1. Introduction. For $|q| < 1$, set

$$(a; q)_\infty := \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$

If $n$ is any positive rational number and $q = \exp(-\pi\sqrt{n})$, Ramanujan's class invariants are defined by

\begin{align*}
G_n & := 2^{-1/4}q^{-1/24}(\ell; q^2)_\infty \\
g_n & := 2^{-1/4}q^{-1/24}(q; q^2)_\infty.
\end{align*}

If $n$ is a positive integer, then $G_n$ and $g_n$ can be roughly viewed as generators of the Hilbert class field of the complex quadratic field $K = \mathbb{Q}(\sqrt{-n})$, or more generally, generators of the ring class field of the order of $K$, that is, $\mathbb{Z}(\sqrt{-n})$, because of their relations with $j(\sqrt{-n})$. For complete accounts, the reader is referred to the important paper of B. Birch [6] and the excellent books of Cox [7] and Lang [8]. In the notation of H. Weber [15], $G_n = 2^{-1/4}f(\sqrt{-n})$ and $g_n = 2^{-1/4}f_1(\sqrt{-n})$ where $f, f_1$ are called Weber’s functions, and $f(\sqrt{-n}), f_1(\sqrt{-n})$ are also called Weber’s invariants. The term “invariant” is due to Weber.

In his monumental book [15], Weber calculated a total of 105 class invariants. Ramanujan [11, 12] calculated a total of 110 class invariants among which 49 are not found in Weber’s book. However, Ramanujan’s approach is still a mystery today. Using Kronecker’s limit formula and modular equa-
tions of degree 3, 5 and 7, we [4, 5] have proved 18 class invariants of Ramanujan which had not been proved in literature.

Watson [14] employed an “empirical process” to establish 16 class invariants: $g_n$ for $n = 66, 114, 126, 138, 154, 238, 310, 522, 630$ and $G_n$ for $n = 333, 465, 765, 777, 897, 1645, 1677$, which were first stated by Ramanujan [11]. In fact, Watson’s “empirical process” is not rigorous. Therefore, it is highly desirable to find rigorous proofs of these class invariants of Ramanujan and Watson. In [16], we have rigorously proved 6 class invariants, namely $g_n$ for $n = 66, 114, 138, 154, 238$ and $310$. Note that in all of these 6 cases, the class number $h_K$ is 8 and the genus number is 4. In [16], we have also pointed out that using Theorems 1 and 2 in [4] one can rigorously prove 5 class invariants of Ramanujan and Watson, namely $g_n$ for $n = 126, 522, 630$ and $G_n$ for $n = 333, 765$. Note that in all of the 5 cases, $n$ is a multiple of 9.

The aim of this paper is to provide rigorous proofs for the remaining 5 class invariants of Ramanujan–Watson, that is, $G_n$ for $n = 465, 777, 897, 1645$ and 1677. Note that in all of these 5 cases, the class number $h_K$ is 16 and the genus number is 8. Therefore, the proofs here are more complicated than those in [5] and [16]. We also point out that this paper is a continuation of our previous work, and the reader is referred to [5] and [16] for more details of the background.

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2. Kronecker’s limit formula and modular equations. Let $Q(u, v) := y^{-1}(u + vz)(u + v\bar{z})$, where $z = x + iy$, with $y > 0$, and $\bar{z}$ is the complex conjugate of $z$. The Epstein zeta-function $\zeta_Q(s)$ is defined for $\sigma = \text{Re } s > 1$ by

$$\zeta_Q(s) := \sum_{u, v}' Q(u, v)^{-s}, \quad (2.1)$$

where $\sum'$ means that the pair $(u, v) = (0, 0)$ is excluded from the summation. It is well known that $\zeta_Q(s)$ can be analytically continued to the entire complex plane with a simple pole at $s = 1$.

The celebrated Kronecker first limit formula can be stated as follows:

$$\lim_{s \to 1} \left( \frac{\zeta_Q(s) - \frac{\pi}{s-1}}{s-1} \right) = 2\pi(\gamma - \log 2 - \log(\sqrt{y|\eta(z)|^2})), \quad (2.2)$$

or equivalently,

$$\zeta_Q(s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{y|\eta(z)|^2})) + O(s-1),$$
where $\gamma$ denotes Euler’s constant and $\eta(z)$ is the Dedekind eta-function defined, for $\text{Im } z > 0$, by

\[(2.3)\quad \eta(z) = q^{1/12}(q^2; q^2)_\infty \quad \text{with } q = e^{\pi iz}.\]

Let $K = \mathbb{Q}(\sqrt{-n})$ be a complex quadratic field with a squarefree positive integer $n$, and let $C_K$ denote the ideal class group of $K$. Then the discriminant of $K$ is given by

\[d := d_K = \begin{cases} -4n & \text{if } -n \equiv 2, 3 \pmod{4}, \\ -n & \text{if } -n \equiv 1 \pmod{4}. \end{cases} \]

