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Nilpotent local class field theory

by

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1. Introduction. Let G be any profinite group and A an abelian profinite group. Let

$$L(G) = \bigoplus_{n=1}^{\infty} G^{(n)}/G^{(n+1)}$$

be the graded Lie algebra associated with G by means of the lower central series $(G^{(n)})_{n\geq 1}$ and let $\mathcal{L}(A) = \bigoplus_{n=1}^{\infty} L_n(A)$ be the universal graded Lie algebra associated with A (see §2 for exact definitions). Any homomorphism φ of A into $G/G^{(2)}$ gives rise to a homomorphism φ_* of $\mathcal{L}(A)$ into $L(G)$.

In this paper we study the special situation where A is the profinite completion $\widehat{K}^\times$ of the multiplicative group $K^\times$ of a local field K, i.e. a field which is complete with respect to a discrete valuation with finite residue class field. The group G is the absolute Galois group G_K of K and φ is the Artin isomorphism of $\widehat{K}^\times$ onto $G_K/G_K^{(2)}$.

The surjectivity of φ implies the same for φ_*. The goal of this paper is the determination of the kernel of φ_*. This is equivalent to the determination of the kernel of the component homomorphisms

$$\varphi_*(l) : \mathcal{L}(A(l)) \to L(G_K(l)),$$

where l is any prime and $B(l)$ is the maximal pro-l quotient of a profinite group B. The difficult case occurs when $l = p$, the residual characteristic of K. If K is of characteristic p, or if K is of characteristic zero and does not contain a primitive pth root of unity, this kernel is zero. So we assume that K is of characteristic zero and contains a primitive p^κth root of unity ζ with κ chosen largest possible. In this case $G_K(p)$ is a Demushkin group so that the cup-product

$$H^1(G_K(p), \mathbb{Z}/p^\kappa Z) \times H^1(G_K(p), \mathbb{Z}/p^\kappa\mathbb{Z}) \to H^2(G_K(p), \mathbb{Z}/p^\kappa\mathbb{Z}) = \mathbb{Z}/p^\kappa\mathbb{Z}$$

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is non-degenerate. We now assume that p is odd. In this case, the form is alternating and so we obtain by duality an element in

$$G_K^{\mathrm{ab}}/(G_K^{\mathrm{ab}})^{p^\kappa} \wedge G_K^{\mathrm{ab}}/(G_K^{\mathrm{ab}})^{p^\kappa}.$$

Using the Artin isomorphism, this determines an element

$$\tau \in \mathcal{L}_2(A(p)) \otimes \mathbb{Z}/p^\kappa\mathbb{Z}$$

which is determined by G_K up to a unit of $\mathbb{Z}/p^\kappa\mathbb{Z}$. Our main result is the following theorem:

THEOREM 1.1. *The kernel of* $\varphi_*(p)$ *is the ideal of* $\mathcal{L}(A(p))$ *generated by the elements of the form* $[\mathrm{ad}(\lambda)(\zeta), \mathrm{ad}(\lambda)(\tau)]$, *where* λ *is an element of the enveloping algebra of* $\mathcal{L}(A(p))$.

E.-W. Zink [Zi1], [Zi2] studied $\varphi_{*2} : \mathcal{L}_2(\widehat{K}^\times) \to L_2(G_K)$ and showed that φ_{*2} is an isomorphism. His main interest in [Zi1], [Zi2] concerns the filtration $(L_2(G_K)^r)_{r\in\mathbb{R}_+}$ of $L_2(G_K) = G_K^{(2)}/G_K^{(3)}$ induced by the ramification groups G_K^r of G_K and the inverse image of this filtration in $\mathcal{L}(\widehat{K}^\times)$. His results were augmented by Cram [Cr] and Kaufhold [Ka]. But the results of these three authors are far from the goal of giving an independent description of $\{\varphi_{*2}^{-1}(L_2(G_K)^r) \mid r \in \mathbb{R}_+\}$. There is of course a corresponding question for $(\varphi_{*n}^{-1}(L_n(G_K)^r))_{r\in\mathbb{R}_+}$, but it will not be considered here.

The present paper originated from the thesis of the second author [Ku], directed by the first, and assisted by important suggestions of the third author. Section 5 was added by the third author.

2. Lie algebras. In this section we introduce the necessary definitions and facts about groups and related Lie algebras.

2.1. Let k be a commutative, associative ring with unity and let A be a k-module. Let $\mathcal{T}(A)$ be the non-associative tensor algebra of A considered as a k-module, i.e.

$$\mathcal{T}(A) := \bigoplus_{n=1}^{\infty} \mathcal{T}_n(A),$$
$$\mathcal{T}_1(A) := A, \quad \mathcal{T}_2(A) := A \otimes_k A,$$
$$\mathcal{T}_n(A) := \bigoplus_{p+q=n} \mathcal{T}_p \otimes_k \mathcal{T}_q.$$

Then we define the Lie algebra $\mathcal{L}(A)$ as the factor algebra of $\mathcal{T}(A)$ by the ideal of $\mathcal{T}(A)$ generated by all elements of the form

$$a \otimes a, \quad (a \otimes b) \otimes c + (b \otimes c) \otimes a + (c \otimes a) \otimes b,$$

with $a, b, c \in \mathcal{T}(A)$. Since this ideal is homogeneous, we have

$$\mathcal{L}(A) = \bigoplus_{n=1}^{\infty} \mathcal{L}_n(A), \qquad \mathcal{L}_n(A) := (\mathcal{T}_n(A) + \mathcal{I}(A))/\mathcal{I}(A),$$

and so $\mathcal{L}$ is a graded Lie algebra over k.

If $\varphi : A \to B$ is a homomorphism of k-modules, then to φ corresponds a homomorphism $\mathcal{L}(\varphi)$ of $\mathcal{L}(A)$ into $\mathcal{L}(B)$ so that $\mathcal{L}$ is a covariant functor from the category of k-modules to the category of graded Lie algebras over k. Moreover, if $L = \bigoplus_{n=1}^{\infty} L_n$ is any graded Lie algebra over k, there is a unique homomorphism ψ of $\mathcal{L}(L_1)$ into L such that

$$\psi(a) = a \qquad \text{for } a \in L_1.$$

In the next section we apply this construction with $k = \mathbb{Z}$ to extend it to the case where A is a profinite abelian group. If A is finitely generated, we recover the above construction with $k = \widehat{\mathbb{Z}}$, the total profinite completion of $\mathbb{Z}$.

2.2. Now let A be a profinite abelian group and $\mathfrak{U}$ the filtration of A given by the set of open subgroups of A. We define $\mathcal{L}_n(A)$ as the projective limit of the groups $\mathcal{L}_n(A/U)$ with $U \in \mathfrak{U}$. Then A and $\mathcal{L}_n(A)$ are $\widehat{\mathbb{Z}}$-modules. In the following *algebra* means always $\widehat{\mathbb{Z}}$-algebra. The product of $a, b \in \mathcal{L}(A)$ is denoted by $[a, b]$. The functor $\mathcal{L}$ is a covariant functor from the category of profinite abelian groups to the category of profinite graded Lie algebras, i.e., graded Lie algebras (over $\widehat{\mathbb{Z}}$) whose homogeneous components are profinite.

2.3. Let $L = \bigoplus_{n=1}^{\infty} L_n$ be any profinite graded Lie algebra. Then we have a natural homomorphism ψ of $\mathcal{L}(L_1)$ into L with $\psi(a) = a$ for $a \in \mathcal{L}_1$.

2.4. The proof of our main result (Theorem 1.1) is based on the comparison of various filtrations of a profinite group G.

A *filtration* of G is a sequence of descending closed subgroups G_i $(i \geq 1)$ such that the following conditions are fulfilled:

(i) $G_1 = G$,
(ii) $[G_i, G_j] \subseteq G_{i+j}$ for $i, j \in \mathbb{N}$,

where $[G_i, G_j]$ denotes the closed subgroup of G generated by the commutators

$$(g, h) := g^{-1}h^{-1}gh \qquad \text{for } g \in G_i,\ h \in G_j.$$

The most interesting filtration is the descending central series $(G^{(i)})$, which is defined by induction:

$$G^{(1)} := G, \qquad G^{(i+1)} := [G, G^{(i)}].$$

One proves by induction that $(G^{(i)})$ is a filtration of G using the following well known rules for commutators (see e.g. [Hl], 10.2), where x^y means $y^{-1}xy$:

$$(h,g)=(g,h)^{-1}, \tag{1}$$

$$h^g=h(h,g), \tag{2}$$

$$(f,gh)=(f,h)(f,g)((f,g),h), \tag{3}$$

$$(fg,h)=(f,h)((f,h),g)(g,h), \tag{4}$$

$$(f^g,(g,h))(g^h,(h,f))(h^f,(f,g))=1, \tag{5}$$

for $f,g,h\in G$.

