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Gauss sums for orthogonal groups over a finite field of characteristic two

by

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1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$. Assume that $q = 2^d$ is a power of two. Then the exponential sum

$$\sum_{w \in G} \lambda(\operatorname{tr} w) \tag{1.1}$$

is considered for each of the groups G, where G is one of the orthogonal or special orthogonal groups $O^+(2n, q), SO^+(2n, q), O^-(2n, q), SO^-(2n, q)$ and $O(2n + 1, q)$.

The purpose of this paper is to find an explicit expression of the sum (1.1), for each of G listed above. It turns out that they can be expressed as polynomials in q with coefficients involving ordinary Kloosterman sums and Gauss sums. In fact, except for the case $O(2n+1, q)$ the expressions for (1.1) are identical to the corresponding ones for q odd (i.e., a power of an odd prime). On the other hand, the expression for $O(2n + 1, q)$ is identical to the one for $SO(2n + 1, q)$ with q odd and differs by a constant from the corresponding one for q odd.

Here it should be stressed that, although our final expressions are (almost) identical to the corresponding ones for q odd, there are many differences between the two cases in many respects.

Similar sums for other classical groups over a finite field have been considered and the results for these sums will appear in various places ([3]–[9]).

We now state some of the main results of this paper. Here again q is a power of two. For some notations, one is referred to the next section.

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Theorem A. *The sum* $\sum_{w\in O(2n+1,q)}\lambda(\operatorname{tr} w)$ *equals*

$$\lambda(1)\sum_{w\in Sp(2n,q)}\lambda(\operatorname{tr} w),$$

so that it is $\lambda(1)$ *times*

$$q^{n^2-1}\sum_{r=0}^{[n/2]}q^{r(r+1)}\begin{bmatrix} n\\ 2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+2)/2]}q^lK(\lambda;1,1)^{n-2r+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where $Sp(2n,q)$ *is the symplectic group over* $\mathbb{F}_q$, $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.21) *and the innermost sum is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\ldots\le j_1\le n-2r+1$.

Theorem B. *The sum* $\sum_{w\in O^+(2n,q)}\lambda(\operatorname{tr} w)$ *is given by*

$$q^{n^2-n-1}\bigg\{\sum_{r=0}^{[n/2]}q^{r(r+1)}\begin{bmatrix} n\\ 2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+2)/2]}q^lK(\lambda;1,1)^{n-2r+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\\ +\sum_{r=0}^{[(n-1)/2]}q^{r(r+1)}\begin{bmatrix} n\\ 2r+1\end{bmatrix}_q\prod_{j=1}^{r+1}(q^{2j-1}-1)\\ \times\sum_{l=1}^{[(n-2r+1)/2]}q^lK(\lambda;1,1)^{n-2r+1-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\bigg\},$$

where the first and second unspecified sums are respectively over all integers $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\ldots\le j_1\le n-2r+1$ *and over the same set of integers satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\ldots\le j_1\le n-2r$.

Theorem C. *The sum* $\sum_{w\in O^-(2n,q)}\lambda(\operatorname{tr} w)$ *is given by*

$$q^{n^2-n-1}\bigg(-\frac{1}{q-1}\sum_{j=1}^{q-1}G(\psi^j,\lambda)^2+q+1\bigg)$$

$$\times \left\{ \sum_{r=0}^{[(n-1)/2]} q^{r(r+3)} \begin{bmatrix} n-1 \\ 2r \end{bmatrix}_q \prod_{j=1}^{r} (q^{2j-1}-1) \right.$$

$$\times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda;1,1)^{n-2r+1-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1)$$

$$- \sum_{r=0}^{[(n-2)/2]} q^{r(r+3)+1} \begin{bmatrix} n-1 \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1} (q^{2j-1}-1)$$

$$\left. \times \sum_{l=1}^{[(n-2r)/2]} q^l K(\lambda;1,1)^{n-2r-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1) \right\},$$

where $G(\psi^j,\lambda)$ *is the usual Gauss sum as in* (2.20) *with* ψ *a multiplicative character of* $\mathbb{F}_q$ *of order* $q-1$, *the first unspecified sum is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r$ *and the second one is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r-1$.

The above Theorems A, B, and C are respectively stated as Theorem 6.1, Theorem 6.3, and Theorem 5.2.

2. Preliminaries. Unless otherwise stated, $\mathbb{F}_q$ will denote the finite field with $q=2^d$ elements. Whenever it is necessary to consider the case $q=p^d$ with p an odd prime, we will say that q is odd. As an excellent background reference for matrix groups over finite fields, one may refer to [11].

Let λ be an additive character of $\mathbb{F}_q$. Then $\lambda=\lambda_a$ for a unique $a\in\mathbb{F}_q$, where, for $\alpha\in\mathbb{F}_q$,

$$\lambda_a(\alpha)=\exp\{\pi i(a\alpha+(a\alpha)^2+\ldots+(a\alpha)^{2^{d-1}})\}.$$

It is nontrivial if $a\ne 0$.

$\operatorname{tr} A$ denotes the trace of A for a square matrix A and ${}^tB$ indicates the transpose of B for any matrix B.

An $n\times n$ matrix $A=(a_{ij})$ over $\mathbb{F}_q$ is called *alternating* if

$$\begin{cases} a_{ii}=0 & \text{for } 1\le i\le n, \\ a_{ij}=-a_{ji}=a_{ji} & \text{for } 1\le i<j\le n. \end{cases} \tag{2.1}$$

In the following discussion, we note that, up to equivalence, $(\mathbb{F}_q^{2n\times 1},\theta^{\pm})$ are all nondegenerate quadratic spaces of dimension $2n$ and $(\mathbb{F}_q^{(2n+1)\times 1},\theta)$ is the only nondegenerate quadratic space of dimension $2n+1$.

Let θ^+ be the nondegenerate quadratic form on the vector space $\mathbb{F}_q^{2n\times 1}$

of all $2n \times 1$ column vectors over $\mathbb{F}_q$, given by

$$\theta^+\Big(\sum_{i=1}^{2n} x_i e^i\Big) = \sum_{i=1}^{n} x_i x_{n+i}, \tag{2.2}$$

where $\{e^1 = {}^t[1\ 0\ \ldots\ 0], e^2 = {}^t[0\ 1\ 0\ \ldots\ 0], \ldots, e^{2n} = {}^t[0\ \ldots\ 0\ 1]\}$ is the standard basis of $\mathbb{F}_q^{2n\times 1}$.

$GL(n,q)$ denotes the group of all $n\times n$ nonsingular matrices with entries in $\mathbb{F}_q$.

Then the group of all isometries of $(\mathbb{F}_q^{2n\times 1}, \theta^+)$ is given by

$$\begin{aligned} O^+(2n,q) &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(2n,q) \;\middle|\; \begin{matrix} {}^tAC \text{ and } {}^tBD \text{ are alternating,} \\ {}^tAD + {}^tCB = 1_n \end{matrix} \right\} \\ &= \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in GL(2n,q) \;\middle|\; \begin{matrix} A{}^tB \text{ and } C{}^tD \text{ are alternating,} \\ A{}^tD + B{}^tC = 1_n \end{matrix} \right\} \end{aligned} \tag{2.3}$$

(cf. (2.1)). Here A, B, C and D are of size n.