Set

\[\omega := \omega_K = \begin{cases} \sqrt{-n} & \text{if } -n \equiv 2, 3 \pmod{4}, \\ \frac{1 + \sqrt{-n}}{2} & \text{if } -n \equiv 1 \pmod{4}. \end{cases} \]

Let $A \in C_K$ and $b = [a, b + \omega]$ be any nonzero integral ideal $\in A^{-1}$. Set $z = x + iy = (b + \omega)/a$, and

\[(2.4)\quad F(A) = |\eta(z)|^2/\sqrt{a}. \]

Note that $F(A)$ depends only on $A$. Genus characters are a special class of characters on the ideal class group $C_K$. A genus character $\chi$ has only values of $\pm 1$ and is associated with a decomposition of $d = d_1d_2$, where $d_1 > 0$, $d$, $d_1$ and $d_2$ are discriminants of $K = \mathbb{Q}(\sqrt{d})$, $K_1 = \mathbb{Q}(\sqrt{d_1})$ and $K_2 = \mathbb{Q}(\sqrt{d_2})$, respectively. Then applying Kronecker’s limit formula, we have the following [13, p. 72]:

**Theorem 2.1.** For a nonprincipal genus character $\chi$,

\[\frac{\nu h_1 h_2 \log \varepsilon_1}{\nu_2} = - \sum_{A \in C_K} \chi(A) \log F(A), \]

or

\[(2.5)\quad \varepsilon_1^{\nu h_1 h_2 / \nu_2} = \prod_{A \in C_K} F(A)^{-\chi(A)}, \]

where $h_1$ and $h_2$ are the class numbers of $K_1$ and $K_2$, respectively, $\varepsilon_1$ is the fundamental unit of $K_1$, and $\nu, \nu_1$ and $\nu_2$ are the numbers of roots of unity in $K, K_1$ and $K_2$, respectively.

We emphasize that (2.5) is the major ingredient in our proofs. For complete accounts, the reader is referred to the great book of Siegel [13].

We also need modular equations of degrees 3, 5, 7, 13 and 21. For brevity, we state them in terms of class invariants. The reader is referred to Berndt’s books [1–3] for details about modular equations.
Theorem 2.2 (modular equation of degree 3; [1, p. 231]).

(2.6) \( \left( \frac{G_n}{G_{n/9}} \right)^6 + \left( \frac{G_{n/9}}{G_n} \right)^6 + 2\sqrt{2} \left( \frac{1}{(G_n G_{n/9})^3} - (G_n G_{n/25})^2 \right) = 0. \)

Theorem 2.3 (modular equation of degree 5; [1, p. 282]).

(2.7) \( \left( \frac{G_n}{G_{n/25}} \right)^3 + \left( \frac{G_{n/25}}{G_n} \right)^3 + 2 \left( \frac{1}{(G_n G_{n/25})^2} - (G_n G_{n/25})^2 \right) = 0. \)

Theorem 2.4 (modular equation of degree 7; [1, p. 315]).

(2.8) \( \left( \frac{G_n}{G_{n/49}} \right)^4 + \left( \frac{G_{n/49}}{G_n} \right)^4 + 7 - 2\sqrt{2} \left( \frac{1}{(G_n G_{n/49})^3} + (G_n G_{n/49})^3 \right) = 0. \)

Theorem 2.5 (modular equation of degree 13; [15, p. 315]). Let

\( A = \frac{G_{n/169}}{G_n} + \frac{G_n}{G_{n/169}} \) and \( B = 8((G_n G_{n/169})^6 - (G_n G_{n/169})^{-6}). \)

Then

(2.9) \( A(A^6 + 6A^4 + A^2 - 20) = B. \)

Theorem 2.6 (modular equation of degree 21; [3, Entry 36, Chapter 36]. Set

\( P = (G_n G_{n/9} G_{n/49} G_{n/441})^3 \) and \( Q = \frac{G_{n/49} G_{n/441}}{G_n G_{n/49} G_{n/441}}. \)

Let \( X = Q + Q^{-1}. \) Then

(2.10) \( X^4 + 7X^3 + 10X^2 = 8(P + P^{-1} - 2). \)

3. Four class invariants of Ramanujan. In this section, we provide rigorous proofs for four class invariants of Ramanujan, namely \( G_n \) for \( n = 465, 1645, 897 \) and 1677. For all of these four cases, as mentioned earlier, the class number \( h_K \) is 16, the genus number is 8 and each genus of \( K \) contains two ideal classes. In the case where each genus of \( K \) contains only one ideal class, \( G_n \) and \( g_n \) are much simpler. They can be found mainly by making use of (2.5). This idea was first developed by Siegel [13] and was utilized by K. Ramachandra [9] and K. G. Ramanathan [10].