We associate with a filtered group G a graded Lie algebra $L(G)$ as follows. By definition, the groups G_i are normal subgroups of G. We put

$$L_n(G):=G_n/G_{n+1}\quad\text{and}\quad[\overline{g},\overline{h}]:=\overline{(g,h)}$$

for $g\in G_n$, $h\in G_m$. It is easy to see that this definition does not depend on the choice of g and h in the classes $\overline{g}\in L_n(G)$ and $\overline{h}\in L_m(G)$ and that it defines the structure of a profinite graded Lie algebra on

$$L(G):=\bigoplus_{n=1}^{\infty}L_n(G)$$

by (1)–(5).

2.5. We now restrict ourselves to the special situation of a free pro-p-group F, where p denotes a prime number (see [Se2] for the definition of F).

Theorem 2.1. *Let $L(F)$ be the Lie algebra associated with the descending central series of F. The natural map $\psi:\mathcal{L}(F/F^{(2)})$ to $L(F)$ is an isomorphism of graded Lie algebras over $\mathbb{Z}_p$.*

Proof. Let F be the free pro-p-group with generator system $\{s_i\mid i\in I\}$ and let S be any finite subset of I. Furthermore, let F_S be the factor group of F with generator system S. Then F is the projective limit of the groups F_S and $\mathcal{L}_n(F/F^{(2)})$ (resp. $L(F)$) is the projective limit of the profinite groups $\mathcal{L}_n(F_S/F_S^{(2)})$ (resp. $L_n(F_S)$). Hence, it is sufficient to prove the theorem for free pro-p-groups F with finite generator rank N.

Let $s_1,\dots,s_N$ be the free generator system of F and let x_i be the class of s_i in $F/F^{(2)}$. Then $F/F^{(2)}$ is the free $\mathbb{Z}_p$-module with generators $x_1,\dots,x_N$ and hence $\mathcal{L}(F/F^{(2)})$ is the free $\mathbb{Z}_p$-Lie algebra with generators $x_1,\dots,x_N$. On the other hand, $L(F)$ as well is the free $\mathbb{Z}_p$-Lie algebra with generators $x_1,\dots,x_N$ as follows from the argument of [Wi1] applied to the embedding of F into the completed group algebra $\mathbb{Z}_p$ (see §2.7). We have

$$\mathrm{rk}_{\mathbb{Z}_p}\mathcal{L}_n(F/F^{(2)}) = \mathrm{rk}_{\mathbb{Z}_p}L_n(F) = \frac{1}{n}\sum_{d|n}\mu(n/d)N^d,$$

where μ denotes the Möbius function.

This completes the proof of Theorem 2.1 since ψ is surjective. ∎

2.6. The special filtrations (G_i) of a pro-p-group G with the property

$$G_i^p \subseteq G_{i+1}$$

are called *p-filtrations*.

If (G_i) is a p-filtration of G, then $L(G)$ is an $\mathbb{F}_p$-Lie algebra with an extra homogeneous operator π of degree 1 defined by

$$\pi(gG_{i+1}) = g^pG_{i+2}, \quad i = 1, 2, \ldots$$

Using induction over s one proves

$$(gh)^s \equiv g^sh^s(g,h)^{s(s-1)/2} \pmod{G_{i+j+1}} \quad \text{for } g \in G_i, h \in G_j.$$

This shows that π is linear for $p > 2$ and for $i > 1$ if $p = 2$. If $p = 2$ and $a, b \in L_1(G)$ one has

$$\pi(a + b) = \pi a + \pi b + [a, b].$$

Using (2), one proves by induction over s that

$$(g^s, h) \equiv (g,h)^s((g,h),g)^{s(s-1)/2} \pmod{G_{2i+j+1}} \quad \text{for } g \in G_i, h \in G_j.$$

This shows that

$$\pi[a, b] = [\pi a, b]$$

if $a \in L_i(G)$, $b \in L_j(G)$ and $p > 2$ or if $i > 1$. Altogether we see that $L(G)$ is a graded $\mathbb{F}_p[\pi]$-Lie algebra in the case where $p > 2$ and (G_n) $(n \geq 1)$ is a p-filtration. If $p = 2$ then $L_{>1}(G) := \bigoplus_{n=2}^{\infty} L_n(G)$ is a graded $\mathbb{F}_p[\pi]$-Lie algebra.

2.7. Let F be a free pro-p-group with generators $s_1, \ldots, s_N$. Beside the filtration $(F^{(i)})_{i\geq 1}$ we need more general filtrations called κ-filtrations, and corresponding p-filtrations called (κ, p)-filtrations. They were introduced in [Lz], II.3.2, in much greater generality, but we restrict ourselves to what will be necessary for our paper.

For the definitions of these filtrations we consider the completed group algebra $A := \mathbb{Z}_p[[F]]$, which is isomorphic to the ring $\mathbb{Z}_p[[X_1, \ldots, X_N]]$ of associative formal power series in the variables $X_1, \ldots, X_N$ with coefficients in $\mathbb{Z}_p$. The isomorphism α is defined by $\alpha(s_i) = 1 + X_i$ ([Se1]). In the following we identify A and $\mathbb{Z}_p[[X_1, \ldots, X_N]]$ by means of α. The restriction of α to F yields the Magnus representation of F.

For any natural number κ we define a valuation v of A in the sense of Lazard ([Lz], I.2.2) by means of

$$v\Big(\sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} X_{i_1}\ldots X_{i_k}\Big) = \inf_{i_1,\ldots,i_k}\{b_{i_1,\ldots,i_k}\}$$

with

$$b_{i_1,\ldots,i_k} = \nu_p(a_{i_1,\ldots,i_k}) + (i_1+\ldots+i_k)\kappa,$$

where ν_p denotes the p-adic (exponential) valuation of $\mathbb{Z}_p$. Then v defines a filtration (A^i) of A with

$$A^i := \{u\in A \mid v(u)\ge i\}.$$

We define the (κ,p)-*filtration* of F by

$$\widehat{F}^{(i)} := \{x\in F \mid v(x-1)\ge i\}.$$

The associated Lie algebra $\widehat{L} = \sum_{n=1}^{\infty}\widehat{L}_n$ is an $\mathbb{F}_p[\pi]$-Lie algebra if $p>2$ or $\kappa>1$. In what follows, we will assume that $p>2$.

In the same way one can define the filtration $(\widetilde{F}^{(n)})$ by means of the valuation w of A which is given by

$$w\Big(\sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} X_{i_1}\ldots X_{i_k}\Big) = \inf_{i_1,\ldots,i_k}\{c_{i_1,\ldots,i_k}\}$$

with

$$c_{i_1,\ldots,i_k} = (i_1+\ldots+i_k)\kappa.$$

We define a filtration (B^i) of A:

$$B^i := \{u\in A \mid w(u)\ge i\}.$$

Then

$$\widetilde{F}^{(i)} := \{x\in F \mid w(x-1)\ge i\}.$$

We denote the associated Lie algebra by $\widetilde{L} = \sum_{n=1}^{\infty}\widetilde{L}_n$. The Lie algebra $\widetilde{L}$ is a free Lie algebra over $\mathbb{Z}_p$ on the images of $s_1,\ldots,s_N$ in $L_\kappa = \widetilde{F}^{(\kappa)}/\widetilde{F}^{(\kappa+1)}$.

Let $\overline{L}$ be the Lie subalgebra of $\widehat{L}$ generated by $\sigma_i := s_i\widehat{F}^{(\kappa+1)}$, $i=1,\ldots,N$, and let

$$\overline{L}_n := \widehat{L}_n\cap\overline{L}, \qquad n=1,2,\ldots$$

Then $\overline{L}_n=\{0\}$ if $n\not\equiv 0 \pmod{\kappa}$.

