A conditional result on Goldbach numbers
in short intervals

by

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1. Introduction. Define a *Goldbach number* (G-number) to be an even number representable as a sum of two primes, and write $L = \log X$. The first result concerning the existence of G-numbers in short intervals is due to Linnik [11] who proved, assuming the Riemann Hypothesis (RH), that for any $\varepsilon > 0$ and X sufficiently large, the interval $[X, X + L^{3+\varepsilon}]$ contains a G-number. Linnik's result was improved by Kátai [8] and, independently, by Montgomery–Vaughan [14] who showed that the interval $[X, X + CL^2]$ contains a G-number for some constant C and X sufficiently large. Other proofs of the Kátai and Montgomery–Vaughan result have recently been obtained by Goldston [3] and Languasco–Perelli [9].

The main aim of this paper is to study the distribution of G-numbers in short intervals under the assumption of RH and Montgomery's pair correlation conjecture, a form of which asserts that

$$(1) \qquad F(X,T) \sim \frac{1}{2\pi} T \log T \quad \text{for } X \to \infty$$

uniformly for $X^\varepsilon \le T \le X$, for every fixed $\varepsilon > 0$, where

$$F(X,T) = 4 \sum_{0 < \gamma_1, \gamma_2 \le T} \frac{X^{i(\gamma_1 - \gamma_2)}}{4 + (\gamma_1 - \gamma_2)^2}$$

and γ_j, $j = 1, 2$, run over the imaginary part of the nontrivial zeros of the Riemann zeta-function $\zeta(s)$.

It is easy to see that

$$(2) \qquad F(X,T) \ll T \log^2 T$$

uniformly in X. Moreover, Montgomery [13] (see also Goldston–Montgomery

[4]) proved, again under RH, that

$$(3) \qquad F(X,T) \sim \frac{1}{2\pi} T \log X \quad \text{for } X \to \infty$$

holds uniformly for $X \leq T$.

In the following we will denote by MC the hypothesis (1) and by WMC a weaker form of it where $\ll$ replaces $\sim$. As a slight generalization of WMC, we state the following

HYPOTHESIS. *Let* $\theta \in [1, 2)$. *For any* $\varepsilon > 0$ *the estimate*

$$(4) \qquad F(X,T) \ll T(\log T)^\theta$$

holds uniformly for $X^\varepsilon \leq T \leq X$.

We will denote the hypothesis above by WMC(θ). Observe that WMC = WMC(1) and that, for $\theta \geq 2$, (4) is implied by (2). Our result is

THEOREM. *Let* $\theta \in [1, 2)$ *be fixed and* $1/(2X) \leq \xi \leq 1/2$. *Assume RH and WMC(θ) uniformly in the range* $2X\xi \leq T \leq X$. *Then*

$$\int_{-\xi}^{\xi} \left| \sum_{X \leq n \leq 2X} (\Lambda(n) - 1)e(n\alpha) \right|^2 d\alpha \ll X\xi L^\theta + \min\left(\frac{L^2}{\xi(\log 2\xi)^2}; X\xi L^{\theta+2} \right),$$

where Λ *is the von Mangoldt function and* $e(x) = \exp(2\pi i x)$.

We remark that this theorem is an analogue of Theorem 3 of Languasco–Perelli [10]. As an application we can obtain the following result on the distribution of Goldbach numbers in short intervals.

COROLLARY. *Let* $\theta \in [1, 2)$ *and* $H = CL^\theta$, *where* $C > 0$ *is a sufficiently large constant. Assume RH and WMC(θ) uniformly in the range* $X/H \leq T \leq X$. *Then, for all sufficiently large* X, *the interval* $[X, X + H]$ *contains a G-number.*

We remark that our Corollary can be obtained using the method of Goldston [3].

We also recall that, under RH and MC, Goldston [3] proved that the interval $[X, X + CL]$ contains a G-number and that Friedlander–Goldston [2] proved that the interval $\left[X, X + C\frac{(\log \log X)^2}{\log \log \log X}\right]$ contains a G-number assuming RH together with a strong form of MC and a suitable form of the Elliott–Halberstam conjecture.

Our second aim is to prove the following result, which may have some independent interest, on the mean-square of the singular series of the Goldbach problem.