Let \( \tau = \sqrt{-n}. \) Then, by (1.1) and (2.3), it is easily seen that

(3.1) \( \left| \eta(\frac{\tau + 1}{\tau}) \right| = \eta(\tau). \)

By a slight abuse of notation, in what follows we use a representative ideal \( a \) to denote the ideal class \( A \) which contains \( a. \) If \( b = [a, b + \omega] = \bar{b} = [a, b + \bar{\omega}], \) then \( b \) is called ambiguous. It is obvious that if an ambiguous ideal \( b \) is in \( A, \) then \( A = A^{-1}. \)
Set $K = \mathbb{Q}(\sqrt{-n})$, where $n$ is a squarefree positive integer divisible by a prime $p$ and $-n \equiv 3 \pmod{4}$. Then $\omega = \sqrt{-n}$, and $[1, \omega], [2, 1 + \omega], [p, \omega]$ and $[2p, p + \omega]$ are ambiguous primitive ideals in $K$. We proved [4] the following theorem.

**Theorem 3.1.** Let $K = \mathbb{Q}(\sqrt{-n})$, where $n$ is a squarefree positive integer with $-n \equiv 3 \pmod{4}$ and is divisible by a prime $p$. Assume that each genus in $K$ contains two ideal classes and $[2, 1 + \omega]$ is not in the principal genus.

(i) If $[p, \omega]$ is in the principal genus, then

$$G_n G_{n/p^2} = \prod_{\chi(2) = -1} \varepsilon_1^{4h_1 h_2/(\nu_2 h)}.$$  

where $\chi$, associated with the decomposition $d = d_1 d_2$, runs through all genus characters with $\chi([2, 1 + \omega]) = \chi(2) = -1$, $h, h_1$ and $h_2$ are the class numbers of $K = \mathbb{Q}(\sqrt{-n}) = \mathbb{Q}(\sqrt{d})$, $K_1 = \mathbb{Q}(\sqrt{d_1})$ and $K_2 = \mathbb{Q}(\sqrt{d_2})$, respectively, $\varepsilon_1$ is the fundamental unit in $K_1$, and $\nu_2$ is the number of roots of unity in $K_2$.

(ii) If $[2p, p + \omega]$ is in the principal genus, then

$$G_n G_{n/p^2} = \prod_{\chi(2) = -1} \varepsilon_1^{4h_1 h_2/(\nu_2 h)}.$$  

In order to make our proofs of these four class invariants of Ramanujan–Watson simpler, we state the following elementary lemmas which are easily verified. So we omit the proofs.

**Lemma 3.1.** Let $a, b$ be positive numbers greater than $1/2$. If $x > 1$ and

$$\frac{1}{2}(x^4 + x^{-4}) = (8a^2 - 1)(8b^2 - 1) + 16ab\sqrt{(4a^2 - 1)(4b^2 - 1)},$$

then

$$x = \left(\sqrt{a + \frac{1}{2}} + \sqrt{a - \frac{1}{2}}\right) \left(\sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}}\right).$$

**Lemma 3.2.** Let $a, b$ be positive numbers greater than $1/2$. If $x > 1$ and

$$x^3 + x^{-3} = (4a - 1)(4b - 1)\sqrt{(2a + 1)(2b + 1)} + (4a + 1)(4b + 1)\sqrt{(2a - 1)(2b - 1)},$$

then

$$x = \left(\sqrt{a + \frac{1}{2}} + \sqrt{a - \frac{1}{2}}\right) \left(\sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}}\right).$$

**Lemma 3.3.** For $B > 0$, the equation $Z(Z^6 + 6Z^4 + Z^2 - 20) = B$ has exactly one root $Z$ with $Z > 1$.

Now we are ready to prove the four class invariants rigorously.
Theorem 3.2.
\[ G_{465} = \frac{1 + \sqrt{5}}{2} \left( 2 + \sqrt{3} \right)^{1/4} (5\sqrt{5} + 2\sqrt{31})^{1/12} \left( \frac{3\sqrt{3} + \sqrt{31}}{2} \right)^{1/4} \]
\[ \times \left( \frac{2 + \sqrt{31}}{4} + \frac{6 + \sqrt{31}}{4} \right)^{1/2} \]
\[ \times \left( \frac{11 + 2\sqrt{31}}{2} + \frac{13 + 2\sqrt{31}}{2} \right)^{1/2}. \]

Proof. We list all information needed in order to apply Theorem 3.1.

1) Set \( K = \mathbb{Q}(\sqrt{-465}) \). Then \( \omega = \sqrt{-465}, d = -1860, h = 16, \) and each genus of \( K \) contains two classes. The principal genus consists of \( A_0 = [1, \omega] \) and \( A'_0 = [10, 5 + \omega] \) while \( A_1 = [2, 1 + \omega] \) and \( A'_1 = [5, \omega] \) form another genus.

2) There are four genus characters \( \chi \) with \( \chi(2) = -1 \), denoted by \( \chi_1, \chi_2, \chi_3 \) and \( \chi_4 \).