We have the following structure theorem for $\widehat{L}$:

THEOREM 2.2. *$\overline{L}$ is the free $\mathbb{F}_p$-Lie algebra with generators $\sigma_1,\ldots,\sigma_N$ and $\widehat{L}$ is the free $\mathbb{F}_p[\pi]$-Lie algebra with generators $\sigma_1,\ldots,\sigma_N$.*

Proof. This result is well known. It is proved in [Lz], II.3.2, and goes already back to A. Skopin ([Sk]). In fact, the assertions follow easily from the embedding of F in the algebra A and the theorem of Witt about Lie polynomials in A ([Wi1]). ∎

2.8. We want to compare the κ- and the (κ,p)-filtration of the free pro-p-group F. For this purpose we introduce filtrations in $\widetilde{L}_n$ and $\widehat{L}_n$. In $\widetilde{L}_n$ our filtration is simply $\widetilde{L}_n^h := p^h\widetilde{L}_n$, $h \geq 1$.

PROPOSITION 2.3.

$$\begin{aligned} p^h\widetilde{L}_n &= (\widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)})\widetilde{F}^{(n+1)}/\widetilde{F}^{(n+1)} \\ &= (\widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)}\widetilde{F}^{(n+1)})/\widetilde{F}^{(n+1)}. \end{aligned}$$

Proof. An element in $p^h\widetilde{L}_n$ has the form $x^{p^h}\widetilde{F}^{n+1}$ with $x \in \widetilde{F}^{(n)}$. Therefore, $x^{p^h} \in \widehat{F}^{(n+h)}$. Let now y be an element of $\widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)}$. We want to show that $y\widetilde{F}^{(n+1)}$ is in $p^h\widetilde{L}_n$.

We assume that $n = \kappa m$ with $m \in \mathbb{N}$. Then

$$y \equiv 1 + y_n \pmod{B^{n+1}},$$

where y_n is a homogeneous polynomial of degree m in A. Furthermore, $y \in \widehat{F}^{(n+h)}$ if and only if $y_n \in A^{n+h}$. This is possible only if each coefficient of the polynomial y_n is divisible by p^h. Hence y has the form

$$y \equiv 1 + p^h z_n \pmod{B^{n+1}}$$

with $z_n \in B^n$. By the theorem of Witt ([Wi1]), z_n is a Lie polynomial in A. Hence, there is a $z \in \widetilde{F}^{(n)}$ such that $z \equiv 1 + z_n \pmod{B^{n+1}}$ and this implies $z^{p^h}\widetilde{F}^{(n+1)} = y\widetilde{F}^{(n+1)} \in p^h\widetilde{L}_n$. ∎

By Theorem 2.2 the group $\widehat{L}_n$ has the form

$$\widehat{L}_n = \bigoplus_{m=0}^{n-1} \pi^m \overline{L}_{n-m}.$$

We define a filtration $(\widehat{L}_n^{(h)})_{1\leq h\leq n}$ of $\widehat{L}_n$ by

$$\widehat{L}_n^{(h)} := \bigoplus_{m=0}^{n-h} \pi^m \overline{L}_{n-m}.$$

PROPOSITION 2.4.

$$\widehat{L}_n^{(h)} = (\widehat{F}^{(n)} \cap \widetilde{F}^{(h)})\widehat{F}^{(n+1)}/\widehat{F}^{(n+1)} = (\widehat{F}^{(n)} \cap \widetilde{F}^{(h)}\widehat{F}^{(n+1)})/\widehat{F}^{(n+1)}.$$

The proof of this proposition is a variation of the proof of Theorem 2.2.

Now we define the following maps $\omega_{h,n}$ from $\widetilde{L}_n^{(h)}$ onto $\pi^h\overline{L}_n$, which allow us to compare $\widetilde{L}$ with $\widehat{L}$:

$$\begin{aligned}\omega_{h,n}:\widetilde{L}_n^{(h)} &= (\widetilde{F}^{(n)}\cap\widehat{F}^{(n+h)})\widetilde{F}^{(n+1)}/\widetilde{F}^{(n+1)}\\ &\to (\widetilde{F}^{(n)}\cap\widehat{F}^{(n+h)})\widetilde{F}^{(n+1)}\widehat{F}^{(n+h+1)}/\widetilde{F}^{(n+1)}\widehat{F}^{(n+h+1)}\\ &\overset{\cong}{\to} (\widetilde{F}^{(n)}\widehat{F}^{(n+h+1)}\cap\widehat{F}^{(n+h)})/(\widetilde{F}^{(n+1)}\widehat{F}^{(n+h+1)}\cap\widehat{F}^{(n+h)})\\ &\overset{\cong}{\to} \widehat{L}_{n+h}^{(n)}/\widehat{L}_{n+h}^{(n+1)}\overset{\cong}{\to}\pi^h\overline{L}_n,\end{aligned}$$

where the arrows denote the corresponding natural maps.

PROPOSITION 2.5. $\ker\omega_{h,n}=\widetilde{L}_n^{(h+1)}$.

Proof. By definition

$$\ker\omega_{h,n}=(\widetilde{F}^{(n)}\cap\widehat{F}^{(n+h+1)}\widetilde{F}^{(n+1)})/\widetilde{F}^{(n+1)}=\widetilde{L}_n^{(h+1)}. \ \blacksquare$$

3. The Artin map. We first recall some facts from class field theory (see e.g. [We]).

3.1. A *local field* is a finite extension of the field $\mathbb{Q}_p$ of rational p-adic numbers (case of characteristic 0) or a finite extension of the field $\mathbb{F}_p((x))$ of power series in the variable x over the field $\mathbb{F}_p$ with p elements (case of characteristic p). A *global field* is a finite extension of the field $\mathbb{Q}$ of rational numbers (case of characteristic 0) or a finite extension of the field $\mathbb{F}_p(x)$ of rational functions with coefficients in $\mathbb{F}_p$.

Let K be a local or global field. Then one has a *formation module* $\mathcal{A}_K$ associated with K, which is the multiplicative group $K^\times$ if K is a local field, and the idele class group of K if K is a global field. Furthermore, let $\widehat{\mathcal{A}}_K$ be the profinite completion of $\mathcal{A}_K$. Then the Artin map is a canonical map from $\mathcal{A}_K$ into the Galois group of the maximal abelian extension K^{ab} of K, which induces an isomorphism ϕ_K from $\widehat{\mathcal{A}}_K$ onto $G(K^{\mathrm{ab}}/K)$. In the following we call ϕ_K the *Artin map*.

3.2. Let $\overline{K}$ be a fixed separable algebraic closure of K and let G_K be the Galois group of $\overline{K}/K$. By 2.1–2.4, the map ϕ_K induces a homomorphism of Lie algebras from $\mathcal{L}(\widehat{\mathcal{A}}_K)$ onto $L(G_K)$, which will be denoted by ϕ_K as well. We let $\phi_{K,n}$ be the component of degree n of ϕ_K. Then $\phi_{K,1}$ is the usual Artin map. We call ϕ_K the *Artin map of* $\mathcal{L}(\widehat{\mathcal{A}}_K)$.

3.3. Let G_K^{nil} be the Galois group of the maximal nilpotent extension of K in $\overline{K}$. Then the kernel of the projection $G_K\to G_K^{\mathrm{nil}}$ is equal to the intersection of the groups $G_K^{(n)}$ for $n\geq 1$. Therefore, one has a natural isomorphism of $L(G_K)$ onto $L(G_K^{\mathrm{nil}})$. Since G_K^{nil} is canonically isomorphic to the product of its l-components $G_K(l)$, this implies that $L(G_K)$ is canonically isomorphic to the direct product of the Lie algebras $L(G_K(l))$, where

l runs through all primes. Similarly, the decomposition of $\widehat{\mathcal{A}}_K$ into the direct product of its l-components $\widehat{\mathcal{A}}_K(l)$ yields a canonical decomposition of $\mathcal{L}(\widehat{\mathcal{A}}_K)$ as the product of the Lie algebras $\mathcal{L}(\widehat{\mathcal{A}}_K(l))$. The study of the Artin map ϕ_K therefore reduces to the study of its l-components

$$\phi_K(l) : \mathcal{L}(\widehat{\mathcal{A}}_K(l)) \to L(G_K(l))$$

as l varies over all primes. The map $\phi_K(p)$ and its $\overline{p}$-component

$$\phi_K(\overline{p}) : \mathcal{L}(\widehat{\mathcal{A}}_K(\overline{p})) \to L(G_K(\overline{p}))$$

with

$$\widehat{\mathcal{A}}_K(\overline{p}) := \prod_{l\neq p} \widehat{\mathcal{A}}_K(l), \quad G_K(\overline{p}) := \prod_{l\neq p} G_K(l)$$

are the subjects of our further investigations.