$P^+(2n,q)$ is the maximal parabolic subgroup of $O^+(2n,q)$ defined by

$$P^+(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; A \in GL(n,q),\ B \text{ alternating} \right\}. \tag{2.4}$$

Let θ^- be the nondegenerate quadratic form on the vector space $\mathbb{F}_q^{2n\times 1}$, given by

$$\theta^-\Big(\sum_{i=1}^{2n} x_i e^i\Big) = \sum_{i=1}^{n-1} x_i x_{n-1+i} + x_{2n-1}^2 + x_{2n-1}x_{2n} + a x_{2n}^2, \tag{2.5}$$

where $\{e^1, \ldots, e^{2n}\}$ is the standard basis of $\mathbb{F}_q^{2n\times 1}$ as above, and a is a fixed element in $\mathbb{F}_q$ such that $z^2 + z + a$ is irreducible over $\mathbb{F}_q$.

Let $\mathcal{P}(x) = x^2 + x$ denote the Artin–Schreier operator in characteristic two. Then the sequence of groups

$$0 \to \mathbb{F}_2^+ \to \mathbb{F}_q^+ \to \mathcal{P}(\mathbb{F}_q) \to 0$$

is exact so that

$$\mathcal{P}(\mathbb{F}_q) = \{b^2 + b \mid b \in \mathbb{F}_q\}, \qquad [\mathbb{F}_q^+ : \mathcal{P}(\mathbb{F}_q)] = 2, \tag{2.6}$$

where the first map is the inclusion from the additive group of the prime subfield of $\mathbb{F}_q$ to that of $\mathbb{F}_q$ and the second one is $x \mapsto \mathcal{P}(x) = x^2 + x$. Moreover, $z^2 + z + a$ is irreducible over $\mathbb{F}_q$ if and only if $a \in \mathbb{F}_q - \mathcal{P}(\mathbb{F}_q)$.

Let δ_a, $\widetilde{\delta}_a$ (with a the fixed element in $\mathbb{F}_q$ as in (2.5)) and η denote the special 2×2 matrices over $\mathbb{F}_q$:

$$\delta_a=\begin{bmatrix}1&1\\0&a\end{bmatrix},\quad \widetilde{\delta}_a=\begin{bmatrix}a&1\\0&1\end{bmatrix},\quad \eta=\begin{bmatrix}0&1\\1&0\end{bmatrix}. \tag{2.7}$$

The group $O^-(2n,q)$ of all isometries of $(\mathbb{F}_q^{2n\times 1},\theta^-)$ consists of all matrices in $GL(2n,q)$,

$$\begin{bmatrix}A&B&e\\C&D&f\\g&h&i\end{bmatrix}, \tag{2.8}$$

satisfying the following relations:

$$\begin{aligned}
&{}^tAC+{}^tg\delta_a g \text{ is alternating},\\
&{}^tBD+{}^th\delta_a h \text{ is alternating},\\
&{}^tef+{}^ti\delta_a i+\delta_a \text{ is alternating},\\
&{}^tAD+{}^tCB+{}^tg\eta h=1_{n-1},\\
&{}^tAf+{}^tCe+{}^tg\eta i=0,\\
&{}^tBf+{}^tDe+{}^th\eta i=0;
\end{aligned} \tag{2.9}$$

or equivalently

$$\begin{aligned}
&A{}^tB+e\widetilde{\delta}_a{}^te \text{ is alternating},\\
&C{}^tD+f\widetilde{\delta}_a{}^tf \text{ is alternating},\\
&g{}^th+i\widetilde{\delta}_a{}^ti+\widetilde{\delta}_a \text{ is alternating},\\
&A{}^tD+B{}^tC+e\eta{}^tf=1_{n-1},\\
&A{}^th+B{}^tg+e\eta{}^ti=0,\\
&C{}^th+D{}^tg+f\eta{}^ti=0.
\end{aligned} \tag{2.10}$$

In (2.8), A, B, C, D are of size $(n-1)\times(n-1)$, e, f are of size $(n-1)\times 2$, g, h are of size $2\times(n-1)$, and i is of size 2×2.

$P^-(2n,q)$ is the maximal parabolic subgroup of $O^-(2n,q)$ given by

$$\begin{aligned}
&P^-(2n,q)\\
&=\left\{\begin{bmatrix}A&0&0\\0&{}^tA^{-1}&0\\0&0&i\end{bmatrix}\begin{bmatrix}1_{n-1}&B&{}^th{}^ti\eta i\\0&1_{n-1}&0\\0&h&1_2\end{bmatrix}\;\middle|\;\begin{array}{l}A\in GL(n-1,q),\\ i\in O^-(2,q),\\ {}^tB+{}^th\delta_a h \text{ is alternating}\end{array}\right\},
\end{aligned} \tag{2.11}$$

where we note that $O^-(2,q)$ is the group of isometries of $(\mathbb{F}_q^{2\times 1},\theta^-)$ with

$$\theta^-(x_1e^1+x_2e^2)=x_1^2+x_1x_2+ax_2^2$$

(cf. (2.5)).

It can be shown that

$$O^-(2,q) = SO^-(2,q) \amalg \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} SO^-(2,q), \tag{2.12}$$

with

$$\begin{aligned} SO^-(2,q) &= \left\{ \begin{bmatrix} d_1 & ad_2 \\ d_2 & d_1+d_2 \end{bmatrix} \,\middle|\, d_1^2 + d_1d_2 + ad_2^2 = 1 \right\} \\ &= \left\{ \begin{bmatrix} d_1 & ad_2 \\ d_2 & d_1+d_2 \end{bmatrix} \,\middle|\, \begin{array}{l} d_1 + d_2 b \in \mathbb{F}_q(b) \text{ with} \\ N_{\mathbb{F}_q(b)/\mathbb{F}_q}(d_1+d_2b) = 1 \end{array} \right\}, \end{aligned} \tag{2.13}$$

where $b \in \overline{\mathbb{F}}_q$ is a root of the irreducible polynomial $z^2 + z + a \in \mathbb{F}_q[z]$. So $SO^-(2,q)$ is a subgroup of index 2 in $O^-(2,q)$, and

$$|SO^-(2,q)| = q+1, \quad |O^-(2,q)| = 2(q+1). \tag{2.14}$$

The reason for defining $SO^-(2,q)$ as in (2.13) will be explained in Section 3.

