5^{k+1}+b)}{(5^{k+1}+b)^{\theta_5}} > \frac{F_5(5^{k+1})}{(5^{k+1}+5^{k-1})^{\theta_5}} = \frac{F_5(25)}{26^{\theta_5}} = .936137\ldots > .758226.$$

Summarizing: in all cases $F_5(n)/n^{\theta_5} > .758226$ for $1 \le n < 2\cdot 5^{k+1}$ provided it is so for all $1 \le n < 2\cdot 5^k$. To start our induction we need to check all n with $1 \le n < 10$ ($k = 1$) since we used at most two digits before the 5^{k+1} to establish the bound for $1 \le n < 2\cdot 5^{k+1}$. A quick check shows $F_5(n)/n^{\theta_5} > .758226$ for all $1 \le n < 10$. ∎

4. Behavior of β_p for large p. Another problem of interest is to investigate the behavior of β_p as p grows. A first step in this direction is the following bounding theorem. It improves on Stein [4], $\beta_p > 1/p$, and on the easily obtainable bound $\beta_p > 2/(p(p+1))$ mentioned in Volodin [7].

THEOREM 2. *For all primes p,*

$$(1-2^{1/(1-\theta_p)})^{\theta_p-1} \le \beta_p < \frac{3-\theta_p}{2(2-\theta_p)^{2-\theta_p}}.$$

Proof. Since $\beta_p = \liminf_{n\to\infty} F_p(n)/n^{\theta_p}$ Lemma 3 says that $\beta_p \le q_0(p)$ for $q_0(p) = \min_{n=1,\ldots,p} F_p(n)/n^{\theta_p}$. Define

$$f_p(x) = \frac{x(x+1)/2}{x^{\theta_p}}.$$

Differentiating $f_p(x)$ we get

$$\frac{df_p}{dx} = \frac{(2-\theta_p)x+(1-\theta_p)}{2x^{\theta_p}}$$

so our lone critical point (a minimum) is at

$$x_{\min} = \frac{\theta_p - 1}{2-\theta_p}.$$

Since there is an integer in $[x_{\min}, 1+x_{\min})$ we know $q_0(p) < f_p(1+x_{\min})$. This leads to

$$\beta_p < f_p(1+x_{\min}) = \frac{3-\theta_p}{2(2-\theta_p)^{2-\theta_p}}.$$

This establishes the upper bound.

We prove the lower bound by induction on k. Suppose

$$\frac{F_p(n)}{n^{\theta_p}} > (1-2^{1/(1-\theta_p)})^{\theta_p-1} \quad \text{for all } 1 \le n < p^k.$$

Then

$$\frac{F_p(ap^k+n)}{(ap^k+n)^{\theta_p}} = \frac{\frac{a(a+1)}{2}\left(\frac{p(p+1)}{2}\right)^k + (a+1)F_p(n)}{(ap^k+n)^{\theta_p}}$$
$$> \frac{\frac{a(a+1)}{2}\left(\frac{p(p+1)}{2}\right)^k + (a+1)(1-2^{1/(1-\theta_p)})^{\theta_p-1}n^{\theta_p}}{(ap^k+n)^{\theta_p}}.$$

We define

$$f_p(n) = \frac{\frac{a(a+1)}{2}\left(\frac{p(p+1)}{2}\right)^k + (a+1)(1-2^{1/(1-\theta_p)})^{\theta_p-1}n^{\theta_p}}{(ap^k+n)^{\theta_p}}.$$

Treating f_p as a continuous function of n we get a single critical point,

$$n_{\min} = p^k \frac{2^{1/(1-\theta_p)}}{1-2^{1/(1-\theta_p)}}.$$

Evaluating f_p at this point we have

$$\frac{F_p(ap^k+n)}{(ap^k+n)^{\theta_p}} > f_p(n_{\min}) = \frac{a+1}{2}\left(a + \frac{2^{1/(1-\theta_p)}}{1-2^{1/(1-\theta_p)}}\right)^{1-\theta_p}.$$