3.4. We now restrict ourselves to the case where K is a local field of residue characteristic p. We denote the ring of integers of K by $\mathfrak{O}_K$ and the maximal ideal of $\mathfrak{O}_K$ by $\mathfrak{p}$. Hence $\mathcal{A}_K = K^\times$ and $\widehat{\mathcal{A}}_K$ is the direct product of a group (π) generated as topological group by a fixed prime element π, the group μ_{q-1} of roots of unity in K of order dividing $q-1$, where q is the number of elements in the residue field, and of the group $1+\mathfrak{p}$ of principal units in K. The group (π) is isomorphic to $\widehat{\mathbb{Z}}$, the total completion of $\mathbb{Z}$, the group μ_{q-1} is cyclic of order $q-1$ and the group $1+\mathfrak{p}$ is a pro-p-group, where p denotes the residue characteristic of K. The group $1+\mathfrak{p}$ is the direct product of a finite cyclic group and a free abelian pro-p-group.

The surjectivity of ϕ_K implies the surjectivity of $\phi_K(p)$ and $\phi_K(\overline{p})$. The main goal of this paper is the determination of the kernel of $\phi_K(p)$ and $\phi_K(\overline{p})$.

3.5. In this section we consider $\phi_K(\overline{p})$. We introduce the following notations: A profinite group G will be called a $\overline{p}$-*group* if G is pro-nilpotent and all finite factor groups of G have order prime to p. Corresponding by a $\overline{p}$-extension of K is a Galois extension of K with Galois group being a $\overline{p}$-group.

PROPOSITION 3.1. *Let σ be an extension of the Frobenius automorphism of the maximal unramified $\overline{p}$-extension of K and let τ be a topological generator of the inertia group of $G_K(\overline{p})$. Then $G_K(\overline{p})$ is generated as $\overline{p}$-group by σ and τ and has one generating relation*

$$(\sigma,\tau)\tau^{q-1} = 1. \tag{6}$$

Let $\overline{\sigma}$ and $\overline{\tau}$ be the images of σ and τ in $L_1(G_K(\overline{p})) = G_K(\overline{p})/G_K(\overline{p})^{(2)}$. If $n \geq 2$, then $L_n(G_K(\overline{p}))$ is a cyclic group of order $q-1$ with generator

$$[\overline{\sigma},[\overline{\sigma},\ldots,[\overline{\sigma},\overline{\tau}]\ldots]]. \tag{7}$$

Proof. The structure of the group $G_K(\overline{p})$ is well known (see e.g. [Ko1], p. 95). The relation (6) implies that any element of the form

$$[a_1,[a_2,\ldots,[a_{n-1},a_n]\ldots]]\in L_n(G_K(\overline{p}))$$

with $a_i\in\{\overline{\sigma},\overline{\tau}\}$ is equal to 0 if at least for two of the $a_1,\ldots,a_n$ one has $a_i=\overline{\tau}$. It follows that

$$[\overline{\sigma},[\overline{\sigma},\ldots,[\overline{\sigma},\overline{\tau}]\ldots]]=\tau^{(1-q)^{n-1}}G_K(\overline{p})^{(n+1)} \tag{8}$$

is a generator of $L_n(G_K(\overline{p}))$ and has order $q-1$. ∎

For the next proposition we introduce some further notation.

If $\alpha\in K^\times$ we denote by $\overline{\alpha}$ the image of α under the map

$$K^\times\to\widehat{K}^\times\to\widehat{K}^\times(\overline{p}).$$

Let μ_{q-1} be the group of roots of unity of order dividing $q-1$ and let ζ be a generator of μ_{q-1}. Furthermore, let π be a prime element of K. Then the pro-$\overline{p}$-group $K^\times(\overline{p})$ is generated by $\overline{\zeta}$ and $\overline{\pi}$. The elements of $K^\times(\overline{p})$ are uniquely represented in the form $\overline{\zeta}^\mu\overline{\pi}^\nu$ with $\mu=0,\ldots,q-2$, $\nu\in\mathbb{Z}_{\overline{p}}$, i.e., $K^\times(\overline{p})\cong\mu_{q-1}\times\mathbb{Z}_{\overline{p}}$. We denote by $\mathcal{M}$ the derived algebra of $\mathcal{L}(K^\times(\overline{p}))$ and by $\mathcal{N}$ the ideal of $\mathcal{L}(K^\times(\overline{p}))$ generated by all the elements of the form

$$[\overline{\zeta},\mathrm{ad}(\overline{\pi})^n\overline{\zeta}]\qquad(n\ge1).$$

Furthermore, let $\mathcal{F}=\mathcal{L}(\mathbb{Z}/(q-1)\mathbb{Z}\oplus\mathbb{Z}/(q-1))$. With these notations we have the following proposition.

Proposition 3.2. *As a graded Lie algebra, $\mathcal{M}$ is isomorphic to the derived algebra of $\mathcal{F}$ and the kernel of the map $\phi_K(\overline{p})$ is $\mathcal{N}$. Furthermore,*

$$\mathcal{M}_n=\mathcal{N}_n\oplus\mathbb{Z}/(q-1)\mathbb{Z}\cdot\mathrm{ad}(\overline{\pi})^n(\overline{\zeta})\qquad(n\ge1).$$

Proof. The natural projection $\widehat{K}^\times(\overline{p})\to\widehat{K}^\times(\overline{p})/\widehat{K}^\times(\overline{p})^{q-1}$ induces a surjective homomorphism ϕ' of graded Lie algebras. Since $\widehat{K}^\times(\overline{p})/\widehat{K}^\times(\overline{p})^{q-1}$ is a free $\mathbb{Z}/(q-1)\mathbb{Z}$-module of rank 2 it follows that the restriction of ϕ' to $\mathcal{M}$ is an isomorphism and $\mathcal{L}(\widehat{K}^\times(\overline{p})/\widehat{K}^\times(\overline{p})^{q-1})$ is the free graded Lie algebra with two generators over the ring $\mathbb{Z}/(q-1)\mathbb{Z}$. Furthermore, we can choose $\overline{\sigma}$ and $\overline{\tau}$ in Proposition 3.1 such that

$$\phi_K(\overline{p})(\overline{\pi})=\overline{\sigma},\qquad\phi_K(\overline{p})(\overline{\zeta})=\overline{\tau}.$$

Proposition 3.1 implies that for $n\ge2$ the group $\mathcal{L}_n(\widehat{K}^\times(\overline{p}))$ is the direct sum of $(\ker\phi_K(\overline{p}))_n$ and the cyclic group of order $q-1$ generated by

$$[\overline{\pi},[\overline{\pi},\ldots,[\overline{\pi},\overline{\zeta}]\ldots]].$$

This proves Proposition 3.2. ∎

4. The map $\phi_K(p)$. It remains to consider $\phi_K(p)$. This is the main goal of the paper. We restrict ourselves to the case $p\ne2$.

The structure of $G_K(p)$ is well known (see e.g. [La0], [Ko1], pp. 96–105): If $\mathrm{Char}\, K = p$, or if K does not contain the pth roots of unity, then $G_K(p)$ is a free pro-p-group and $\phi_K(p)$ is an isomorphism.