Let θ be the nondegenerate quadratic form on the vector space $\mathbb{F}_q^{(2n+1)\times 1}$ of all $(2n+1)\times 1$ column vectors over $\mathbb{F}_q$, given by

$$\theta\Big(\sum_{i=1}^{2n+1} x_i e^i\Big) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2, \tag{2.15}$$

where $\{e^1 = {}^t[1\ 0\ \dots\ 0], e^2 = {}^t[0\ 1\ 0\ \dots\ 0], \dots, e^{2n+1} = {}^t[0\ \dots\ 0\ 1]\}$ is the standard basis of $\mathbb{F}_q^{(2n+1)\times 1}$.

The group of all isometries of $(\mathbb{F}_q^{(2n+1)\times 1}, \theta)$ is given by

$$\begin{aligned} &O(2n+1,q) \\ &= \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ g & h & 1 \end{bmatrix} \in GL(2n+1,q) \,\middle|\, \begin{array}{l} {}^tAC + {}^tgg \text{ and } {}^tBD + {}^thh \\ \text{are alternating,} \\ {}^tAD + {}^tCB = 1_n \end{array} \right\} \\ &= \left\{ \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ g & h & 1 \end{bmatrix} \in GL(2n+1,q) \,\middle|\, \begin{array}{l} A{}^tB + B{}^tgg{}^tB + A{}^thh{}^tA \text{ and} \\ C{}^tD + D{}^tgg{}^tD + C{}^thh{}^tC \text{ are} \\ \text{alternating, } A{}^tD + B{}^tC = 1_n \end{array} \right\}. \end{aligned} \tag{2.16}$$

Here A, B, C, D are of size $n \times n$ and g, h are $1 \times n$ matrices.

It is worth observing, for example, that ${}^tAC + {}^tgg$ is alternating if and only if ${}^tAC = {}^tCA$ and $g = \sqrt{\operatorname{diag}({}^tAC)}$, where the meaning of the latter condition is as follows. Recall that every element in $\mathbb{F}_q$ can be written as α^2 for a unique $\alpha \in \mathbb{F}_q$. Now,

(2.17) $\sqrt{\operatorname{diag}({}^tAC)}$ indicates the $1 \times n$ matrix $[\alpha_1\ \alpha_2\ \dots\ \alpha_n]$ if the diagonal entries of ${}^tAC$ are given by

$$({}^tAC)_{11} = \alpha_1^2,\ \dots,\ ({}^tAC)_{nn} = \alpha_n^2 \quad \text{for } \alpha_i \in \mathbb{F}_q.$$

As is well known or can be checked immediately, there is an isomorphism of groups

$$\iota : O(2n+1,q) \to Sp(2n,q), \tag{2.18}$$

given by

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ g & h & 1 \end{bmatrix} \mapsto \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Let $P(2n+1,q)$ be the maximal parabolic subgroup of $O(2n+1,q)$ given by

$$\begin{aligned} &P(2n+1,q) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1_n & B & 0 \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \;\middle|\; \begin{array}{l} A \in GL(n,q), \\ B + {}^thh \text{ is alternating} \end{array} \right\}. \end{aligned} \tag{2.19}$$

For a multiplicative character χ of $\mathbb{F}_q$ and an additive character λ of $\mathbb{F}_q$, $G(\chi,\lambda)$ denotes the Gauss sum defined by

$$G(\chi,\lambda) = \sum_{\alpha \in \mathbb{F}_q^\times} \chi(\alpha)\lambda(\alpha). \tag{2.20}$$

For a nontrivial additive character λ of $\mathbb{F}_q$, and $a, b \in \mathbb{F}_q$, $K(\lambda; a, b)$ is the Kloosterman sum defined by

$$K(\lambda; a, b) = \sum_{\alpha \in \mathbb{F}_q^\times} \lambda(a\alpha + b\alpha^{-1}). \tag{2.21}$$

The order of the group $GL(n,q)$ is given by

$$g_n = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1). \tag{2.22}$$

Then we have, for integers n, r with $0 \le r \le n$,

$$\frac{g_n}{g_{n-r}g_r} = q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q, \tag{2.23}$$

where $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]_q$ is as in (2.24) just below.

From now on till the end of this section, q will denote not just a power of 2 but also an indeterminate.

For integers n, r with $0 \le r \le n$, the q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} (q^{n-j} - 1)/(q^{r-j} - 1). \tag{2.24}$$

For x an indeterminate, n a nonnegative integer,

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}).$$

Then the q-binomial theorem says

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\binom{r}{2}} x^r = (x;q)_n. \tag{2.25}$$

Finally, $[y]$ denotes the largest integer $\le y$, for a real number y.

3. Bruhat decompositions. In this section, we discuss the Bruhat decompositions of the orthogonal groups $O^+(2n,q), O^-(2n,q)$ and $O(2n+1,q)$, respectively, with respect to the maximal parabolic subgroups $P^+(2n,q)$, $P^-(2n,q)$ and $P(2n+1,q)$.

As simple applications, we will show that these decompositions, when combined with the q-binomial theorem, can be used to derive the orders of those orthogonal groups.

Let $\mathbb{F}_2^+$ be the additive group of the prime subfield of $\mathbb{F}_q$. Then there are epimorphisms $\delta^+ : O^+(2n,q) \to \mathbb{F}_2^+$ and $\delta^- : O^-(2n,q) \to \mathbb{F}_2^+$, which are respectively related to the Clifford algebras $C(\mathbb{F}_q^{2n\times 1}, \theta^+)$ and $C(\mathbb{F}_q^{2n\times 1}, \theta^-)$. Explicit expressions for δ^+ and δ^- can be obtained so that $SO^+(2n,q) := \operatorname{Ker}\delta^+$, $SO^-(2n,q) := \operatorname{Ker}\delta^-$ are determined in the form of certain decompositions (cf. (3.46), (3.52)).

The Bruhat decomposition of $O^+(2n,q)$ with respect to $P^+ = P^+(2n,q)$ is given by

$$O^+(2n,q) = \coprod_{r=0}^{n} P^+\sigma_r^+ P^+, \tag{3.1}$$

where

$$\sigma_r^+ = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in O^+(2n,q). \tag{3.2}$$

This can be proved in exactly the same manner as in the proof of Theorem 3.1 of [9].

Write, for each r $(0 \le r \le n)$,

$$A_r^+ = \{w \in P^+(2n,q) \mid \sigma_r^+ w (\sigma_r^+)^{-1} \in P^+(2n,q)\}. \tag{3.3}$$

By expressing $O^+(2n,q)$ as a disjoint union of right cosets of $P^+ = P^+(2n,q)$, the Bruhat decomposition in (3.1) can be written as

$$O^+(2n,q) = \coprod_{r=0}^{n} P^+\sigma_r^+ (A_r^+ \backslash P^+). \tag{3.4}$$

Write $w \in P^+(2n,q)$ as

$$w = \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix}, \tag{3.5}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ {}^tB_{12} & B_{22} \end{bmatrix}, \tag{3.6}$$

B_{11} and B_{22} alternating.

Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r \times r$, $r \times (n-r)$, $(n-r) \times r$, and $(n-r) \times (n-r)$, and similarly for ${}^tA^{-1}$ and B.

