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On the diophantine equation $n(n+1)\ldots(n+k-1) = bx^l$

by

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1. Introduction. In 1975, P. Erdős and J. L. Selfridge [4] proved the following remarkable theorem.

Theorem A (P. Erdős and J. L. Selfridge [4]). *The equation*

$$n(n+1)\ldots(n+k-1) = x^l \quad \text{in positive integers } n,k,x,l \text{ with } k \ge 2,\ l \ge 2 \tag{1}$$

has no solution.

For $k \ge 4$, the next theorem (cf. [5], Theorem 2) was established by P. Erdős [3] in 1951. The case $k = 2$ is a consequence of a recent result of H. Darmon and L. Merel [1], while the case $k = 3$ has recently been proved by the present author [5].

Theorem B (P. Erdős, case $k \ge 4$; H. Darmon and L. Merel, case $k = 2$; K. Győry, case $k = 3$). *Apart from the case $k = l = 2$, the equation*

$$\binom{n+k-1}{k} = x^l \quad \text{in positive integers } n,k,x,l \text{ with } k \ge 2,\ n \ge k+1,\ l \ge 2 \tag{2}$$

has only the solution $(n,k,x,l) = (48,3,140,2)$.

It is clear that for $k = l = 2$ equation (2) has infinitely many solutions. In view of $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ Theorem B furnishes the solutions of (2) for $n \le k$ as well.

Denote by $P(b)$ the greatest prime factor of an integer $b > 1$, and write $P(1) = 1$. As a common generalization of (1) and (2) consider the equation

$$n(n+1)\ldots(n+k-1) = bx^l \quad \text{in positive integers } n,k,b,x,l \text{ with } k \ge 2,\ l \ge 2,\ P(b) \le k,\ b\ l\text{th power free.} \tag{3}$$

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This equation as well as certain further generalizations of it (e.g. with $n(n+d)\dots(n+(k-1)d)$ in place of $n(n+1)\dots(n+k-1)$) were extensively studied by T. N. Shorey, R. Tijdeman, N. Saradha and others; see e.g. [15], [13], [17], [14], [11], [12], [18] and the references given there.

For given k, (3) is solvable with $P(x) \le k$, and all such solutions can be determined. Indeed, a solution of (3) has the property $P(x) \le k$ if and only if $n \in \{1, 2, \dots, p^{(k)} - k\}$, where $p^{(k)}$ denotes the least prime satisfying $p^{(k)} > k$. This is a consequence of a theorem of J. J. Sylvester [16] which says that if $P(n(n+1)\dots(n+k-1)) \le k$ then $n \le k$. Hence more interesting are those solutions of (3) for which $P(x) > k$.

N. Saradha [11] has recently established some non-existence results for a more general version of (3) which imply Theorem A and, for $k \ge 4$, Theorem B. For equation (3), her result gives the following.

Theorem C (N. Saradha [11]). *For $k \ge 4$, equation (3) has no solution with $P(x) > k$.*

The purpose of our paper is to extend Theorem C to the cases $k = 2$ and $k = 3$. The methods of [3] and [11] cannot be applied to $k = 2$ and 3. Using some recent results of K. A. Ribet [9] and H. Darmon and L. Merel [1] on equations of the form

(4) $\quad x^l + y^l = 2^\alpha z^l \quad$ in non-zero relatively prime integers x, y, z

where $l \ge 3$, $\alpha \ge 1$ are given integers, we prove the following.

Theorem 1. *Apart from the case $k = b = l = 2$, for $k \le 3$ equation (3) has only the solution $(n, k, b, x, l) = (48, 3, 6, 140, 2)$ with $P(x) > k$.*

For $k = b = l = 2$ equation (3) has infinitely many solutions. The case $k = 3$, $l = 2$ of Theorem 1 is a consequence of some old diophantine results (cf. Section 2). This case has been settled independently by N. Saradha [12].

By using the above remark on the solutions with $P(x) \le k$ it is easy to verify that, for $k \le 3$, $(1, 2, 2, 1, l \ge 2)$, $(1, 3, 6, 1, l \ge 2)$, $(2, 3, 24, 1, l \ge 4)$, $(2, 3, 6, 2, 2)$ and $(2, 3, 3, 2, 3)$ are the only solutions (n, k, b, x, l) of (3) with $P(x) \le k$.

Together with Theorem C, Theorem 1 provides a complete solution of equation (3) under the assumption $P(x) > k$.