(i) For \( \chi_1 \) associated with the decomposition \(-1860 = 5(-372), h_1 = 1, \)
\( \varepsilon_1 = (1 + \sqrt{5})/2 \) and \( h_2 = 4, \nu_2 = 2. \)

(ii) For \( \chi_2 \) associated with the decomposition \(-1860 = 12(-155), h_1 = 1, \)
\( \varepsilon_1 = 2 + \sqrt{3} \) and \( h_2 = 4, \nu_2 = 2. \)

(iii) For \( \chi_3 \) associated with the decomposition \(-1860 = 93(-20), h_1 = 1, \)
\( \varepsilon_1 = (29 + 3\sqrt{93})/2 \) and \( h_2 = 2, \nu_2 = 2. \)

(iv) For \( \chi_4 \) associated with the decomposition \(-1860 = 620(-3), h_1 = 2, \)
\( \varepsilon_1 = 249 + 20\sqrt{155} \) and \( h_2 = 1, \nu_2 = 6. \)

Applying (3.3) with \( p = 5 \), we find that

\[ G_{465} \]
\[ \frac{G_{465}}{G_{93/5}} \]
\[ = \left( \frac{1 + \sqrt{5}}{2} \right)^{1/2} (2 + \sqrt{3})^{1/2} (249 + 20\sqrt{155})^{1/12} \left( \frac{29 + 3\sqrt{93}}{2} \right)^{1/4}. \]

It follows that

\[ Q = \left( \frac{G_{465}}{G_{93/5}} \right)^6 = (2 + \sqrt{5})(26 + 15\sqrt{3})(5\sqrt{5} + 2\sqrt{31})(45\sqrt{3} + 14\sqrt{31}) \]

and

\[ Q^{-1} = \left( \frac{G_{93/5}}{G_{465}} \right)^6 \]
\[ = (-2 + \sqrt{5})(26 - 15\sqrt{3})(5\sqrt{5} - 2\sqrt{31})(-45\sqrt{3} + 14\sqrt{31}). \]
By (2.7) with \( n = 465 \) and simple algebra, one can see that

\[
(3.7) \quad (G_{465}G_{93/5})^4 + (G_{465}G_{93/5})^{-4} = \frac{1}{4}(Q + Q^{-1} + 10).
\]

Set \( X = G_{465}G_{93/5} \), by (3.5), (3.6) and (3.7), we find that

\[
(3.8) \quad \frac{X^4 + X^{-4}}{2} = \frac{1}{2}(47883 + 12360\sqrt{15} + 8600\sqrt{31} + 2220\sqrt{465}).
\]

Set \( A = (4 + \sqrt{31})/4 \) and \( B = 6 + \sqrt{31} \). It is elementary to see that

\[
(3.9) \quad (8A^2 - 1)(8B^2 - 1) + 16AB\sqrt{(4A^2 - 1)(4B^2 - 1)} = \frac{1}{2}(47883 + 12360\sqrt{15} + 8600\sqrt{31} + 2220\sqrt{465}).
\]

It is obvious that \( X = G_{465}G_{93/5} > 1 \). By Lemma 3.1, we find that

\[
(3.10) \quad G_{465}G_{93/5} = \left(\sqrt{\frac{2 + \sqrt{31}}{4}} + \sqrt{\frac{6 + \sqrt{31}}{4}}\right) \times \left(\sqrt{\frac{11 + 2\sqrt{31}}{2}} + \sqrt{\frac{13 + 2\sqrt{31}}{2}}\right).
\]

Therefore, by (3.4) and (3.10), we complete the proof.

**Theorem 3.3.**

\[
G_{1645} = (2 + \sqrt{5})^{1/2}(3 + \sqrt{7})^{1/4} \left(\frac{7 + \sqrt{47}}{2}\right)^{1/4} \left(\frac{73\sqrt{5} + 9\sqrt{329}}{2}\right)^{1/8}
\]

\[
\times \left(\sqrt{\frac{119 + 7\sqrt{329}}{8}} + \sqrt{\frac{127 + 7\sqrt{329}}{8}}\right)^{1/2}
\]

\[
\times \left(\sqrt{\frac{743 + 41\sqrt{329}}{8}} + \sqrt{\frac{751 + 41\sqrt{329}}{8}}\right)^{1/2}.
\]

**Proof.** We record all information needed in order to apply Theorem 3.1.

1) Set \( K = \mathbb{Q}(\sqrt{-1645}) \). Then \( \omega = \sqrt{-465} \), \( d = -6580 \), \( h = 16 \), and each genus of \( K \) contains two classes. The principal genus consists of \( A_0 = [1, \omega] \) and \( A'_0 = [14, 7 + \omega] \) while \( A_1 = [2, 1 + \omega] \) and \( A'_1 = [7, \omega] \) form another genus.

2) There are four genus characters \( \chi \) with \( \chi(2) = -1 \), denoted by \( \chi_1, \chi_2, \chi_3 \) and \( \chi_4 \).