PROPOSITION 4.1. *Let K be a local field of characteristic p or of characteristic 0 and not containing the pth roots of unity. Then $\phi_K(p)$ is an isomorphism of $\mathcal{L}(\widehat{K}^\times(p))$ onto $L(G_K(p))$.* ∎

Now let K be a local field of characteristic 0 which contains the pth roots of unity. Then $\widehat{K}^\times(p)$ is isomorphic to $\mu_{p^\kappa} \times \mathbb{Z}_p^{N-1}$, where $N = [K : \mathbb{Q}_p] + 2$ and κ is the natural number such that $\mu_{p^\kappa} \subset K$ but $\mu_{p^{\kappa+1}} \not\subset K$. Then $G_K(p)$ is a Demushkin group and so is a group with N generators $s_1, \dots, s_N$ and one generating relation r. One can choose $s_1, \dots, s_N$ such that

$$r = s_1^{p^\kappa}(s_1, s_2)(s_3, s_4)\dots(s_{N-1}, s_N).$$

In the following we identify $G_K(p)$ with F/R, where F is the free pro-p-group with generators $s_1, \dots, s_N$ and R is the closed normal subgroup of F generated by r. The projection $F \to G_K(p)$ induces a surjective homomorphism

$$\theta : L(F) \to L(G_K(p)).$$

We let ψ be the unique homomorphism of $L(F)$ onto $\mathcal{L}(\widehat{K}^\times(p))$ such that

$$\theta = \phi_K(p)\psi.$$

We first study θ. With the identification

$$G_K(p) = F/R, \qquad R = (r)$$

this study is a question of group theory. We introduce the following notations:

$$\begin{aligned} R^{(n)} &:= R \cap F^{(n)}, \\ \mathcal{N}_n(R) &:= R^{(n)}F^{(n+1)}/F^{(n+1)}, \\ \mathcal{N}(R) &:= \sum_{n=1}^{\infty} \mathcal{N}_n(R). \end{aligned}$$

PROPOSITION 4.2. *$\mathcal{N}_n(R)$ is the kernel of $\theta_n : L_n(F) \to L_n(F/R)$.*

Proof. We have

$$\begin{aligned} L_n(F/R) &= (F/R)^{(n)}/(F/R)^{(n+1)} \\ &\cong F^{(n)}R/F^{(n+1)}R \cong F^{(n)}/F^{(n+1)}(F^{(n)} \cap R). \end{aligned}$$

Hence, $\ker \theta_n = F^{(n+1)}(F^{(n)} \cap R)/F^{(n+1)}$. ∎

Let U be the enveloping algebra of $L(F)$ ([Se1]). Since the $\mathbb{Z}_p$-Lie algebra $L(F)$ is a free algebra generated by

$$\{\sigma_i := s_i F^{(2)} \mid i = 1, \dots, N\}$$

we can identify U with the ring of polynomials in the non-commutative indeterminants $\sigma_1, \dots, \sigma_N$ with coefficients in $\mathbb{Z}_p$. The ring U operates on $L(F)$ by adjoint action such that

$$\mathrm{ad}(\alpha)\beta = [\alpha, \beta],$$
$$\mathrm{ad}(\lambda_1\lambda_2)\alpha = \mathrm{ad}(\lambda_1)\mathrm{ad}(\lambda_2)\alpha$$

and

$$\mathrm{ad}(\lambda_1 + \lambda_2)\alpha = \mathrm{ad}(\lambda_1)\alpha + \mathrm{ad}(\lambda_2)\alpha$$

for $\alpha, \beta \in L(F)$, $\lambda_1, \lambda_2 \in U$.

We put

$$t := (s_1, s_2) \dots (s_{N-1}, s_N), \qquad \tau := tF^{(3)} \in L_2(F).$$

Let $\mathcal{N}'(R)$ be the ideal of $\mathcal{N}(R)$ generated by the elements

$$p^\kappa \sigma_1, \quad [\mathrm{ad}(\lambda)\sigma_1, \mathrm{ad}(\lambda)\tau] \quad (\lambda \in U). \tag{9}$$

Then $\mathcal{N}'(R)$ is generated as a $\mathbb{Z}_p$-module by the element $p^\kappa \sigma_1$ together with the elements

$$[\mathrm{ad}(\lambda)\sigma_1, \mathrm{ad}(\lambda)\tau], \tag{10}$$
$$[\mathrm{ad}(\lambda)\sigma_1, \mathrm{ad}(\mu)\tau] + [\mathrm{ad}(\mu)\sigma_1, \mathrm{ad}(\lambda)\tau] \tag{11}$$

with λ, μ homogeneous elements of U. The goal of this section is the proof of the following theorem:

THEOREM 4.3. *$\mathcal{N}(R) = \mathcal{N}'(R)$.*

COROLLARY 4.4. *The subalgebra of $L(G_K(p))$ generated by $\sigma_2, \dots, \sigma_N$ is a free Lie algebra over $\mathbb{Z}_p$ on these generators.*

The corollary follows immediately from the fact that $\mathcal{N}(R)$ is a subset of the ideal of L generated by σ_1.

To prove the theorem we first show that $\mathcal{N}'(R) \subseteq \mathcal{N}(R)$. Firstly,

$$rF^{(2)} = s_1^{p^\kappa} F^{(2)} = p^\kappa \sigma_1,$$

and, to show that the elements of the form (10), (11) lie in $\mathcal{N}(R)$, we may assume that

$$\lambda = \sigma_{i_1} \dots \sigma_{i_l} \quad \text{and} \quad \mu = \sigma_{j_1} \dots \sigma_{j_k}.$$

Then

$$[\mathrm{ad}(\lambda)\sigma_1, \mathrm{ad}(\lambda)\tau] = ((s_{i_1}, \dots, (s_{i_l}, s_1) \dots), (s_{i_1}, \dots, (s_{i_l}, t) \dots))F^{(2l+4)}.$$

Since $r = s_1^{p^\kappa} t$, we have

$$(s_{i_l}, t) \in (s_{i_l}, s_1^{-p^\kappa} r)F^{(4)}R = (s_{i_l}, s_1^{-p^\kappa})F^{(4)}R$$

and

$$(s_{i_l}, s_1^{-p^\kappa})F^{(4)} = (s_{i_l}, s_1)^{-p^\kappa}((s_{i_l}, s_1), s_1)^{-p^\kappa(-p^\kappa+1)/2}F^{(4)}$$
$$= (s_{i_l}, s_1)^{-p^\kappa}((s_{i_l}, s_1), r)^{(p^\kappa-1)/2}F^{(4)}.$$

We get

$$((s_{i_1}, \ldots, (s_{i_l}, s_1)\ldots), (s_{i_1}, \ldots, (s_{i_l}, t)\ldots))F^{(2l+4)} \in RF^{(2l+4)} \cap F^{(2l+3)}$$

and this implies

$$[\mathrm{ad}(\lambda)\sigma_1, \mathrm{ad}(\lambda)\tau] \in \mathcal{N}_{2l+3}(R),$$

which shows that elements of the form (10) belong to $\mathcal{N}(R)$. In a similar manner one shows that the elements of the form (11) also belong to $\mathcal{N}(R)$.

To show that $\mathcal{N}(R) \subseteq \mathcal{N}'(R)$ we use a technique of [La3] consisting in the comparison of the κ- and (κ, p)-filtrations of F, where now κ is equal to the κ appearing in the defining relation $r = s_1^{p^\kappa}(s_1, s_2)\ldots(s_{N-1}, s_N)$ of $G_K(p)$.

We introduce the following notation as supplement to the notation in 2.7–2.8:

$$\widetilde{\sigma}_i := s_i\widetilde{F}^{(\kappa+1)} \in \widetilde{L}_\kappa, \quad i = 1, \ldots, m+2,$$
$$\widetilde{\tau} := (s_1, s_2)\ldots(s_{N-1}, s_N)\widetilde{F}^{(2\kappa+1)} \in \widetilde{L}_{2\kappa},$$
$$\widetilde{\mathcal{N}}_n(R) := (R \cap \widetilde{F}^{(n)})\widetilde{F}^{(n+1)}/\widetilde{F}^{(n+1)},$$
$$\widetilde{\mathcal{N}}(R) := \sum_{n=1}^{\infty} \widetilde{\mathcal{N}}_n(R).$$

Then $\widetilde{\mathcal{N}}'(R)$ is the ideal of $\widetilde{L}$ generated by $p^\kappa\widetilde{\sigma}_1$ and $\mathrm{ad}(\lambda)\widetilde{\sigma}_1 \wedge \mathrm{ad}(\lambda)\widetilde{\tau}$ for $\lambda \in \widetilde{U}$, where $\widetilde{U}$ denotes the enveloping algebra of $\widetilde{L}$. Set

$$\widehat{\sigma}_i := s_i\widehat{F}^{(\kappa+1)} \in \widehat{L}_\kappa, \quad i = 1, \ldots, m+2,$$
$$\widehat{\tau} := (s_1, s_2)\ldots(s_{N-1}, s_N)\widehat{F}^{(2\kappa+1)} \in \widehat{L}_{2\kappa},$$
$$\widehat{\mathcal{N}}_n(R) := (R \cap \widehat{F}^{(n)})\widehat{F}^{(n+1)}/\widehat{F}^{(n+1)},$$
$$\widehat{\mathcal{N}}(R) := \bigoplus_{n=1}^{\infty} \widehat{\mathcal{N}}_n(R).$$

The homogeneous component $\widehat{\mathcal{N}}_{2\kappa}(R)$ contains the element

$$r\widehat{F}^{(2\kappa+1)} = \pi^\kappa\widehat{\sigma}_1 + \widehat{\tau}$$

and by Theorem 4′ of [La1], $\widehat{\mathcal{N}}(R)$ is even generated as an ideal of $\widehat{L}$ by $\pi^\kappa\widehat{\sigma}_1 + \widehat{\tau}$. This is the initial point of our proof.