PROPOSITION. *Let*

$$\mathfrak{S}(n) = \begin{cases} 2\displaystyle\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\displaystyle\prod_{\substack{p|n \\ p>2}}\left(\frac{p-1}{p-2}\right) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\sum_{n \le X} \mathfrak{S}(n)^2 = 2X \prod_{p>2}\left(1 + \frac{1}{(p-1)^3}\right) - \frac{1}{4}L^2 + O(L^{5/3}).$$

The Proposition is a sharper version of Lemma 2 of Goldston [3], and its proof is based on the argument in Friedlander–Goldston [2].

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2. Some lemmas. Now we state two lemmas whose proofs follow the lines of Languasco–Perelli [10] (see also Heath-Brown [6]).

LEMMA 1. *Writing*

$$\Sigma(X,T,v) = \sum_{0 < \gamma \le T} X^{i\gamma} e^{iv\gamma}$$

we have

$$F(X,T) = \int_{-\infty}^{\infty} |\Sigma(X,T,v)|^2 e^{-2|v|}\,dv.$$

LEMMA 2. *Let* $\alpha = \alpha(X)$ *and* $\beta = \beta(X)$ *be real numbers satisfying* $c \le \alpha < \beta \le C$ *for some absolute constants* $c, C > 0$. *Let* $T > U \ge 0$. *Then*

$$\int_{\alpha X}^{\beta X} \left| \sum_{0 < \gamma \le T} y^{i\gamma} \right|^2 dy \ll X F(X,T)$$

and

$$\sum_{U < \gamma \le T} X^{i\gamma} \ll T^{1/2}\big(\max_{U \le u \le T} F(X,u)\big)^{1/2}.$$

The next lemma is an analogue of Lemma 2 in which we insert a factor $1/\varrho$ in the summation on the γ.

LEMMA 3. *Let* $\varrho = 1/2 + i\gamma$, $\alpha = \alpha(X)$ *and* $\beta = \beta(X)$ *be real numbers satisfying* $c \le \alpha < \beta \le C$ *for some absolute constants* $c, C > 0$. *Let* $T > U \ge 0$. *Then*

$$\int\limits_{\alpha X}^{\beta X} \Big| \sum_{U < \gamma \le T} \frac{y^{i\gamma}}{\varrho} \Big|^2 dy \ll X \left(\frac{F(X,T)}{T^2} + \frac{F(X,U)}{U^2} + \frac{1}{U^{1/2}} \int\limits_{U}^{T} F(X,u) \frac{du}{u^{5/2}} \right)$$

and

$$\sum_{U < \gamma \le T} \frac{X^{i\gamma}}{\varrho} \ll \left(\frac{1}{T} \max_{U \le u \le T} F(X,u) \right)^{1/2} + \int\limits_{U}^{T} (\max_{U \le v \le u} F(X,v))^{1/2} \frac{du}{u^{3/2}}.$$

P r o o f. By partial summation

$$\sum_{U < \gamma \le T} \frac{X^{i\gamma}}{\varrho} \ll \frac{1}{T} \Big| \sum_{U < \gamma \le T} X^{i\gamma} \Big| + \int\limits_{U}^{T} \Big| \sum_{U < \gamma \le u} X^{i\gamma} \Big| \frac{du}{u^2}.$$

The second estimate above follows immediately from the second estimate in Lemma 2. For the first estimate we have, by the Cauchy–Schwarz inequality and Lemma 2,

$$\int\limits_{\alpha X}^{\beta X} \Big| \sum_{U < \gamma \le T} \frac{y^{i\gamma}}{\varrho} \Big|^2 dy$$

$$\ll \frac{1}{T^2} \int\limits_{\alpha X}^{\beta X} \Big| \sum_{U < \gamma \le T} y^{i\gamma} \Big|^2 dy + \int\limits_{\alpha X}^{\beta X} \Big| \int\limits_{U}^{T} \Big| \sum_{U < \gamma \le u} y^{i\gamma} \Big| \frac{du}{u^2} \Big|^2 dy$$

$$\ll \frac{X}{T^2} (F(X,T) + F(X,U)) + \int\limits_{\alpha X}^{\beta X} \left(\int\limits_{U}^{T} \Big| \sum_{U < \gamma \le u} y^{i\gamma} \Big|^2 \frac{du}{u^{5/2}} \right) \left(\int\limits_{U}^{T} \frac{du}{u^{3/2}} \right) dy$$

$$\ll \frac{X}{T^2} (F(X,T) + F(X,U)) + \frac{1}{U^{1/2}} \int\limits_{U}^{T} X(F(X,u) + F(X,U)) \frac{du}{u^{5/2}}$$

$$\ll X \left(\frac{F(X,T)}{T^2} + \frac{F(X,U)}{U^2} + \frac{1}{U^{1/2}} \int\limits_{U}^{T} F(X,u) \frac{du}{u^{5/2}} \right).$$