(i) For \( \chi_1 \) associated with the decomposition \(-6580 = 5(-1316)\), \( h_1 = 1, \varepsilon_1 = (1 + \sqrt{5})/2 \) and \( h_2 = 4, \nu_2 = 2 \).

(ii) For \( \chi_2 \) associated with the decomposition \(-6580 = 28(-235)\), \( h_1 = 1, \varepsilon_1 = 8 + 3\sqrt{7} \) and \( h_2 = 2, \nu_2 = 2 \).

(iii) For \( \chi_3 \) associated with the decomposition \(-6580 = 188(-35)\), \( h_1 = 1, \varepsilon_1 = 48 + 7\sqrt{47} \) and \( h_2 = 2, \nu_2 = 2 \).
(iv) For \( \chi_4 \) associated with the decomposition \(-6580 = 1645(-4)\), \( h_1 = 2, \) \( \varepsilon_1 = (26647 + 657\sqrt{1645})/2 \) and \( h_2 = 1, \) \( \nu_2 = 4. \)

Applying (3.3) with \( p = 7, \) we find that

\[
\frac{G_{1645}}{G_{235/7}} = \left( \frac{1 + \sqrt{5}}{2} \right)^3 (8 + 3\sqrt{7})^{1/4} (48 + 7\sqrt{47})^{1/4} \left( \frac{26647 + 657\sqrt{1645}}{2} \right)^{1/8}.
\]

It follows that

\[
(3.11) \quad \frac{G_{1645}}{G_{235/7}} = (2 + \sqrt{5})(3 + \sqrt{7})^{1/2} \left( \frac{7 + \sqrt{47}}{2} \right)^{1/12} \left( \frac{73\sqrt{5} + 9\sqrt{329}}{2} \right)^{1/4},
\]

\[
(3.12) \quad Q := \left( \frac{G_{1645}}{G_{235/7}} \right)^4 = (161 + 72\sqrt{5})(8 + 3\sqrt{7})(48 + 7\sqrt{47})\left( \frac{73\sqrt{5} + 9\sqrt{329}}{2} \right)
\]

and

\[
(3.13) \quad Q^{-1} = \left( \frac{G_{235/7}}{G_{1645}} \right)^4 = (161 - 72\sqrt{5})(8 - 3\sqrt{7})(48 - 7\sqrt{47})\left( \frac{-73\sqrt{5} + 9\sqrt{329}}{2} \right).
\]

Set \( X = G_{1645}G_{235/7}. \) By (2.8) with \( n = 1645 \) and simple algebra, one can see that

\[
(3.14) \quad X^3 + X^{-3} = \frac{1}{2\sqrt{2}}(Q + Q^{-1} + 7)
\]

\[
= 5025667\sqrt{2} + 849492\sqrt{70}
\]

\[
+ 327838\sqrt{470} + 277074\sqrt{658}.
\]

Next we apply Lemma 3.2. Set \( A = (123 + 7\sqrt{329})/8 \) and \( B = 747 + 41\sqrt{329}. \) It is elementary to see that

\[
\sqrt{(2A + 1)(2B + 1)} = \frac{1}{2} \sqrt{47450 + 2616\sqrt{329}} = \frac{1}{2} (109\sqrt{2} + 6\sqrt{658}),
\]

\[
(4A - 1)(4B - 1)\sqrt{(2A + 1)(2B + 1)} = 5025667 + 277074\sqrt{658}.
\]

It is also elementary to show that

\[
(4A + 1)(4B + 1)\sqrt{(2A - 1)(2B - 1)} = 849492\sqrt{70} + 327838\sqrt{470}.
\]

Therefore, we have

\[
X^3 + X^{-3} = (4A - 1)(4B - 1)\sqrt{(2A + 1)(2B + 1)}
\]

\[
\]
By Lemma 3.2, we find that
\[(3.15) \quad X = G_{1645G_{235/7}} \]
\[= \left( \frac{\sqrt{119 + 7\sqrt{329}}}{8} + \frac{\sqrt{127 + 7\sqrt{329}}}{8} \right) \]
\[\times \left( \frac{\sqrt{743 + 41\sqrt{329}}}{8} + \frac{\sqrt{751 + 41\sqrt{329}}}{8} \right).\]

Therefore, by (3.11) and (3.15), we have proved the theorem.

Theorem 3.4.
\[G_{897} = (2 + \sqrt{3})^{1/2} \left( \frac{3 + \sqrt{13}}{2} \right)^{1/2} \left( \frac{3\sqrt{3} + \sqrt{23}}{2} \right)^{1/4} (4\sqrt{13} + 3\sqrt{23})^{1/12} \]
\[\times \left( \frac{60 + 9\sqrt{39}}{4} + \sqrt{56 + 9\sqrt{39}} \right)^{1/2} \]
\[\times \left( \frac{8 + \sqrt{39}}{4} + \sqrt{4 + \sqrt{39}} \right)^{1/2}.\]

Proof. We record all information needed in order to apply Theorem 3.1.