Now we show $\widetilde{\mathcal{N}}'(R) = \widetilde{\mathcal{N}}(R)$. The proof of $\widetilde{\mathcal{N}}'(R) \subseteq \widetilde{\mathcal{N}}(R)$ is similar to the proof of $\mathcal{N}'(R) \subseteq \mathcal{N}(R)$.

Let $\overline{U}$ be the enveloping algebra of $\overline{L}$. Then $\overline{U}$ can and will be identified with the $\mathbb{F}_p$-subalgebra of the enveloping algebra $\widehat{U}$ of $\widehat{L}$ generated by $\widehat{\sigma}_1, \ldots, \widehat{\sigma}_N$. Any non-zero homogeneous element λ of $\widehat{L}$ can be uniquely written in the form

$$\lambda = \lambda_0 + \pi\lambda_1 + \ldots + \pi^l \lambda_l \tag{12}$$

with $\lambda_0, \lambda_1, \ldots, \lambda_l \in \overline{U}$ and $\lambda_l \neq 0$. Since $\deg(\lambda_l) \equiv 0 \pmod{\kappa}$ and $\deg(\lambda_{l-i})$
$= \deg(\lambda_l) + i$, we have $\lambda_i = 0$ if $i \not\equiv l \pmod{\kappa}$.

Let $\overline{\mathcal{I}}$ be the ideal of $\overline{L}$ generated by $\widehat{\sigma}_1$ and let $\overline{\mathcal{N}}$ be the ideal of $\overline{L}$ generated by the elements of the form

$$[\mathrm{ad}(\lambda)\widehat{\sigma}_1, \mathrm{ad}(\lambda)\widehat{\tau}] \tag{13}$$

with $\lambda \in \overline{U}$.

Lemma 4.5.

$$\widehat{\mathcal{N}}_m(R) \cap \widehat{L}_m^{(m-j)} \subseteq \begin{cases} \pi^j \overline{\mathcal{N}}_{m-j} + \widehat{L}_m^{(m-j+1)} & \textit{if } j < \kappa, \quad (14) \\ \pi^j \overline{\mathcal{I}}_{m-j} + \widehat{L}_m^{(m-j+1)} & \textit{if } j \geq \kappa. \quad (15) \end{cases}$$

Proof. Any element ϱ of $\widehat{\mathcal{N}}_m(R)$ has the form $\mathrm{ad}(\lambda)(\pi^\kappa \widehat{\sigma}_1 + \widehat{\tau})$ with λ as above. If $l = d\kappa + e$ with $0 \leq e < \kappa$, we have

$$\varrho = \pi^e \mathrm{ad}(\lambda_e)\widehat{\tau} + \sum_{j=1}^{d} \pi^{e+j\kappa}(\mathrm{ad}(\lambda_{e+(j-1)\kappa})\widehat{\sigma}_1 + \mathrm{ad}(\lambda_{e+j\kappa})\widehat{\tau}) + \pi^{l+\kappa}\mathrm{ad}(\lambda_l)\widehat{\sigma}_1.$$

If $\mathrm{ad}(\lambda_l)\widehat{\sigma}_1 \neq 0$, we have

$$\varrho \in \pi^{l+\kappa}\overline{\mathcal{I}}_{m-(l+\kappa)} + \widehat{L}_m^{(m-(l+\kappa)+1)},$$

which yields the required result.

Now suppose that $\mathrm{ad}(\lambda_l)\widehat{\sigma}_1 = 0$. Then λ_l lies in the annihilator of $\widehat{\sigma}_1$. By [La4], Theorem 2, the annihilator of $\widehat{\sigma}_1$ consists of the elements $u \in \overline{U}$ of the form

$$u = \sum_{v \in \overline{U}} a_v (\mathrm{ad}(v)\widehat{\sigma}_1) v \qquad (a_v \in \overline{U}).$$

Therefore λ_l has this form. If $l < \kappa$, we have

$$\varrho = \pi^l \sum_{v \in \overline{U}} a_v [\mathrm{ad}(v)\widehat{\sigma}_1, \mathrm{ad}(v)\widehat{\tau}] \in \pi^l \overline{\mathcal{N}}_{m-l}$$

as required. If $l \geq \kappa$ and $\mathrm{ad}(\lambda_{l-\kappa})\widehat{\sigma}_1 + \mathrm{ad}(\lambda_l)\widehat{\tau} \neq 0$, we have

$$\varrho \in \pi^l \overline{\mathcal{I}}_{m-l} + \widehat{L}_m^{(m-l+1)}$$

as required. If $\mathrm{ad}(\lambda_{l-\kappa})\widehat{\sigma}_1 + \mathrm{ad}(\lambda_l)\widehat{\tau} = 0$ we get

$$\begin{aligned}\mathrm{ad}(\lambda_{l-\kappa})\widehat{\sigma}_1 &= -\mathrm{ad}(\lambda_l)\widehat{\tau} = -\sum_{v\in\overline{U}} \mathrm{ad}(a_v(\mathrm{ad}(v)\widehat{\sigma}_1)v)\widehat{\tau}\\ &= \sum_{v\in\overline{U}} \mathrm{ad}(a_v(\mathrm{ad}(v)\widehat{\tau})v)\widehat{\sigma}_1\end{aligned}$$

since

$$\begin{aligned}\mathrm{ad}((\mathrm{ad}(v)\widehat{\sigma}_1)v)\widehat{\tau} &= \mathrm{ad}(\mathrm{ad}(v)\widehat{\sigma}_1)\mathrm{ad}(v)\widehat{\tau} = [\mathrm{ad}(v)\widehat{\sigma}_1, \mathrm{ad}(v)\widehat{\tau}]\\ &= -[\mathrm{ad}(v)\widehat{\tau}, \mathrm{ad}(v)\widehat{\sigma}_1] = -\mathrm{ad}((\mathrm{ad}(v)\widehat{\tau})v)\widehat{\sigma}_1.\end{aligned}$$

Hence

$$\lambda_{l-\kappa} - \sum_{v\in\overline{U}} a_v(\mathrm{ad}(v)\widehat{\tau})v$$

is in the annihilator of $\widehat{\sigma}_1$. Therefore,

$$\lambda_{l-\kappa} \in \mathrm{ann}(\widehat{\sigma}_1) + \mathrm{ann}(\widehat{\tau}).$$

If $\mathrm{ad}(\lambda_{l-(j+1)\kappa})\widehat{\sigma}_1 + \mathrm{ad}(\lambda_{l-j\kappa})\widehat{\tau} = 0$ for $1 \le j \le d$ then, repeating the above argument, we get

$$\lambda_e \in \mathrm{ann}(\widehat{\sigma}_1) + \mathrm{ann}(\widehat{\tau}),$$

which yields $\varrho \in \pi^e\overline{\mathcal{N}}_{m-e}$. Otherwise, there is a j such that

$$\mathrm{ad}(\lambda_{l-(j-1)\kappa})\widehat{\sigma}_1 + \mathrm{ad}(\lambda_{l-j\kappa})\widehat{\tau} \ne 0$$

and

$$\varrho \in \pi^{l-j\kappa}\overline{\mathcal{I}}_{m-(l-j\kappa)} + \widehat{L}_m^{(m-(l-j\kappa)+1)}. \ \blacksquare$$

R e m a r k. Lemma 4.5 deals with the ideal $\widehat{\mathcal{N}}(R)$ of the graded $\mathbb{F}_p$-algebra $\widehat{L}$ generated by $\pi^\kappa\widehat{\sigma}_1 + \widehat{\tau}$. It is easy to be seen that Lemma 4.5 is valid in the case $p = 2$ as well. This will be used in the proof of Theorem 5.1.

COROLLARY 4.6. $\widehat{\mathcal{N}}(R) \cap \overline{L} = \overline{\mathcal{N}}$.