3. Proof of the Theorem and of the Corollary. Writing Selberg's integral

$$J(X, 2X, H) = \int\limits_{X}^{2X} |\psi(t+H) - \psi(t) - H|^2 dt,$$

where $\psi(t) = \sum_{n \le t} \Lambda(n)$, we get, by Gallagher's lemma (see, e.g., Montgomery [12], Lemma 1.9),

(5) $\displaystyle\int_{-\xi}^{\xi} \Big| \sum_{X \le n \le 2X} (\Lambda(n) - 1)e(n\alpha) \Big|^2 d\alpha$

$$\ll \xi^2 \Bigg[\int_{X-1/(2\xi)}^{X} \Big| \psi\Big(t + \frac{1}{2\xi}\Big) - \psi(X) - \Big(t - X + \frac{1}{2\xi}\Big) \Big|^2 dt$$

$$+ J\Big(X, 2X - \frac{1}{2\xi}, \frac{1}{2\xi}\Big) + \int_{2X-1/(2\xi)}^{2X} |\psi(2X) - \psi(t) - (2X - t)|^2 dt \Bigg]$$

$$+ X\xi^2 + \xi$$

$$= \xi^2 (I_1 + I_2 + I_3) + O(X\xi^2 + \xi),$$

say. We remark that the term $O(X\xi^2 + \xi)$ in (5) arises from the $O(1)$ term in the estimate $\sum_{a < n < b} 1 = b - a + O(1)$ applied in the above integrals.

Using the explicit formula (see Davenport [1], Ch. 17)

$$\psi(x) = x - \sum_{|\gamma| \le K} \frac{x^\varrho}{\varrho} + O\Big(\frac{x(\log xK)^2}{K}\Big) + O\Big(\log x \min\Big(1; \frac{x}{K\|x\|}\Big)\Big),$$

where $\|x\| = \min_{n \in \mathbb{N}} |x - n|$, with $K = XL^2\xi^{1/2}$, we get, by (5),

(6) $\displaystyle\int_{-\xi}^{\xi} \Big| \sum_{X \le n \le 2X} (\Lambda(n) - 1)e(n\alpha) \Big|^2 d\alpha \ll \xi^2 (J_1 + J_2 + J_3) + O(X\xi),$

where

$$J_1 = \int_{X-1/(2\xi)}^{X} \Big| \sum_{0 < \gamma \le K} \frac{(y + 1/(2\xi))^\varrho - X^\varrho}{\varrho} \Big|^2 dy,$$

$$J_2 = \int_{X}^{2X-1/(2\xi)} \Big| \sum_{0 < \gamma \le K} \frac{(y + 1/(2\xi))^\varrho - y^\varrho}{\varrho} \Big|^2 dy,$$

$$J_3 = \int_{2X-1/(2\xi)}^{2X} \Big| \sum_{0 < \gamma \le K} \frac{(2X)^\varrho - y^\varrho}{\varrho} \Big|^2 dy.$$

We first consider the terms in $J_1 + J_2 + J_3$ where $0 < \gamma \le 2X\xi$, and show that they make a contribution

(7) $$\ll \frac{1}{X\xi^2} \int_{\alpha X}^{\beta X} \Big| \sum_{0 < \gamma \le 2X\xi} y^{i\gamma} \Big|^2 dy.$$

Using the Cauchy–Schwarz inequality with $0 < V < W$ we get

$$\left| \sum_{0 < \gamma \leq U} \frac{W^\varrho - V^\varrho}{\varrho} \right|^2 = \left| \int_V^W \sum_{0 < \gamma \leq U} u^{\varrho-1} \, du \right|^2 \leq |W - V| \int_V^W \left| \sum_{0 < \gamma \leq U} u^{i\gamma} \right|^2 \frac{du}{u}.$$

Applying this estimate in J_1, J_2 and J_3 we obtain (7). By Lemma 2 and WMC(θ) the right hand side of (7) is $\ll F(X, 2X\xi)/\xi^2 \ll XL^\theta/\xi$, which, by (6), makes the contribution $X\xi L^\theta$.