Theorem 2 (N. Saradha, case $k \ge 4$; K. Győry, case $k \le 3$). *Apart from the case $k = b = l = 2$, equation (3) has only the solution $(n, k, b, x, l) = (48, 3, 6, 140, 2)$ with $P(x) > k$.*

As will be seen in Section 2, Theorems A and B can be easily deduced from Theorem 2.

2. Proofs. For $k = 3$ Theorem 1 can be proved by means of the tools applied in [5], in the proof of the case $k = 3$ of Theorem B. Our proof in [5] depends among other things on Baker's method concerning linear forms in logarithms. We give here a different proof which involves Lemma 1 below.

LEMMA 1 (K. A. Ribet [9]). *Let* $l \geq 3$ *be a prime and* α *an integer with* $2 \leq \alpha < l$. *Then equation* (4) *has no solution.*

Proof. This is the first part of Theorem 3 of Ribet [9]. ∎

LEMMA 2 (H. Darmon and L. Merel [1]). *Let* $l \geq 3$ *be an integer. Then for* $\alpha = 1$ *equation* (4) *has only trivial solutions for which* $xyz = \pm 1$.

Proof. This is the first part of the Main Theorem in [1]. ∎

In the proofs of Lemmas 1 and 2 the authors combined various powerful results and methods in number theory, including Wiles' proof of most cases of the Shimura–Taniyama conjecture.

LEMMA 3. *Let* $l \geq 3$ *and* $\alpha \geq 0$ *be integers. The equation*

$$u^l + 1 = 2^\alpha v^l \quad \text{in positive integers } u, v \tag{5a}$$

is solvable only if $\alpha = 1$, *when* $(u, v) = (1, 1)$ *is the only solution. Further, the equation*

$$u^l - 1 = 2^\alpha v^l \quad \text{in positive integers } u, v \tag{5b}$$

has no solution.

Proof. We may suppose without loss of generality that $0 \leq \alpha < l$. For $\alpha = 0$, equations (5a) and (5b) are not solvable. If $\alpha = 1$ then Lemma 2 implies that (5a) has the only solution $(u, v) = (1, 1)$ and, for l odd, (5b) has no solution.

Consider now the case when $\alpha \geq 2$ and l has an odd prime divisor p. If p divides α then neither (5a) nor (5b) is solvable. For $\alpha \equiv 1 \pmod{p}$, (5a) and (5b) have no solution by Lemma 2. In the remaining cases equations (5a) and (5b) are not solvable by Lemma 1.

If $\alpha \geq 2$, l is even and (u, v) satisfies (5a) then the left-hand side of (5a) is congruent to 1 or 2 $\pmod 4$, while the right-hand side is divisible by 4. Hence in this case (5a) has no solution.

There remains the case when in equation (5b), $\alpha \geq 1$ and l is even. We prove by induction on α that under this assumption (5b) has no solution. Assume that for some $\alpha \geq 1$ equation (5b) has a solution (u, v), and that (5b) is not solvable for any $\alpha' < \alpha$. Then $l = 2m$ with an integer $m > 1$, and (5b) can be written in the form

$$(u^m + 1)(u^m - 1) = 2^\alpha v^l.$$

It follows that there are positive integers u_1, v_1 such that either

$$u^m + 1 = 2u_1^{2m}, \qquad u^m - 1 = 2^{\alpha-1}v_1^{2m}, \tag{6}$$

or

$$u^m + 1 = 2^{\alpha-1}v_1^{2m}, \quad u^m - 1 = 2u_1^{2m}. \tag{7}$$

For $\alpha = 1$ this cannot hold, hence we assume that $\alpha \geq 2$. From (6) and (7) we obtain

$$u_1^l - 1 = 2^{\alpha-2}v_1^l \tag{8}$$

and

$$u_1^l + 1 = 2^{\alpha-2}v_1^l, \tag{9}$$

respectively. Equation (8) is not solvable by the induction hypothesis. Further, our result proved above for (5a) implies that equation (9) is solvable only if $\alpha - 2 = 1$, when $(u_1, v_1) = (1, 1)$ is the only solution. However, in this case it follows from (7) that $u^m = 3$, which is impossible. ∎

Proof of Theorem 1. First consider the case $k = 2$. Then equation (3) takes the form

(3a) $n(n+1) = bx^l$ in positive integers n, b, x, l with $l \geq 2$, $P(b) \leq 2$, b lth power free.

For $l = 2$, b can take only the values 1 and 2. But the case $b = 2$ has been excluded, while for $b = 1$ equation (3a) is not solvable.