1) Set \(K = \mathbb{Q}(-897).\) Then \(\omega = \sqrt{-897}, d = -3588, h = 16,\) and each genus of \(K\) contains two classes. The principal genus consists of \(A_0 = [1, \omega]\) and \(A'_0 = [13, \omega]\) while \(A_1 = [2, 1 + \omega]\) and \(A'_1 = [26, 13 + \omega]\) form another genus.

2) There are four genus characters \(\chi\) with \(\chi(2) = -1,\) denoted by \(\chi_1, \chi_2, \chi_3\) and \(\chi_4.\)
   (i) For \(\chi_1\) associated with the decomposition \(-3588 = 13(-276), h_1 = 1, \varepsilon_1 = (3 + \sqrt{13})/2\) and \(h_2 = 4, \nu_2 = 2.\)
   (ii) For \(\chi_2\) associated with the decomposition \(-3588 = 69(-52), h_1 = 1, \varepsilon_1 = (25 + 3\sqrt{69})/2\) and \(h_2 = 2, \nu_2 = 2.\)
   (iii) For \(\chi_3\) associated with the decomposition \(-3588 = 12(-299), h_1 = 1, \varepsilon_1 = 2 + \sqrt{3}\) and \(h_2 = 8, \nu_2 = 2.\)
   (iv) For \(\chi_4\) associated with the decomposition \(-3588 = 1196(-3), h_1 = 2, \varepsilon_1 = 415 + 24\sqrt{299}\) and \(h_2 = 1, \nu_2 = 6.\)

Applying (3.2) with \(p = 13,\) we find that
\[(3.16) \quad G_{897G_{69/13}} \]
\[= \left( \frac{3 + \sqrt{13}}{2} \right) \left( \frac{25 + 3\sqrt{69}}{2} \right)^{1/4} (2 + \sqrt{3}) (415 + 24\sqrt{299})^{1/12} \]
\[= \left( \frac{3 + \sqrt{13}}{2} \right) \left( \frac{3\sqrt{3} + \sqrt{23}}{2} \right)^{1/2} (2 + \sqrt{3}) (4\sqrt{13} + 3\sqrt{23})^{1/6}.\]
It follows that
\begin{align*}
P &:= (G_{897}G_{69/13})^6 \\
&= (649 + 180\sqrt{13})(36\sqrt{3} + 13\sqrt{23}) \\
&\quad \times (1351 + 780\sqrt{3})(4\sqrt{13} + 3\sqrt{23}),
\end{align*}
(3.17)
\begin{align*}
P^{-1} &:= (649 - 180\sqrt{13})(36\sqrt{3} - 13\sqrt{23}) \\
&\quad \times (1351 - 780\sqrt{3})(4\sqrt{13} - 3\sqrt{23}).
\end{align*}
(3.18)

Therefore, we find that
\begin{align*}
B &:= 8(P - P^{-1}) \\
&= 64(227328075\sqrt{3} + 109204875\sqrt{13} + 47401173\sqrt{69} + 22770787\sqrt{299}).
\end{align*}
(3.19)

Set
\[ A = \frac{11}{2}\sqrt{3} + 3\sqrt{13} + \sqrt{69} + \frac{1}{2}\sqrt{299}. \]

Then, by elementary algebra, we find that
\[ A(A^6 + 6A^4 + A^2 - 20) = B. \]

From Theorem 2.5 and Lemma 3.3, we find that
\[ A = \frac{G_{897}}{G_{69/13}} + \frac{G_{69/13}}{G_{897}} = \frac{11}{2}\sqrt{3} + 3\sqrt{13} + \sqrt{69} + \frac{1}{2}\sqrt{299} \]
and
\begin{align*}
\frac{G_{897}}{G_{69/13}} &= \left(\sqrt{\frac{60 + 9\sqrt{39}}{4}} + \sqrt{\frac{56 + 9\sqrt{39}}{4}}\right) \\
&\quad \times \left(\sqrt{\frac{8 + \sqrt{39}}{4}} + \sqrt{\frac{4 + \sqrt{39}}{4}}\right).
\end{align*}
(3.20)

Now, the theorem follows from (3.16) and (3.20).

**Theorem 3.5.**
\[ G_{1677} = (4414\sqrt{13} + 2427\sqrt{43})^{1/12}\left(\frac{3 + \sqrt{13}}{2}\right)^{3/4} \]
\[ \times (\sqrt{13} + 2\sqrt{3})^{1/4}\left(\frac{\sqrt{43} + \sqrt{39}}{2}\right)^{1/4} \]
\[ \times \left(\sqrt{\frac{355 + 54\sqrt{43}}{4}} + \sqrt{\frac{351 + 54\sqrt{43}}{4}}\right)^{1/2} \]
\[ \times \left(\sqrt{\frac{17 + 2\sqrt{43}}{4}} + \sqrt{\frac{13 + 2\sqrt{43}}{4}}\right)^{1/2}. \]

**Proof.** We list all information needed in order to apply Theorem 3.1.
1) Set $K = \mathbb{Q}(\sqrt{-1677})$. Then $\omega = \sqrt{-1677}$, $d = -6708$, $h = 16$, and each genus of $K$ contains two classes. The principal genus consists of $A_0 = [1, \omega]$ and $A'_0 = [13, \omega]$ while $A_1 = [2, 1+\omega]$ and $A'_1 = [26, 13+\omega]$ form another genus.