Now we consider the contribution from the terms $2X\xi < \gamma \leq K$ in $J_1 + J_2 + J_3$. We see immediately that this contribution is

$$\ll X \int_{\alpha X}^{\beta X} \left| \sum_{2X\xi < \gamma \leq K} \frac{y^{i\gamma}}{\varrho} \right|^2 dy + \frac{X}{\xi} \left(\left| \sum_{2X\xi < \gamma \leq K} \frac{X^{i\gamma}}{\varrho} \right|^2 + \left| \sum_{2X\xi < \gamma \leq K} \frac{(2X)^{i\gamma}}{\varrho} \right|^2 \right).$$

By Lemma 3 and WMC(θ) the first term is $\ll XL^\theta/\xi$. The other terms on the right come from J_1 and J_3. By Lemma 3 and WMC(θ) they contribute $\ll XL^{2+\theta}/\xi$; alternatively they are bounded by $I_1 + I_3$, which by the Brun–Titchmarsh theorem is $\ll L^2/(\xi^3 (\log 2\xi)^2)$. These estimates, combined with (6), give the second error term in the Theorem.

Now we prove the Corollary. Let $H = [CL^\theta]$, where $C \geq 1$ is a constant. Define

$$L(\alpha) = \left| \sum_{m=1}^H e(-m\alpha) \right|^2 = \sum_{m=-H}^H a(m) e(-m\alpha),$$

where $a(m) = H - |m|$,

$$R(n) = \sum_{h+k=n} \Lambda(h)\Lambda(k), \quad S(\alpha) = \sum_{X \leq n \leq 2X} \Lambda(n)e(n\alpha),$$

$$T(\alpha) = \sum_{X \leq n \leq 2X} e(n\alpha), \quad E(\alpha) = S(\alpha)^2 - T(\alpha)^2.$$

We have

$$\begin{aligned}
(8) \quad \sum_{n=X-H}^{X+H} a(n-X)R(n) &= \int_{-1/2}^{1/2} S(\alpha)^2 L(\alpha) e(-X\alpha) \, d\alpha \\
&= \int_{-1/2}^{1/2} T(\alpha)^2 L(\alpha) e(-X\alpha) \, d\alpha \\
&\quad + \int_{-1/2}^{1/2} E(\alpha)L(\alpha)e(-X\alpha) \, d\alpha = A + B,
\end{aligned}$$

say. It is easy to prove that

$$(9) \qquad A = \sum_{n=X-H}^{X+H} a(n-X) \sum_{h+k=n} 1 = H^2 X + O(H^3).$$

Now we proceed to estimate B. Using

$$T(\alpha) \ll \min(X; 1/|\alpha|) \quad \text{for } |\alpha| \le 1/2$$

we get

$$(10) \qquad \int_{-\xi}^{\xi} |T(\alpha)|^2 \, d\alpha \begin{cases} \ll X^2 \xi & \text{if } 0 < \xi < 1/X, \\ = X + O(1/\xi) & \text{if } 1/X \le \xi \le 1/2. \end{cases}$$

Hence, using the identity

$$f^2 - g^2 = 2f(f-g) - (f-g)^2,$$

the Cauchy–Schwarz inequality and (10), we have

$$(11) \quad \int_{-\xi}^{\xi} |E(\alpha)| \, d\alpha \ll \left(X \int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \, d\alpha \right)^{1/2} + \int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \, d\alpha$$

provided $1/X \le \xi \le 1/2$.