Then, by multiplying out, we see that $\sigma_r^+ w (\sigma_r^+)^{-1} \in P^+(2n,q)$ if and only if $A_{12} = 0$, $B_{11} = 0$. Hence

$$|A_r^+| = g_r g_{n-r} q^{\binom{n}{2}} q^{r(2n-3r+1)/2}, \tag{3.7}$$

where g_n is as in (2.22). Also, we have

$$|P^+(2n,q)| = q^{\binom{n}{2}} g_n. \tag{3.8}$$

From (3.7), (3.8) and (2.23), we get

$$|A_r^+ \backslash P^+(2n,q)| = \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}}, \tag{3.9}$$

$$|P^+(2n,q)|^2 |A_r^+|^{-1} = q^{\binom{n}{2}} g_n \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}}. \tag{3.10}$$

Since we have, from (3.4),

$$|O^+(2n,q)| = \sum_{r=0}^{n} |P^+(2n,q)|^2 \, |A_r^+|^{-1}, \tag{3.11}$$

(3.10) and (3.11), on applying the q-binomial theorem (2.25) with $x = -1$, yield

$$|O^+(2n,q)| = 2q^{n^2-n}(q^n - 1) \prod_{j=1}^{n-1} (q^{2j} - 1). \tag{3.12}$$

Note here that (3.7), (3.8), and hence (3.9) and (3.12) are the same as the corresponding formulas in [9] for q odd.

Next, the Bruhat decomposition of $O^-(2n,q)$ with respect to $P^- = P^-(2n,q)$ is

$$O^-(2n,q) = \coprod_{r=0}^{n-1} P^- \sigma_r^- P^-, \tag{3.13}$$

where

$$\sigma_r^- = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-1-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 \\ 0 & 0 & 0 & 0 & 1_2 \end{bmatrix} \in O^-(2n,q). \tag{3.14}$$

(3.13) can be shown in an exactly analogous manner to the proof of Theorem 3.1 in [5].

For each r ($0 \le r \le n-1$), put

$$A_r^- = \{w \in P^-(2n,q) \mid \sigma_r^- w (\sigma_r^-)^{-1} \in P^-(2n,q)\}. \tag{3.15}$$

Then the Bruhat decomposition in (3.13) can be written, expressed as a disjoint union of right cosets of $P^- = P^-(2n,q)$, as

$$O^-(2n,q) = \coprod_{r=0}^{n-1} P^- \sigma_r^- (A_r^- \backslash P^-). \tag{3.16}$$

Write $w \in P^-(2n,q)$ as

$$w = \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & {}^th\,{}^ti\eta i \\ 0 & 1_{n-1} & 0 \\ 0 & h & 1_2 \end{bmatrix}, \tag{3.17}$$

with

$$\begin{gathered} A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \\ h = [h_1\ h_2], \quad {}^tB + {}^th\delta_a h \text{ alternating} \end{gathered} \tag{3.18}$$

(cf. (2.7)). Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r \times r$, $r \times (n-1-r)$, $(n-1-r) \times r$, and $(n-1-r) \times (n-1-r)$, similarly for ${}^tA^{-1}$, B, and h_1 is of size $2 \times r$. Then $\sigma_r^- w (\sigma_r^-)^{-1} \in P^-(2n,q)$ if and only if $A_{12} = 0$, $B_{11} = 0$, $h_1 = 0$. So, recalling the order of $O^-(2,q)$ from (2.14), we get

$$|A_r^-| = 2(q+1) g_r g_{n-1-r} q^{(n-1)(n+2)/2} q^{r(2n-3r-5)/2}, \tag{3.19}$$

where g_n is as in (2.22). Also,

$$|P^-(2n,q)| = 2(q+1) g_{n-1} q^{(n-1)(n+2)/2}. \tag{3.20}$$

From (3.19), (3.20) and (2.23), we get

$$|A_r^- \backslash P^-(2n,q)| = \begin{bmatrix} n-1 \\ r \end{bmatrix}_q q^{r(r+3)/2}, \tag{3.21}$$

$$|P^-(2n,q)|^2 |A_r^-|^{-1} = 2(q+1) q^{n^2-n} \prod_{j=1}^{n-1} (q^j - 1) \begin{bmatrix} n-1 \\ r \end{bmatrix}_q q^{\binom{r}{2}} q^{2r}. \tag{3.22}$$

Note that we have, from (3.16),

$$|O^-(2n,q)| = \sum_{r=0}^{n-1} |P^-(2n,q)|^2\, |A_r^-|^{-1}. \tag{3.23}$$

From (3.22), (3.23) and applying the q-binomial theorem (2.25) with $x = -q^2$, we get

$$|O^-(2n,q)| = 2q^{n^2-n}(q^n+1)\prod_{j=1}^{n-1}(q^{2j}-1). \tag{3.24}$$

Again, we see that (3.19), (3.20), and hence (3.21) and (3.24) are the same as the corresponding formulas in [5] for q odd.

Finally, the Bruhat decomposition of $O(2n+1,q)$ with respect to $P = P(2n+1,q)$ is

$$O(2n+1,q) = \coprod_{r=0}^{n} P\sigma_r P, \tag{3.25}$$

where

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in O(2n+1,q). \tag{3.26}$$

The decomposition in (3.25) can be proved, for example, by using the isomorphism ι in (2.18) and the well known Bruhat decomposition

$$Sp(2n,q) = \coprod_{r=0}^{n} P'\sigma_r' P', \tag{3.27}$$

where

$$\begin{aligned} P' &= P'(2n,q) \\ &= \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \,\middle|\, A \in GL(n,q),\ {}^tB = B \right\} \end{aligned} \tag{3.28}$$

is a maximal parabolic subgroup of $Sp(2n,q)$, and

$$\sigma_r' = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in Sp(2n,q). \tag{3.29}$$

As usual, (3.25) and (3.27) can be rewritten respectively as

$$O(2n+1,q) = \coprod_{r=0}^{n} P\sigma_r (A_r \backslash P) \tag{3.30}$$

and

$$Sp(2n,q)=\coprod_{r=0}^{n} P'\sigma_r'(A_r'\backslash P'), \tag{3.31}$$

where, for each r $(0 \le r \le n)$,

$$A_r=\{w\in P(2n+1,q)\mid \sigma_r w\sigma_r^{-1}\in P(2n+1,q)\}, \tag{3.32}$$

$$A_r'=\{w\in P'(2n,q)\mid \sigma_r' w(\sigma_r')^{-1}\in P'(2n,q)\}. \tag{3.33}$$

Write $w\in P(2n+1,q)$ as

$$w=\begin{bmatrix} A & 0 & 0\\ 0 & {}^tA^{-1} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1_n & B & 0\\ 0 & 1_n & 0\\ 0 & h & 1\end{bmatrix}, \tag{3.34}$$

with

$$\begin{gathered} A=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\quad {}^tA^{-1}=\begin{bmatrix} E_{11} & E_{12}\\ E_{21} & E_{22}\end{bmatrix},\quad B=\begin{bmatrix} B_{11} & B_{12}\\ {}^tB_{12} & B_{22}\end{bmatrix},\\ B_{11}={}^tB_{11},\qquad B_{22}={}^tB_{22},\qquad h=[h_1\ h_2]=\sqrt{\operatorname{diag} B}\end{gathered} \tag{3.35}$$

(cf. (2.17)). Here A_{11}, A_{12}, A_{21}, and A_{22} are respectively of sizes $r\times r$, $r\times(n-r)$, $(n-r)\times r$, $(n-r)\times(n-r)$, similarly for ${}^tA^{-1}$ and B, and h_1 is of size $1\times r$.