2) There are four genus characters $\chi$ with $\chi(2) = -1$, denoted by $\chi_1, \chi_2, \chi_3$ and $\chi_4$.

   (i) For $\chi_1$ associated with the decomposition $-6708 = 2236(-3)$, $h_1 = 2$, $\varepsilon_1 = 506568295 + 21425556\sqrt{559}$ and $h_2 = 1$, $\nu_2 = 6$.

   (ii) For $\chi_2$ associated with the decomposition $-6708 = 13(-516)$, $h_1 = 1$, $\varepsilon_1 = (3 + \sqrt{13})/2$ and $h_2 = 12$, $\nu_2 = 2$.

   (iii) For $\chi_3$ associated with the decomposition $-6708 = 156(-43)$, $h_1 = 2$, $\varepsilon_1 = 25 + 4\sqrt{39}$ and $h_2 = 1$, $\nu_2 = 2$.

   (iv) For $\chi_4$ associated with the decomposition $-6708 = 1677(-4)$, $h_1 = 4$, $\varepsilon_1 = (41 + \sqrt{1677})/2$ and $h_2 = 1$, $\nu_2 = 4$.

Applying (3.2) with $p = 13$, we find that

\[ G_{1677}G_{129/13} = (506568295 + 21425556\sqrt{559})^{1/12} \times \left( \frac{3 + \sqrt{13}}{2} \right)^{3/2} (25 + 4\sqrt{39})^{1/4} (41 + \sqrt{1677})^{1/4} \]

or

\[ G_{1677}G_{129/13} = (4414\sqrt{13} + 2427\sqrt{43})^{1/6} \left( \frac{3 + \sqrt{13}}{2} \right)^{3/2} \times \left( \sqrt{13} + 2\sqrt{3} \right)^{1/2} \left( \frac{\sqrt{43} + \sqrt{39}}{2} \right)^{1/2}. \]

It follows that

\[ P := (G_{1677}G_{129/13})^6 \]

\[ = (4414\sqrt{13} + 2427\sqrt{43})(23382 + 6485\sqrt{13}) \times (102\sqrt{3} + 49\sqrt{13})(20\sqrt{43} + 21\sqrt{39}), \]

\[ P^{-1} = (4414\sqrt{13} - 2427\sqrt{43})(-23382 + 6485\sqrt{13}) \times (-102\sqrt{3} + 49\sqrt{13})(20\sqrt{43} - 21\sqrt{39}). \]

Therefore, we find that

\[ B := 8(P - P^{-1}) \]

\[ = 32(8621996645262 + 4977912082935\sqrt{3} + 1314842161815\sqrt{43} + 759124475889\sqrt{129}). \]

Set

\[ A = 37 + \frac{39}{2} \sqrt{3} + \frac{11}{2} \sqrt{43} + 3\sqrt{129}. \]
Then, by an elementary algebra, we find that
\[ A(A^6 + 6A^4 + A^2 - 20) = B. \]
From Theorem 2.5 and Lemma 3.3, we find that
\[ A = \frac{G_{1677}}{G_{129/13}} + \frac{G_{129/13}}{G_{1677}} = 37 + \frac{39}{2}\sqrt{3} + \frac{11}{2}\sqrt{43} + 3\sqrt{129} \]
and
\[
(3.25) \quad \frac{G_{1677}}{G_{129/13}} = \left( \sqrt{\frac{355 + 54\sqrt{43}}{4}} + \sqrt{\frac{351 + 54\sqrt{43}}{4}} \right) \times \left( \sqrt{\frac{17 + 2\sqrt{43}}{4}} + \sqrt{\frac{13 + 2\sqrt{43}}{4}} \right). \]
Now, the theorem follows from (3.21) and (3.25).

4. Class invariant \(G_{777}\). In this section, we shall give a rigorous proof of \(G_{777}\). As we will see, the proof is different from the proofs of the other four invariants and does not use Theorem 3.1.