Since

$$(12) \qquad L(\alpha) \ll \min(H^2; 1/|\alpha|^2) \quad \text{for } |\alpha| \le 1/2,$$

we have

$$(13) \qquad B \ll H^2 \int_{-1/H}^{1/H} |E(\alpha)| \, d\alpha + \int_{1/H}^{1/2} |E(\alpha)| \frac{d\alpha}{\alpha^2}.$$

From the Theorem and (11) we get

$$(14) \qquad H^2 \int_{-1/H}^{1/H} |E(\alpha)| \, d\alpha \ll H^{3/2} X L^{\theta/2}.$$

By partial integration, the Theorem and (11) we obtain

$$(15) \qquad \int_{1/H}^{1/2} |E(\alpha)| \frac{d\alpha}{\alpha^2} \ll H^{3/2} X L^{\theta/2}.$$

Hence from (13)–(15) we have

$$(16) \qquad B \ll H^{3/2} X L^{\theta/2},$$

and from (8), (9) and (16) we get

$$(17) \qquad \sum_{n=X-H}^{X+H} a(n-X)R(n) \gg H^2 X$$

provided that C is sufficiently large. Thus the Corollary follows.

4. Proof of the Proposition. Let

$$\mathfrak{S} = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right).$$

We have

$$(18) \qquad \sum_{n \leq X} \mathfrak{S}(n)^2 = 4\mathfrak{S}^2 \sum_{\substack{2n \leq X \\ p \mid n \\ p>2}} \prod \left(\frac{p-1}{p-2} \right)^2$$

$$= 4\mathfrak{S}^2 \sum_{\substack{2n \leq X \\ p \mid n \\ p>2}} \prod \left(1 + \frac{2p-3}{(p-2)^2} \right)$$

$$= 4\mathfrak{S}^2 \sum_{n \leq X/2} \sum_{j \mid n} f(j),$$

where

$$f(j) = \begin{cases} \mu^2(j) \prod_{p \mid j} \dfrac{2p-3}{(p-2)^2} & \text{if } j \text{ is odd,} \\[2mm] 0 & \text{if } j \text{ is even.} \end{cases}$$

Then, changing the order of summation in (18), we obtain

$$(19) \qquad \sum_{n \leq X} \mathfrak{S}(n)^2 = 4\mathfrak{S}^2 \sum_{j \leq X/2} f(j) \left[\frac{X}{2j} \right]$$

$$= 2\mathfrak{S}^2 X \sum_{j=1}^{\infty} \frac{f(j)}{j} - 2\mathfrak{S}^2 X \sum_{j>X/2} \frac{f(j)}{j}$$

$$- 2\mathfrak{S}^2 \sum_{j \leq X/2} f(j) - 4\mathfrak{S}^2 \sum_{j \leq X/2} f(j) P\left(\frac{X}{2j} \right),$$

where $P(u) = u - [u] - 1/2$.

By straightforward computations we get

$$(20) \qquad 2\mathfrak{S}^2 \sum_{j=1}^{\infty} \frac{f(j)}{j} = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right)^2 \left(1 + \frac{2p-3}{p(p-2)^2} \right)$$

$$= 2 \prod_{p>2} \left(1 + \frac{1}{(p-1)^3} \right).$$

Next, we will show

(21)
$$\sum_{j \le U} f(j) = \frac{1}{8 \mathfrak{S}^2} \log^2 U + B \log U + O(1),$$

where B is a constant. By partial summation this implies

(22)
$$\sum_{j > U} \frac{f(j)}{j} = \frac{1}{4 \mathfrak{S}^2} \cdot \frac{\log U}{U} + O\left(\frac{1}{U}\right).$$

The Proposition follows from (19)–(22) together with the estimate

(23)
$$\sum_{j \le X} f(j) P(X/j) \ll L^{5/3}.$$

Now we prove (21). Writing

$$H(s) = \sum_{m=1}^{\infty} f(m) m^{-s} = \prod_{p > 2} \left(1 + \frac{2p-3}{p^s (p-2)^2}\right)$$

we see that $H(s)$ is an analytic function for $\mathrm{Re}\, s = \sigma > 0$ and, using the Perron formula with error term (see, e.g., Lemma 3.12 of Titchmarsh [15]), we obtain

$$\sum_{j \le U} f(j) = \frac{1}{2\pi i} \int_{c-iZ}^{c+iZ} H(s) \frac{U^s}{s}\, ds + O\left(U^c \sum_{m=1}^{\infty} \frac{|f(m)|}{m^c (1 + Z|\log(U/m)|)}\right),$$

where $\varepsilon > 0$ is a fixed constant, $c = \varepsilon + 1/\log U < 1/4$ and Z will be chosen later on.