Then $\sigma_r w\sigma_r^{-1}\in P(2n+1,q)$ if and only if $A_{12}=0, B_{11}=0$. Thus

$$|A_r|=g_r g_{n-r} q^{\binom{n+1}{2}} q^{r(2n-3r-1)/2}, \tag{3.36}$$

where g_n is as in (2.22). Also,

$$|P(2n+1,q)|=g_n q^{\binom{n+1}{2}}. \tag{3.37}$$

From (3.36), (3.37) and (2.23), we get

$$|A_r\backslash P(2n+1,q)|=\begin{bmatrix} n\\ r\end{bmatrix}_q q^{\binom{r+1}{2}}, \tag{3.38}$$

$$|P(2n+1,q)|^2|A_r|^{-1}=q^{n^2}\prod_{j=1}^{n}(q^j-1)\begin{bmatrix} n\\ r\end{bmatrix}_q q^{\binom{r}{2}}q^r. \tag{3.39}$$

Since $|O(2n+1,q)|=\sum_{r=0}^{n}|P(2n+1,q)|^2\,|A_r|^{-1}$ from (3.30), by applying the q-binomial theorem (2.25) with $x=-q$ we get

$$|O(2n+1,q)|=q^{n^2}\prod_{j=1}^{n}(q^{2j}-1). \tag{3.40}$$

Note here again that (3.36), (3.37), and hence (3.38) and (3.40) are the same as the corresponding formulas in [4] for q odd.

In order to define $SO^+(2n,q)$ and $SO^-(2n,q)$, we turn our attention to the δ-function defined on the group of isometries of an even-dimensional nondegenerate quadratic space over a finite field of characteristic two.

Let $(V,\widetilde{\theta})$ be a vector space V over $\mathbb{F}_q$, of dimension $2n$, together with the nondegenerate quadratic form $\widetilde{\theta}$. Then the epimorphism $\delta : O(V,\widetilde{\theta}) \to \mathbb{F}_2^+$ can be described as follows, where $\mathbb{F}_2^+$ is the additive group of the prime subfield of $\mathbb{F}_q$. Assume that

$$V = \langle e_1, f_1\rangle \perp \ldots \perp \langle e_n, f_n\rangle, \tag{3.41}$$

where $\widetilde{\beta}(e_i, f_i) = 1$ $(i = 1,\ldots,n)$ for the associated symmetric bilinear form $\widetilde{\beta}$ of $\widetilde{\theta}$, and the orthogonality in (3.41) is with respect to $\widetilde{\beta}$. Then, for $w \in O(V,\widetilde{\theta})$,

$$\delta(w) = \sum_{i,j=1}^{n} (a_{ij}b_{ij}\widetilde{\theta}(e_i) + c_{ij}d_{ij}\widetilde{\theta}(f_i) + b_{ij}c_{ij}), \tag{3.42}$$

where

$$[w]_{\mathcal{B}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3.43}$$

is the matrix of w relative to the ordered basis $\mathcal{B} = (e_1,\ldots,e_n,f_1,\ldots,f_n)$, i.e., the columns of (3.43) are the "coordinate matrices" relative to $\mathcal{B}$ of the images under w of the vectors in $\mathcal{B}$, with $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ $n \times n$ matrices.

It is known that δ is independent of a choice of basis as in (3.41). The explicit formula of δ in (3.42) can be obtained from the fact that, for each $w \in O(V,\widetilde{\theta})$, $\delta(w) \in \mathbb{F}_q$ satisfies

$$\sum_{i=1}^{n} e_i f_i = \sum_{i=1}^{n} (we_i)(wf_i) + \delta(w)$$

in the Clifford algebra $C(V,\widetilde{\theta})$ of $(V,\widetilde{\theta})$.

Writing

$$\mathbb{F}_q^{2n\times 1} = \langle e^1, e^{n+1}\rangle \perp \ldots \perp \langle e^n, e^{2n}\rangle,$$

we see from (3.42) that $\delta^+ : O^+(2n,q) \to \mathbb{F}_2^+$ is given by

$$\delta^+(w) = \mathrm{tr}(B^tC), \tag{3.44}$$

where

$$w = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in O^+(2n,q)$$

(cf. (2.3)).

On the other hand, writing

$$\mathbb{F}_q^{2n\times 1} = \langle e^1, e^n\rangle \perp \langle e^2, e^{n+1}\rangle \perp \ldots \perp \langle e^{n-1}, e^{2n-2}\rangle \perp \langle e^{2n-1}, e^{2n}\rangle,$$

we see, from (3.42) again, that $\delta^- : O^-(2n,q) \to \mathbb{F}_2^+$ is given, for $w \in O^-(2n,q)$, by

$$\delta^-(w) = \operatorname{tr}({}^th\delta_a g) + \operatorname{tr}\left(e\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}{}^tf\right) + \operatorname{tr}(B{}^tC) + {}^ti^2\delta_a i^1, \tag{3.45}$$

where δ_a is as in (2.7), $i = [i^1\ i^2]$ with i^1, i^2 respectively denoting the first and second columns of i, and

$$w = \begin{bmatrix} A & B & e\\ C & D & f\\ g & h & i\end{bmatrix} \in O^-(2n,q)$$

(cf. (2.8)–(2.10)).

Using (3.44), we see that $\delta^+(w) = 0$ for $w \in P^+(2n,q)$ (cf. (2.4)), $\delta^+(\sigma_r^+) = 0$ for r even, and $\delta^+(\sigma_r^+) = 1$ for r odd (cf. (3.2)). So, from (3.4), we see that $SO^+(2n,q) := \operatorname{Ker}\delta^+$ is given by

$$SO^+(2n,q) = \coprod_{\substack{0\le r\le n\\ r\ \text{even}}} P^+\sigma_r^+(A_r^+\backslash P^+). \tag{3.46}$$

On the other hand, we see, by exploiting (3.45), that $\delta^-(\sigma_r^-) = 0$ for r even and $\delta^-(\sigma_r^-) = 1$ for r odd (cf. (3.14)). Further, for $w \in P^-(2n,q)$ we have $\delta^-(w) = {}^ti^2\delta_a i^1$ in the notation of w in (2.8). Here $i = [i^1\ i^2] \in O^-(2,q)$. Thus, from (2.12) and (2.13), we see that $\delta^-(w) = 0$ for $i \in SO^-(2,q)$ and that $\delta^-(w) = 1$ for $i \in \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]SO^-(2,q)$.