To determine which value of a minimizes this expression we differentiate and find the only critical point (a minimum) is at

$$a_{\min} = \frac{\theta_p - \frac{1}{1-2^{1/(1-\theta_p)}}}{2-\theta_p}.$$

It can be shown that $a_{\min} < 1$ so among the values $1, \ldots, p-1$, the minimum for $f_p(n_{\min})$ occurs at $a = 1$, i.e.,

$$\frac{F_p(ap^k+n)}{(ap^k+n)^{\theta_p}} > \frac{1+1}{2}\left(1 + \frac{2^{1/(1-\theta_p)}}{1-2^{1/(1-\theta_p)}}\right)^{1-\theta_p} = (1-2^{1/(1-\theta_p)})^{\theta_p-1}. \quad \blacksquare$$

Theorem 2 can be used to say what happens to β_p for large values of p.

THEOREM 3.

$$\lim_{p\to\infty} \beta_p = .5.$$

Proof. By Theorem 2 and the Squeeze Theorem it is enough to show

$$\lim_{p\to\infty} (1-2^{1/(1-\theta_p)})^{\theta_p-1} = .5$$

and

$$\lim_{p\to\infty} \frac{3-\theta_p}{2(2-\theta_p)^{2-\theta_p}} = .5.$$

These are both easily verified. ∎

5. Conjectures. In [2] it was conjectured that $\beta_2 = \lim_{r\to\infty} q_2(r)$ (where $q_r = F_2(n_r)/n_r^{\theta_2}$ for $n_0 = 1, n_r = n_{r-1} \pm 1$ for $r \geq 1$, with sign chosen to minimize q_r). To generalize this conjecture for an odd prime p, choose $n_p(0)$, $n_p(1)$ to minimize $F_p(n)/n^{\theta_p}$ on $\{1,\ldots,p\}$ and $\{p+1,\ldots,p^2\}$ respectively, $n_p(r) = pn_p(r-1) \pm (p\pm 1)/2$ for $r \geq 2$ with signs chosen independently to minimize $F_p(n_p(r))/n_p(r)^{\theta_p}$. Let $q_p(r) = F_p(n_p(r))/n_p(r)^{\theta_p}$.

Conjecture 1.

$$\beta_p = \lim_{r\to\infty} q_p(r).$$

It was in using such a sequence for $p = 5$ that led to our checking $F_5(n)/n^{\theta_5}$ at $n = 2929687$ in the proof of Theorem 1. Similarly such sequences for $p < 20$ give the values of β_p at least to 6 decimal places. For the primes $p = 3, 5, 7, 17, 19$ the choice always seems to be $+(p-1)/2$ but the author has no proof of this at present. More generally, we have Conjecture 2.

Conjecture 2. *For p an odd prime there exists b such that*

$$n_p(r) = pn_p(r-1) + \frac{p-1}{2} \quad \textit{for all } r \geq b.$$

For example the minimum for $F_{11}(n)/n^{\theta_{11}}$ for $11^k \leq n < 11^{k+1}$ seems to be at $n = 1455\ldots55_{(11)}$ for all $k \geq 3$. The base p expansions for the minima for $p = 5, 7, 11$ lead us to Conjecture 3.

Conjecture 3.

$$\beta_5 = \left(\frac{3}{2}\right)^{1-\theta_5}, \quad \beta_7 = \left(\frac{3}{2}\right)^{1-\theta_7}, \quad \beta_{11} = \frac{59/44}{(31/22)^{\theta_{11}}}.$$

By Theorem 1 the values in Conjecture 3 are correct to at least six decimal places. Volodin [7] conjectured $\beta_3 = 2^{\log_3 2 - 1} = (3/2)^{1-\theta_3}$ but conjectured incorrect values for β_5 and β_7. The simple form for the liminf, $\beta_p = (3/2)^{1-\theta_p}$, can hold for at most finitely many p by Theorem 2 and the fact that $\lim_{p\to\infty} \theta_p = 2$.

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Department of Mathematics
SUNY College at Brockport
Brockport, New York 14420
U.S.A.
E-mail: bwilson@acspr1.acs.brockport.edu

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