Suppose now that $l \geq 3$. We determine all solutions of (3a) without any assumption on $P(x)$. This result will be used in the proof of the case $k = 3$. Let (n, b, x, l) be a solution of (3a) with $l \geq 3$. Then $b = 2^\alpha$ for some integer α with $0 \leq \alpha < l$. We deduce from (3a) that $n = u^l$, $n + 1 = 2^\alpha v^l$ or $n = 2^\alpha v^l$, $n + 1 = u^l$ with some positive integers u, v. In the second case we infer that $u^l - 1 = 2^\alpha v^l$, which is not possible by Lemma 3. In the first case we have $u^l + 1 = 2^\alpha v^l$, whence we conclude by Lemma 3 that $\alpha = 1$ and $u = v = 1$. Then it follows that $(n, b, x, l) = (1, 2, 1, l \geq 3)$ is the only solution of (3a) with $l \geq 3$. This implies that under the assumption $P(x) > 2$ equation (3a) has no solution, which proves Theorem 1 for $k = 2$.

Next let $k = 3$. In this case equation (3) can be written in the form

(3b) $n(n+1)(n+2) = bx^l$ in positive integers n, b, x, l with $l \geq 2$, $P(b) \leq 3$, b lth power free.

First assume that $l = 2$. Then $b = 1, 2, 3$ or 6. It was proved by A. J. J. Meyl [8] and G. N. Watson [19] (see also W. Ljunggren [6]) that for $b = 6$ the only solutions (n, x) are $(1, 1)$, $(2, 2)$ and $(48, 140)$. This implies that if $b = 6$ then (3b) has only the solution $(n, b, x, l) = (48, 6, 140, 2)$ with $P(x) > 3$. We show that for the remaining values of b, (3b) is not possible. For $b = 1$ this assertion is a special case of a result of P. Erdős [2] and O. Rigge [10].

For $b=2$ and 3, (3b) can be written as $y(y^2-1)=bx^2$ where $y=n+1>1$. Since y^2-1 cannot be a perfect square, we infer that $y=t^2$, $y^2-1=bs^2$, whence $bs^2=t^4-1$ with positive integers s, t. However, for $b=3$ this is impossible by a theorem of W. Ljunggren [7]. If $b=2$, it follows that t^2+1 or t^2-1 must be the square of a positive integer, which is also impossible. This proves our claim.

In what follows assume that $l\ge 3$, and let (n,b,x,l) be a solution of (3b) with $P(x)>3$. If $3\mid n+2$ then we infer from (3b) that $n(n+1)=2^{\alpha_1}x_1^l$ with some integers $\alpha_1\ge 0$, $x_1\ge 1$. Then our result proved above for equation (3a) implies that $n=1$. It now follows from (3b) that $x=1$, which is excluded.

If $3\mid n$, then (3b) gives $(n+1)(n+2)=2^{\alpha_2}x_2^l$ with some integers $\alpha_2\ge 0$, $x_2\ge 1$. However, this is impossible by our above result on equation (3a).

Finally, if $3\mid n+1$, it follows from (3b) that $n(n+2)=2^{\alpha_3}x_3^l$ with integers $\alpha_3\ge 0$, $x_3\ge 1$. This can hold only if n is even, when $n=2u^l$, $n+2=2^{\alpha_4}v^l$ or $n=2^{\alpha_4}v^l$, $n+2=2u^l$ for some positive integers α_4, u, v with $\alpha_4\ge 2$. This implies that (u,v) is a solution of equation (5a) or (5b) with $\alpha=\alpha_4-1$. Using Lemma 3, we deduce that $u=1$, $n=2u^l=2$. Now from (3b) we find that $x=1$ or 2, which is excluded. This completes the proof of Theorem 1. ∎

We now deduce Theorems A and B from Theorem 2.

P r o o f o f T h e o r e m A. For $k=l=2$, equation (1) is not solvable. Further, by Theorem 2 equation (1) has no solution with $P(x)>k$. If (n,k,x,l) is a solution of (1) with $P(x)\le k$ then, by Sylvester's theorem [16], we have $n\le k$, whence $n\le (n+k)/2$. By Chebyshev's theorem there exists a prime p with $(n+k)/2\le p\le n+k-1$, and this prime divides $n(n+1)\dots(n+k-1)$ to the first power only. This proves that (1) has no solution. ∎

P r o o f o f T h e o r e m B. We write equation (2) in the form

$$n(n+1)\dots(n+k-1)=k!x^l.$$

By assumption $n\ge k+1$ holds, hence Sylvester's theorem implies that $P(x)>k$. Now Theorem B follows immediately from Theorem 2. ∎

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Corrections to [5]

P. 294, line 14: For "Satz 8" read "Satz 7", and for "equation (10)" read "equation (13)".

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