Put

$$Q^- = Q^-(2n,q) = \left\{ \begin{bmatrix} A & 0 & 0\\ 0 & {}^tA^{-1} & 0\\ 0 & 0 & i\end{bmatrix}\begin{bmatrix} 1_{n-1} & B & {}^th{}^ti\eta i\\ 0 & 1_{n-1} & 0\\ 0 & h & 1_2\end{bmatrix} \;\middle|\; \begin{array}{l} A \in GL(n-1,q),\\ i \in SO^-(2,q),\\ {}^tB + {}^th\delta_a h \text{ is alternating}\end{array} \right\}, \tag{3.47}$$

which is a subgroup of index 2 in $P^- = P^-(2n,q)$. Then the Bruhat decomposition in (3.13) can be modified to give

$$O^-(2n,q) = \coprod_{r=0}^{n-1} P^-\sigma_r^- Q^-. \tag{3.48}$$

Also, we put, for each r $(0 \le r \le n-1)$,

$$B_r^- = \{w \in Q^-(2n,q) \mid \sigma_r^- w(\sigma_r^-)^{-1} \in P^-(2n,q)\}. \tag{3.49}$$

It is a subgroup of index 2 in A_r^- (cf. (3.15)), and (3.48) can be rewritten as

$$O^-(2n,q) = \coprod_{r=0}^{n-1} P^-\sigma_r^-(B_r^-\backslash Q^-). \tag{3.50}$$

Moreover,

$$|B_r^-\backslash Q^-| = |A_r^-\backslash P^-|. \tag{3.51}$$

Now, from the above observation about the values of δ^- and (3.50), $SO^-(2n,q) := \operatorname{Ker}\delta^-$ is given by

$$SO^-(2n,q) = \Big(\coprod_{\substack{0\le r\le n-1\\ r \text{ even}}} Q^-\sigma_r^-(B_r^-\backslash Q^-)\Big) \amalg \Big(\coprod_{\substack{0\le r\le n-1\\ r \text{ odd}}} \varrho Q^-\sigma_r^-(B_r^-\backslash Q^-)\Big), \tag{3.52}$$

where

$$\varrho = \begin{bmatrix} 1_{n-1} & 0 & 0 & 0\\ 0 & 1_{n-1} & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 \end{bmatrix} \in P^-(2n,q) \tag{3.53}$$

(cf. (2.11)).

4. Certain propositions

PROPOSITION 4.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then*:

(a) *For any positive integer r*,

$$\sum_{h\in\mathbb{F}_q^{r\times 2}} \lambda(\operatorname{tr}\delta_a{}^t hh) = (-q)^r. \tag{4.1}$$

(b) *For any positive even integer r*,

$$\sum_{h\in\mathbb{F}_q^{r\times 2}} \lambda(\operatorname{tr}\delta_a{}^t hNh) = q^r. \tag{4.2}$$

Here δ_a is as in (2.7), *and N is the $r\times r$ matrix*

$$N = \begin{bmatrix} 0 & 1_{r/2}\\ 1_{r/2} & 0 \end{bmatrix}. \tag{4.3}$$

Proof. It is easily seen that the LHS of (4.1) equals

$$\Big(\sum_{x,y\in\mathbb{F}_q} \lambda(x^2+xy+ay^2)\Big)^r,$$

where

$$\sum_{x,y\in\mathbb{F}_q} \lambda(x^2+xy+ay^2) = \sum_{y\in\mathbb{F}_q^\times}\sum_{x\in\mathbb{F}_q} \lambda(x^2+xy+ay^2). \tag{4.4}$$

Here one notes that $\sum_{x\in\mathbb{F}_q}\lambda(x^2) = \sum_{x\in\mathbb{F}_q}\lambda(x) = 0$.

For each fixed $y \in \mathbb{F}_q^\times$,

$$\sum_{x\in\mathbb{F}_q} \lambda(x^2+xy+ay^2) + \sum_{x\in\mathbb{F}_q} \lambda(x^2+xy) = \sum_{x\in\mathbb{F}_q} \lambda(y^2(x^2+x+a)) + \sum_{x\in\mathbb{F}_q} \lambda(y^2(x^2+x)) = 2\Big\{\sum_{t\in\mathcal{P}(\mathbb{F}_q)} \lambda(y^2(t+a)) + \sum_{t\in\mathcal{P}(\mathbb{F}_q)} \lambda(y^2t)\Big\} = 2\sum_{x\in\mathbb{F}_q} \lambda(y^2x) = 2\sum_{x\in\mathbb{F}_q} \lambda(x) = 0$$

(cf. (2.6)).

Thus (4.4) equals

$$-\sum_{y\in\mathbb{F}_q^\times}\sum_{x\in\mathbb{F}_q} \lambda(x(x+y)) = -\sum_{x\in\mathbb{F}_q}\sum_{y\in\mathbb{F}_q} \lambda(x(x+y)) = -\sum_{x,y\in\mathbb{F}_q} \lambda(xy) = -\Big\{\sum_{x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q} \lambda(y) + \sum_{y\in\mathbb{F}_q} 1\Big\} = -q.$$

This shows (a). (b) is easy to see. ∎

The following proposition was proved in [1] and mentioned in [2, Theorems 2.3 and 2.4].

Proposition 4.2. (a) *If B is an $r\times r$ alternating matrix of rank p over $\mathbb{F}_q$, then there exists $A \in GL(r,q)$ such that*

$$B = {}^tA \begin{bmatrix} 0 & 1_s & 0 \\ 1_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A \qquad (2s = p).$$

(b) *If B is an $r\times r$ symmetric, nonalternating matrix of rank p over $\mathbb{F}_q$, then there exists $A \in GL(r,q)$ such that*

$$B = {}^tA \begin{bmatrix} 1_p & 0 \\ 0 & 0 \end{bmatrix} A.$$

The next proposition contains special cases of Theorems 2 and 3 of [10].