We now consider the homomorphism $\omega_{0,n}$ of $\widetilde{L}_n$ onto $\overline{L}_n$. By Proposition 2.5 its kernel is $p\widetilde{L}_n$. Furthermore, $\omega_{0,n}$ maps $\widetilde{\mathcal{N}}_n(R)$ onto $\overline{L}_n \cap \widehat{\mathcal{N}}_n(R) = \overline{\mathcal{N}}_n$. Hence

$$\widetilde{\mathcal{N}}(R) \subseteq \widetilde{\mathcal{N}}'(R) + p\widetilde{L}. \tag{16}$$

More generally, we prove by induction

$$\widetilde{\mathcal{N}}(R) \subseteq \widetilde{\mathcal{N}}'(R) + p^{1+h}\widetilde{L}, \qquad h = 0, 1, \ldots, \tag{17}$$

using the homomorphisms $\omega_{h,n}$.

LEMMA 4.7. $\widetilde{\mathcal{N}}_n(R) \cap p^h\widetilde{L}_n = (R \cap \widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)})\widetilde{F}^{(n+1)}/\widetilde{F}^{(n+1)}$.

Proof. Let $\eta \in \widetilde{\mathcal{N}}_n(R) \cap p^h \widetilde{L}_n$. Then $\eta = yF^{(n+1)}$ with $y = u^{p^h} v$, $u \in \widetilde{F}_n$, $v \in \widetilde{F}_{n+1}$. Since $\widetilde{F}^{(n+1)} \subseteq \widehat{F}^{(n+1)}$, we have $v \in \widetilde{F}^{(n+1)} \cap \widehat{F}^{(n+1)}$. Let $l \geq 1$ be largest such that there exists $s \in R \cap \widetilde{F}^{(n+1)}$ with $vs \in \widehat{F}^{(n+l)} \cap \widetilde{F}^{(n+1)}$. Assume that $\delta < h$ and let δ be the image of vs in $(\widehat{F}^{(n+l)} \cap \widetilde{F}^{(n+1)})\widehat{F}^{(n+l+1)}/\widehat{F}^{(n+l+1)}$. Then

$$\delta \in \widehat{\mathcal{N}}_{n+l}(R) \cap \widehat{L}_{n+l}^{(l-m-1)}$$

for some integer m with $0 \leq m \leq l$, which we can assume is maximal and $\neq l$. By 4.5, we have $\delta = \delta_1 + \delta_2$ where $\delta_2 \in \widehat{L}_{n+l}^{(l-m-2)}$ and

$$\delta_1 \in \begin{cases} \pi^{l-1}\overline{\mathcal{N}}_{n+1} & \text{if } l \leq \kappa, \\ \pi^{l-1}\overline{\mathcal{I}}_{n+1} + \pi^{l-1}\overline{\mathcal{N}}_{n+1} & \text{if } l > \kappa, \end{cases}$$

where $\overline{\mathcal{I}}$ is the ideal of $\overline{L}$ generated by $\widehat{\sigma}_1$. It follows that there is an element $y_1 \in \widehat{F}^{(n+l)} \cap \widetilde{F}^{(n+1)}$ with $\delta_1 = y_1\widehat{F}^{(n+l+1)}$. But then $vyy_1^{-1} = \delta_2$ contradicting the maximality of m. ∎

Now, since

$$\begin{aligned} &\omega_{h,n}((R \cap \widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)})\widetilde{F}^{(n+1)}/\widetilde{F}^{(n+1)}) \\ &\qquad = ((R \cap \widetilde{F}^{(n)} \cap \widehat{F}^{(n+h)})\widehat{F}^{(n+h+1)}/\widehat{F}^{(n+h+1)})\widehat{L}_{n+h}^{(n+1)}/\widehat{L}_{n+h}^{(n+1)} \\ &\qquad = (\widehat{\mathcal{N}}_{n+h}(R) \cap \widehat{L}_{n+h}^{(n)}) + \widehat{L}_{n+h}^{(n+1)}/\widehat{L}_{n+h}^{(n+1)}, \end{aligned}$$

we have

$$\omega_{h,n}(\widetilde{\mathcal{N}}_n(R) \cap p^h \widetilde{L}_n) = (\widehat{\mathcal{N}}_{n+h}(R) \cap \widehat{L}_{n+h}^{(n)}) + \widehat{L}_{n+h}^{(n+1)}/\widehat{L}_{n+h}^{(n+1)}.$$

Assume that we proved

$$\widetilde{\mathcal{N}}(R) \subseteq \widetilde{\mathcal{N}}'(R) + p^h\widetilde{L}$$

for a certain h. We want to show

$$\widetilde{\mathcal{N}}(R) \subseteq \widetilde{\mathcal{N}}'(R) + p^{h+1}\widetilde{L}.$$

Let $\xi \in \widetilde{\mathcal{N}}_n(R)$. Then there exists $\xi' \in \widetilde{\mathcal{N}}'_n$ such that $\xi'' = \xi - \xi' \in p^h\widetilde{L}_n$. It follows that $\xi'' \in \widetilde{\mathcal{N}}_n(R) \cap p^h\widetilde{L}_n$. By Lemma 4.7 we have

$$\omega_{h,n}(\xi'') \in \begin{cases} \pi^h\overline{\mathcal{N}}_n & \text{if } h < \kappa, \quad (18) \\ \pi^h\overline{\mathcal{I}}_n + \pi^h\overline{\mathcal{N}}_n & \text{if } h \geq \kappa. \quad (19) \end{cases}$$

Hence there exists $\delta \in \widetilde{\mathcal{N}}'_n(R)$ such that $\omega_{h,n}(\xi'') = \omega_{h,n}(\delta)$, which implies that $\xi'' - \delta \in p^{h+1}\widetilde{L}_n$ and hence that $\xi - (\delta + \xi') \in p^{h+1}$ which completes the inductive step. It follows that $\widetilde{\mathcal{N}}(R) \subseteq \widetilde{\mathcal{N}}'(R)$ and hence that $\widetilde{\mathcal{N}}(R) = \widetilde{\mathcal{N}}'(R)$.

Since the grading of $\widetilde{L}$ is only a rescaling of the grading of L it follows immediately that $\mathcal{N}(R) = \mathcal{N}'(R)$, i.e. Theorem 4.3.

Now we consider the map

$$\psi : L(F) \to \mathcal{L}(\widehat{\mathcal{A}}_K(p))$$

in connection with $\theta : L(F) \to L(G_K(p))$. Let s_iR, $1 \le i \le m+2$, be a system of generators of $F/R = G_K(p)$ such that $R = (r)$ with

$$r = s_1^{p^\kappa}[s_1, s_2] \ldots [s_{N-1}, s_N].$$

Then (see e.g. [Ko1], Lemma 10.7) there are elements $\alpha_1, \ldots, \alpha_N$ in $K^\times$ such that

$$\left(\frac{\alpha_i}{K^{\mathrm{ab}}(p)/K}\right) = s_i R F^{(2)},$$

and α_1 is a root of unity of order p^κ in K.

Since the kernel of $\psi : L_1(F) \to \widehat{K}^\times(p)$ is generated by $p^\kappa \sigma_1$, where $\sigma_i := s_i F^{(2)}$, the kernel of ψ, as an ideal of $L(F)$, is generated by $p^\kappa\sigma_1$ as well.

Combining our knowledge of θ and ψ we get the following result about the kernel of the Artin map.

THEOREM 4.8. *The kernel of the Artin map $\phi_K(p)$ is generated as an ideal of $\mathcal{L}(\widehat{K}^\times(p))$ by the elements of the form*

$$[\mathrm{ad}(\lambda)\alpha_1, \mathrm{ad}(\lambda)\beta] \quad (\lambda \in U),$$

where U is the enveloping algebra of $\mathcal{L}(\widehat{K}^\times(p))$ and

$$\beta = [\alpha_1, \alpha_2] + \ldots + [\alpha_{N-1}, \alpha_N].$$

This yields Theorem 1.1 since, by Satz 7.23 of [Ko1], the image of β in $(\mathcal{L}_2(\widehat{K}^\times(p))) \otimes \mathbb{Z}/p^\kappa\mathbb{Z}$ is equal to τ.

COROLLARY 4.9. *The kernel of $\phi_K(p)$ is p^κ-torsion and, modulo torsion, $\mathcal{L}(\widehat{K}^\times(p))$ is a free Lie algebra over $\mathbb{Z}_p$ with basis the images of $\alpha_2, \ldots, \alpha_N$.*

Actually, as we shall see in Section 5, the kernel of $\phi_K(p)$ is a free $\mathbb{Z}/p^\kappa\mathbb{Z}$-module as is the torsion submodule of $\mathcal{L}(G_K(p))$. We shall, moreover, give formulae for the ranks of these free modules.