The error term can be estimated using

$$f(m) \ll \frac{d(m)}{m} \sum_{h \mid m} \frac{d^2(h)}{h}$$

(see (30) of Goldston [3]), where $d(m)$ is the divisor function, and the classical estimates $\sum_{v \le m} d(v)^q \ll m (\log m)^{2^q - 1}$ (see, e.g., Theorem 5.3 of Hua [7]) and $d(m) \ll m^{\varepsilon}$ (see, e.g., Theorem 315 of Hardy–Wright [5]). So we have

(24)
$$\sum_{j \le U} f(j) = \frac{1}{2\pi i} \int_{c-iZ}^{c+iZ} H(s) \frac{U^s}{s}\, ds + O\left(\frac{U^c}{Z}\right).$$

Now we observe that

$$H(s) = \left(1 - \frac{1}{2^{s+1}}\right)^2 \zeta(s+1)^2 g(s),$$

where

$$g(s) = \prod_{p > 2} \left(1 - \frac{1}{p^{s+1}}\right)^2 \left(1 + \frac{2p-3}{p^s (p-2)^2}\right),$$

which converges absolutely and is analytic for $\sigma > -1/2$. So $H(s)U^s/s$ has a triple pole at $s = 0$ with residue

$$\frac{1}{8\mathfrak{S}^2} \log^2 U + B \log U + O(1).$$

Consider a rectangular contour with right side $s = c + it$, $t \in [-Z, Z]$, and left side $-1/4 + it$, $t \in [-Z, Z]$. The contribution of the top, bottom and left sides of the contour can be estimated using $\zeta(\sigma + it) \ll t^{1/6}$ for $\sigma \geq 1/2$ (see, e.g., Titchmarsh [15], p. 115). Hence we have

$$\sum_{j \leq U} f(j) = \frac{1}{8\mathfrak{S}^2} \log^2 U + B \log U + O(Z^{-2/3}U^c + Z^{1/3}U^{-1/4}) + O(1).$$

Choosing $Z = U^{3c}$, we obtain (21).

Now we prove (23). For j odd we have

$$f(j) = \mu^2(j) \prod_{p|j} \left(\frac{2}{p-2}\right)\left(\frac{p-3/2}{p-2}\right) = \frac{\mu^2(j)d(j)}{\varphi_2(j)} \prod_{p|j}\left(1 + \frac{1}{2(p-2)}\right)$$

$$= \frac{\mu^2(j)d(j)}{\varphi_2(j)} \sum_{\delta|j} \frac{\mu^2(\delta)}{2^{\omega(\delta)}\varphi_2(\delta)},$$

where $\omega(n)$ is the number of distinct prime factors of n, $\varphi_2(p) = p - 2$ and φ_2 is extended to square-free numbers by multiplicativity.

Hence, interchanging the order of summation, we obtain

$$\sum_{j \leq X} f(j)P\left(\frac{X}{j}\right) = \sum_{\substack{\delta \leq X \\ (\delta,2)=1}} \frac{\mu^2(\delta)d(\delta)}{2^{\omega(\delta)}(\varphi_2(\delta))^2}\left(\sum_{\substack{k \leq X/\delta \\ (k,2)=1}} \frac{\mu^2(k)d(k)}{\varphi_2(k)}P\left(\frac{X/\delta}{k}\right)\right).$$

Using the argument in (2.9)–(2.13) of Friedlander–Goldston [2], we find that the inner sum can be estimated by

$$\sum_{n \leq X} \frac{d(n)}{n}P(X/n),$$

which is $\ll L^{5/3}$ by the remark at the end of Section 2 of [2]. Using this estimate we obtain

$$\sum_{j \leq X} f(j)P\left(\frac{X}{j}\right) \ll L^{5/3} \sum_{\substack{\delta \leq X \\ (\delta,2)=1}} \frac{\mu^2(\delta)d(\delta)}{(\varphi_2(\delta))^2}$$

and hence (23) follows from the convergence of the series

$$\sum_{\substack{\delta=1 \\ (\delta,2)=1}}^{\infty} \frac{\mu^2(\delta)d(\delta)}{(\varphi_2(\delta))^2}.$$

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