Proposition 4.3. *Let s_r and n_r denote respectively the number of $r\times r$ nonsingular symmetric matrices over $\mathbb{F}_q$ and that of $r\times r$ nonsingular alternating matrices over $\mathbb{F}_q$. So $s_r - n_r$ equals the number of $r\times r$ nonsingular symmetric, nonalternating matrices over $\mathbb{F}_q$. Then $s_r, n_r, s_r - n_r$ are respec-*

tively given by:

$$(4.5)\qquad s_r = \begin{cases} q^{r(r+2)/4}\prod_{j=1}^{r/2}(q^{2j-1}-1) & \text{for } r \text{ even},\\ q^{(r^2-1)/4}\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1) & \text{for } r \text{ odd},\end{cases}$$

$$(4.6)\qquad n_r = \begin{cases} q^{r(r-2)/4}\prod_{j=1}^{r/2}(q^{2j-1}-1) & \text{for } r \text{ even},\\ 0 & \text{for } r \text{ odd},\end{cases}$$

$$(4.7)\qquad s_r - n_r = \begin{cases} q^{r(r-2)/4}(q^r-1)\prod_{j=1}^{r/2}(q^{2j-1}-1) & \text{for } r \text{ even},\\ q^{(r^2-1)/4}\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1) & \text{for } r \text{ odd}.\end{cases}$$

PROPOSITION 4.4. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. For each positive integer r, let Ω_r be the set of all $r\times r$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then*:

$$(4.8)\qquad b_r(\lambda) = \sum_{B\in\Omega_r}\sum_{h\in\mathbb{F}_q^{r\times 2}}\lambda(\operatorname{tr}\delta_a{}^t hBh)$$

$$= \begin{cases} q^{r(r+6)/4}\prod_{j=1}^{r/2}(q^{2j-1}-1) & \text{for } r \text{ even},\\ -q^{(r^2+4r-1)/4}\prod_{j=1}^{(r+1)/2}(q^{2j-1}-1) & \text{for } r \text{ odd}.\end{cases}$$

Proof. In view of Proposition 4.2 and with the notations of Proposition 4.3, $b_r(\lambda)$ can be written as

$$b_r(\lambda) = \begin{cases} n_r\sum_{h\in\mathbb{F}_q^{r\times 2}}\lambda(\operatorname{tr}\delta_a{}^t hNh) + (s_r-n_r)\sum_{h\in\mathbb{F}_q^{r\times 2}}\lambda(\operatorname{tr}\delta_a{}^t hh) & \text{for } r \text{ even},\\ (s_r-n_r)\sum_{h\in\mathbb{F}_q^{r\times 2}}\lambda(\operatorname{tr}\delta_a{}^t hh) & \text{for } r \text{ odd},\end{cases}$$

where δ_a and N are respectively as in (2.7) and (4.3).

Now, our result follows from (4.1), (4.2), (4.6) and (4.7). ∎

Remark. It is amusing to note that the formula of $b_r(\lambda)$ in (4.8) coincides with that of the corresponding sum in (4.6) of [5] for q odd.

PROPOSITION 4.5. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then*

$$\sum_{w\in SO^-(2,q)} \lambda(\operatorname{tr} w) = -\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2, \tag{4.9}$$

$$\sum_{w\in SO^-(2,q)} \lambda(\operatorname{tr} \delta_1 w) = q+1, \tag{4.10}$$

where ψ is a multiplicative character of $\mathbb{F}_q$ of order $q-1$ and

$$\delta_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{4.11}$$

Proof. (4.10) is clear from (2.13) and (2.14), since $\lambda(\operatorname{tr}\delta_1 w) = \lambda(0) = 1$ for each $w \in SO^-(2,q)$.

Let $b \in \overline{\mathbb{F}}_q$ be a root of the irreducible polynomial $z^2+z+a \in \mathbb{F}_q[z]$ (with a as in (2.5)). Then, for the quadratic extension $K = \mathbb{F}_q(b)$ of $\mathbb{F}_q$ and

$$w = \begin{bmatrix} d_1 & ad_2 \\ d_2 & d_1+d_2 \end{bmatrix} \in SO^-(2,q)$$

(cf. (2.13)), we have

$$\operatorname{tr} w = d_2 = \operatorname{tr}_{K/\mathbb{F}_q}(d_1+d_2 b).$$

Thus the LHS of (4.9) can be rewritten as

$$\sum_{\alpha\in K,\, N_{K/\mathbb{F}_q}(\alpha)=1} \lambda\circ \operatorname{tr}_{K/\mathbb{F}_q}(\alpha).$$

Now, (4.9) follows by using the same argument as in the proof of Proposition 4.5 of [5]. ∎

Remark. As in the odd q case ([5], Remark after Proposition 4.5), (4.9) yields the estimate

$$\Big|\sum_{w\in SO^-(2,q)} \lambda(\operatorname{tr} w)\Big| \le q-1.$$

5. $O^-(2n,q)$ case. In this section, we will consider the sum

$$\sum_{w\in G} \lambda(\operatorname{tr} w)$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and $G = O^-(2n,q)$ or $SO^-(2n,q)$, and find explicit expressions for these by using the decompositions in (3.50) and (3.52).

In view of (3.50), the sum $\sum_{w\in O^-(2n,q)} \lambda(\operatorname{tr} w)$ can be written as

$$\sum_{r=0}^{n-1} |B_r^- \backslash Q^-| \sum_{w\in P^-} \lambda(\operatorname{tr} w\sigma_r^-). \tag{5.1}$$

Here one has to observe that, for each $y \in Q^-$,

$$\sum_{w\in P^-} \lambda(\operatorname{tr} w\sigma_r^- y) = \sum_{w\in P^-} \lambda(\operatorname{tr} yw\sigma_r^-) = \sum_{w\in P^-} \lambda(\operatorname{tr} w\sigma_r^-).$$

Write $w \in P^-(2n,q)$ as in (3.17) with $A, {}^tA^{-1}$, B, h as in (3.18). Note here that B and h are subject to the condition

$${}^tB + {}^th\delta_a h \text{ is alternating,}$$

which is equivalent to the conditions:

$$\begin{cases} {}^tB_{11} + {}^th_1\delta_a h_1 \text{ is alternating,} \\ {}^tB_{22} + {}^th_2\delta_a h_2 \text{ is alternating,} \\ {}^tB_{12} + {}^th_2\delta_a h_1 = {}^tB_{21} + {}^th_1\delta_a h_2. \end{cases} \tag{5.2}$$

Now,

$$\begin{aligned} \sum_{w\in P^-} \lambda(\operatorname{tr} w\sigma_r^-) = \sum_{i\in O^-(2,q)} &\lambda(\operatorname{tr} i) \sum_{A,h} \lambda(\operatorname{tr} A_{22} + \operatorname{tr} E_{22}) \\ &\times \sum_{B} \lambda(\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{12}B_{21}). \end{aligned} \tag{5.3}$$

For each fixed A, h and taking the last condition in (5.2) into consideration, the last sum in (5.3) is over all B_{11}, B_{21}, B_{22} satisfying the first and second conditions in (5.2), so that it equals

$$q^{\binom{n-1-r}{2}} \sum_{B_{11}} \lambda(\operatorname{tr} A_{11}B_{11}) \sum_{B_{21}} \lambda(\operatorname{tr} A_{12}B_{21}). \tag{5.4}$$

The inner sum in (5.4) is nonzero if and only if $A_{12} = 0$, in which case it equals $q^{r(n-1-r)}$. On the other hand, the sum over B_{11} in (5.4) is nonzero if and only if A_{11} is symmetric, in which case it equals $q^{\binom{r}{2}}\lambda(\operatorname{tr} \delta_a h_1 A_{11} {}^th_1)$. To see this, we let

$$A_{11} = (\alpha_{ij}), \quad B_{11} = (\beta_{ij}), \quad h = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1r} \\ h_{21} & h_{22} & \dots & h_{2r} \end{bmatrix}.$$