5. The module structure of $\mathcal{L}(G_K(p))$ and $\ker\phi_K(p)$. Let $L_\mathbb{Z} = L_\mathbb{Z}(x_1, \ldots, x_N)$ be the free Lie algebra over $\mathbb{Z}$ on the elements $x_1, \ldots, x_N$ and let U be its enveloping algebra. Assume that N is even, let

$$y = [x_1, x_2] + [x_3, x_4] + \ldots + [x_{N-1}, x_N],$$

and let $\mathcal{N}$ be the ideal of $L_\mathbb{Z}$ generated by the elements of the form $[\mathrm{ad}(u)x_1, \mathrm{ad}(u)y]$ with $u \in U$. Then, by Theorem 4.8, $\mathcal{L}(G_K(p))$ is isomorphic to $(L_\mathbb{Z}/\mathcal{N}) \otimes \mathbb{Z}_p$ modulo the ideal generated by $p^\kappa x_1$.

THEOREM 5.1. *$L_\mathbb{Z}/\mathcal{N}$ is a free $\mathbb{Z}$-module.*

Proof. Since the homogeneous components of $L_{\mathbb{Z}}/\mathcal{N}$ are finitely generated, it suffices to prove that, for each prime l, the homogeneous components of $(L_{\mathbb{Z}}/\mathcal{N})\otimes\mathbb{F}_l$ have ranks which are independent of l. Let $\overline{L}=L_{\mathbb{Z}}\otimes\mathbb{F}_l$ and let $\overline{\mathcal{N}}$ be the image of $\mathcal{N}$ in $\overline{L}$. Then

$$\overline{L}/\overline{\mathcal{N}}=(L_{\mathbb{Z}}/\mathcal{N})\otimes\mathbb{F}_l.$$

Let $\widehat{L}=\overline{L}\otimes\mathbb{F}_l[\pi]$, where π is an indeterminate over $\mathbb{F}_l$ of degree 1 and let $\widehat{\mathcal{N}}(R)$ as in Section 4 ($\kappa=1$) be the ideal of $\widehat{L}$ generated by $\pi\overline{x}_1+\overline{y}$, where $\overline{x}_i$ is the image of x_i in $\widehat{L}$ and

$$\overline{y}=[\overline{x}_1,\overline{x}_2]+[\overline{x}_3,\overline{x}_4]+\ldots+[\overline{x}_{N-1},\overline{x}_N].$$

Then, since $\widehat{L}$ is the free Lie algebra over $\mathbb{F}_l[\pi]$ on $\overline{x}_1,\ldots,\overline{x}_N$, we have $\widehat{\mathcal{N}}(R)\cap\overline{L}=\overline{\mathcal{N}}$ by Corollary 4.6, which implies that

$$\overline{L}/\overline{\mathcal{N}}\cong(\overline{L}+\widehat{\mathcal{N}}(R))/\widehat{\mathcal{N}}(R).$$

By Lemma 4.5 the initial form $\pi^s\lambda_0$ of a homogeneous element

$$\lambda=\pi^s\lambda_0+\pi^{s-1}\lambda_1+\ldots+\lambda_s,\qquad\lambda_0\neq0,$$

of $\widehat{\mathcal{N}}(R)$ is in the ideal of $\widehat{L}$ generated by $\overline{x}_1$. Hence

$$(\widehat{\mathcal{N}}(R)+\overline{L})\cap\pi\widehat{L}(\overline{x}_2,\ldots,\overline{x}_N)=0,$$

which implies that $\widehat{L}$ is the direct sum of $\widehat{\mathcal{N}}(R)+\overline{L}$ and $\pi\widehat{L}(\overline{x}_2,\ldots,\overline{x}_N)$ and hence that

$$\begin{aligned}\dim(\overline{L}/\overline{N})_n&=\dim(\widehat{L}/\widehat{\mathcal{N}}(R))_n-\dim(\pi\widehat{L}(\overline{x}_2,\ldots,\overline{x}_N))_n\\&=\dim(\widehat{L}/\widehat{\mathcal{N}}(R))_n-\dim(\widehat{L}(\overline{x}_2,\ldots,\overline{x}_N))_n\\&\quad+\dim(\overline{L}(\overline{x}_2,\ldots,\overline{x}_N))_n.\end{aligned}$$

Now, by Théorème 3 of [La1], $\widehat{L}/\widehat{\mathcal{N}}(R)$ is a free graded $\mathbb{F}_l[\pi]$-module and so

$$\widehat{L}/\widehat{\mathcal{N}}(R)\cong(\overline{L}/(\overline{y}))\otimes_{\mathbb{F}_l}\mathbb{F}_l[\pi]$$

as graded $\mathbb{F}_l$-modules. Now, by [La1], Théorème 2, the Poincaré series of the enveloping algebra of $\overline{L}/(\overline{y})$ is

$$\frac{1}{1-Nt+t^2}=\frac{1}{(1-\beta_1t)(1-\beta_2t)},$$

where $\beta_1+\beta_2=N$, $\beta_1\beta_2=1$. If $a_n=\dim_{\mathbb{F}_l}(\overline{L}/(\overline{y}))_n$, we have (by the Birkhoff–Witt Theorem)

$$\prod_{n\geq1}\frac{1}{(1-t^n)^{a_n}}=\frac{1}{1-Nt+t^2},$$

which yields by a standard calculation (see [Se1], LA 4.5–4.6),

$$a_n = \frac{1}{n}\sum_{d|n}\mu(n/d)(\beta_1^d + \beta_2^d).$$

It follows that

$$\dim(\widehat{L}/\widehat{\mathcal{N}}(R))_n = \sum_{k=1}^{n}\frac{1}{k}\sum_{d|k}\mu(k/d)(\beta_1^d + \beta_2^d)$$

and hence that

$$\dim(\overline{L}/\overline{\mathcal{N}})_n = \sum_{k=1}^{n}\frac{1}{k}\sum_{d|k}\mu(k/d)(\beta_1^d + \beta_2^d - (N-1)^d) + \frac{1}{n}\sum_{d|n}\mu(n/d)(N-1)^d \tag{20}$$

is independent of l. ■

THEOREM 5.2. *We have*

$$\ker\phi_K(p)_n \cong (\mathbb{Z}/p^\kappa\mathbb{Z})^{c_n}, \qquad \mathcal{L}_n(G_K(p)) \cong (\mathbb{Z}/p^\kappa\mathbb{Z})^{b_n} \oplus \mathbb{Z}_p^{d_n},$$

where

$$b_n = \sum_{k=1}^{n}\frac{1}{k}\sum_{d|k}\mu(k/d)(\beta_1^d + \beta_2^d - (N-1)^d),$$

$$c_n = \sum_{k=1}^{n-1}\frac{1}{k}\sum_{d|k}\mu(k/d)((N-1)^d - \beta_1^d - \beta_2^d) + \frac{1}{n}\sum_{d|n}\mu(n/d)(N^d - \beta_1^d - \beta_2^d),$$

$$d_n = \frac{1}{n}\sum_{d|n}\mu(n/d)(N-1)^d.$$

Proof. By Theorem 4.8 we have $\ker\phi_K(p)$ isomorphic to $\mathcal{N}\otimes\mathbb{Z}/p^\kappa\mathbb{Z}$, which gives the first isomorphism since the $\mathbb{Z}/p^\kappa\mathbb{Z}$-rank of $\mathcal{N}_n\otimes\mathbb{Z}/p^\kappa\mathbb{Z}$ is the dimension of $\overline{\mathcal{N}}_n$ over $\mathbb{F}_p$, which in turn equals c_n.

Again, by Theorem 4.8, the torsion submodule of $L_n(G_K(p))$ is isomorphic to $((x_1)/\mathcal{N})\otimes\mathbb{Z}/p^\kappa\mathbb{Z}$, and $L_n(G_K(p))$ modulo torsion is isomorphic to the free Lie algebra over $\mathbb{Z}_p$ on $N-1$ generators. This yields the second isomorphism since the $\mathbb{Z}/p^\kappa\mathbb{Z}$-rank of $((x_1)/\mathcal{N})\otimes\mathbb{Z}/p^\kappa\mathbb{Z}$ is the dimension of $(\overline{x}_1)/\overline{\mathcal{N}}$ over $\mathbb{F}_p$, which in turn is equal to b_n. ■

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