**Theorem 4.1.**
\[
G_{777} = (2 + \sqrt{3})^{1/4}(6 + \sqrt{37})^{1/4}\left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^{1/4} (246\sqrt{7} + 107\sqrt{37})^{1/12} \\
\times \left( \sqrt{\frac{6 + 3\sqrt{7}}{4}} + \sqrt{\frac{10 + 3\sqrt{7}}{4}} \right)^{1/2} \\
\times \left( \sqrt{\frac{15 + 6\sqrt{7}}{2}} + \sqrt{\frac{17 + 6\sqrt{7}}{2}} \right)^{1/2}. 
\]

**Proof.** We list some information needed.

I) Set \(K = \mathbb{Q}(\sqrt{-777})\). Then \(\omega = \sqrt{-777}, \ d = -3108, \ h = 16, \) and each genus of \(K\) contains two classes. The genus structure and class group \(C_K\) are as follows:
1) The principal genus, \(G_0\) consists of \(A_0 = [1, \omega]\) and \(A'_0 = [21, \omega]\).
2) The genus \(G_1\) consists of \(A_1 = [2, 1 + \omega]\) and \(A'_1 = [42, 21 + \omega]\).
3) The genus \(G_2\) consists of \(A_2 = [3, \omega]\) and \(A'_2 = [7, \omega]\).
4) The genus \(G_3\) consists of \(A_3 = [6, 3 + \omega]\) and \(A'_3 = [14, 7 + \omega]\).
5) The genus \(G_4\) consists of \(A_4 = [11, 2 + \omega]\) and \(A'_4 = [11, -2 + \omega]\).
6) The genus \(G_5\) consists of \(A_5 = [13, 4 + \omega]\) and \(A'_5 = [13, -4 + \omega]\).
7) The genus \(G_6\) consists of \(A_6 = [22, 9 + \omega]\) and \(A'_6 = [22, -9 + \omega]\).
8) The genus \(G_7\) consists of \(A_7 = [26, 9 + \omega]\) and \(A'_7 = [26, -9 + \omega]\).

II) There are two genus characters \(\chi\) that we need here. We denote them by \(\chi_1, \chi_2\).
(i) For $\chi_1$ associated with the decomposition $-3108 = 1036(-3)$, $h_1 = 2$, $\varepsilon_1 = 847225 + 52644\sqrt{259}$ and $h_2 = 1$, $\nu_2 = 6$. It is evident that $\chi_1(A_j) = 1$ for $j = 0, 2, 5, 6$ and $\chi_1(A_j) = -1$ for $j = 1, 3, 4, 7$.

(ii) For $\chi_2$ associated with the decomposition $-3108 = 37(-84)$, $h_1 = 1$, $\varepsilon_1 = 6 + \sqrt{37}$ and $h_2 = 4$, $\nu_2 = 2$. It is also clear that $\chi_2(A_j) = 1$ for $j = 0, 2, 4, 7$ and $\chi_2(A_j) = -1$ for $j = 1, 3, 5, 6$.

It follows from Theorem 2.1 that

$$
\prod_{\chi = \chi_1, \chi_2} \prod_{A \in \mathcal{C}_K} F(A)^{-\chi(A)} = \prod_{\chi = \chi_1, \chi_2} \varepsilon_1^{h_1 h_2 / \nu_2}.
$$

Therefore, we have

$$
\left( \frac{F(A_1)F(A_1')F(A_3)F(A_3')}{F(A_0)F(A_0')F(A_2)F(A_2')} \right)^2 = (847225 + 52644\sqrt{259})^{2/3}(6 + \sqrt{37})^4.
$$

By (2.4) and (3.1), we find that

$$
(G_{777}G_{37/21}G_{259/3}G_{111/7})^4 = (847225 + 52644\sqrt{259})^{2/3}(6 + \sqrt{37})^4,
$$

and

(4.1) $P := G_{777}G_{37/21}G_{259/3}G_{111/7} = (246\sqrt{7} + 107\sqrt{37})^{1/3}(6 + \sqrt{37})$.

Employing Theorem 2.6 with $n = 777$ and by a laborious calculation, one can see that

(4.2) $Q := \frac{G_{777}G_{259/3}}{G_{111/7}G_{37/21}} = \left( \sqrt{\frac{15 + 6\sqrt{7}}{2}} + \sqrt{\frac{17 + 6\sqrt{7}}{2}} \right)^2$.

By (4.1) and (4.2), we find that

(4.3) $G_{777}G_{259/3} = (246\sqrt{7} + 107\sqrt{37})^{1/6}(6 + \sqrt{37})^{1/2}$

$$
\times \left( \sqrt{\frac{15 + 6\sqrt{7}}{2}} + \sqrt{\frac{17 + 6\sqrt{7}}{2}} \right).
$$

Making use of a modular equation of degree 3 (Theorem 2.2) with $n = 777$ and again by a lengthy calculation, we find that

(4.4) $\frac{G_{777}}{G_{259/3}} = (2 + \sqrt{3})^{1/2}\left( \frac{\sqrt{7} + \sqrt{3}}{2} \right)^{1/2}\left( \sqrt{\frac{6 + 3\sqrt{7}}{4}} + \sqrt{\frac{10 + 3\sqrt{7}}{4}} \right)$.

Hence, the theorem follows from (4.3) and (4.4).

References


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