Then ${}^tB_{11} + {}^th_1\delta_a h_1$ is alternating if and only if

$$\begin{cases} \beta_{ii} = h_{1i}^2 + h_{1i}h_{2i} + ah_{2i}^2 & \text{for } 1 \le i \le r, \\ \beta_{ij} = \beta_{ji} + h_{1i}h_{2j} + h_{1j}h_{2i} & \text{for } 1 \le i < j \le r. \end{cases} \tag{5.5}$$

Using these relations, we see that

$$\operatorname{tr} A_{11}B_{11} = \sum_{i=1}^{r} \alpha_{ii}(h_{1i}^2 + h_{1i}h_{2i} + ah_{2i}^2) + \sum_{1\le i<j\le r} \alpha_{ij}(h_{1i}h_{2j} + h_{1j}h_{2i}) + \sum_{1\le i<j\le r} (\alpha_{ij}+\alpha_{ji})\beta_{ij}. \tag{5.6}$$

Thus the sum over B_{11} in (5.4) is nonzero if and only if $\alpha_{ij} = \alpha_{ji}$ for $1 \le i < j \le r$, i.e., A_{11} is symmetric. Moreover, in that case (5.6) can be rewritten as $\operatorname{tr} \delta_a h_1 A_{11} {}^t h_1$, so that

$$\sum_{B_{11}} \lambda(\operatorname{tr} A_{11}B_{11}) = q^{\binom{r}{2}} \lambda(\operatorname{tr} \delta_a h_1 A_{11} {}^t h_1).$$

We have shown that (5.4) is nonzero if and only if $A = \left[\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$ with A_{11} nonsingular symmetric, in which case it equals

$$q^{\binom{n-1-r}{2}+\binom{r}{2}+r(n-1-r)} \lambda(\operatorname{tr} \delta_a h_1 A_{11} {}^t h_1) = q^{\binom{n-1}{2}} \lambda(\operatorname{tr} \delta_a h_1 A_{11} {}^t h_1).$$

For such an $A = \left[\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$,

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} {}^t A_{11}^{-1} & * \\ 0 & {}^t A_{22}^{-1} \end{bmatrix}.$$

So the sum in (5.3) can be written as

$$\begin{aligned} &q^{\binom{n-1}{2}} \sum_{i\in O^-(2,q)} \lambda(\operatorname{tr} i) \sum_{A_{21},h_2} \sum_{A_{11},h_1} \lambda(\operatorname{tr} \delta_a h_1 A_{11} {}^t h_1) \sum_{A_{22}} \lambda(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) \\ &= q^{\binom{n-1}{2}+2(n-1-r)+r(n-1-r)} \sum_{i\in O^-(2,q)} \lambda(\operatorname{tr} i) b_r(\lambda) K_{GL(n-1-r,q)}(\lambda;1,1) \\ &= q^{(n-1)(n+2)/2+r(n-r-3)} \sum_{i\in O^-(2,q)} \lambda(\operatorname{tr} i) b_r(\lambda) K_{GL(n-1-r,q)}(\lambda;1,1), \end{aligned}$$

where $b_r(\lambda)$ is as in (4.8), and in [8], for $a, b \in \mathbb{F}_q$, $K_{GL(t,q)}(\lambda; a, b)$ is defined as

$$K_{GL(t,q)}(\lambda; a, b) = \sum_{w\in GL(t,q)} \lambda(a \operatorname{tr} w + b \operatorname{tr} w^{-1}). \tag{5.7}$$

Putting everything together, the sum in (5.1) can be written as

$$q^{(n-1)(n+2)/2} \sum_{i\in O^-(2,q)} \lambda(\operatorname{tr} i) \times \sum_{r=0}^{n-1} |B_r^- \backslash Q^-| q^{r(n-r-3)} b_r(\lambda) K_{GL(n-1-r,q)}(\lambda;1,1). \tag{5.8}$$

An explicit expression for (5.7) was obtained in [8].

THEOREM 5.1. *For integers $t \geq 1$ and nonzero elements a, b of $\mathbb{F}_q$, the Kloosterman sum $K_{GL(t,q)}(\lambda; a, b)$ is given by*

$$(5.9)\qquad K_{GL(t,q)}(\lambda; a, b) = q^{(t-2)(t+1)/2} \sum_{l=1}^{[(t+2)/2]} q^l K(\lambda; a, b)^{t+2-2l} \times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),$$

where $K(\lambda; a, b)$ is the usual Kloosterman sum as in (2.21) *and the inner sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \leq j_{l-1} \leq j_{l-2} \leq \ldots \leq j_1 \leq t + 1$. Here we agree that the inner sum is* 1 *for $l = 1$.*

Remark. The inner sum in (5.9) is equivalently given by

$$\sum \prod_{\nu=1}^{l-1} (q^{j_\nu} - 1),$$

where the sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 3 \leq j_1 \leq t - 1$, $2l - 5 \leq j_2 \leq j_1 - 2, \ldots, 1 \leq j_{l-1} \leq j_{l-2} - 2$ (with the understanding $j_0 = t + 1$ for $l = 2$).

In view of (2.12), (4.9), (4.10), (3.51), (3.21), (4.8) and (5.9), we get the following theorem from (5.8).

THEOREM 5.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then the Gauss sum over $O^-(2n, q)$,*

$$\sum_{w \in O^-(2n,q)} \lambda(\operatorname{tr} w),$$

is given by

$$(5.10)\qquad q^{n^2-n-1} \Big(-\frac{1}{q-1} \sum_{j=1}^{q-1} G(\psi^j, \lambda)^2 + q + 1 \Big) \times \Big\{ \sum_{r=0}^{[(n-1)/2]} q^{r(r+3)} \begin{bmatrix} n-1 \\ 2r \end{bmatrix}_q \prod_{j=1}^{r} (q^{2j-1} - 1) \times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda; 1, 1)^{n-2r+1-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1)$$

$$- \sum_{r=0}^{[(n-2)/2]} q^{r(r+3)+1} \begin{bmatrix} n-1 \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1} (q^{2j-1}-1)$$

$$\times \sum_{l=1}^{[(n-2r)/2]} q^l K(\lambda;1,1)^{n-2r-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu}-1)\Big\},$$

where $G(\psi^j,\lambda)$ *is the usual Gauss sum as in* (2.20) *with* ψ *a multiplicative character of* $\mathbb{F}_q$ *of order* $q-1$, *and* $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.21). *In addition, the first unspecified sum in* (5.10) *is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r$ *and the second one is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r-1$.

As to the Gauss sum $\sum_{w\in SO^-(2n,q)} \lambda(\operatorname{tr} w)$, we may write it, using the decomposition in (3.52), as

$$\text{(5.11)} \quad \sum_{w\in SO^-(2n,q)} \lambda(\operatorname{tr} w) = \sum_{\substack{0\le r\le n-1\\ r \text{ even}}} |B_r^-\backslash Q^-| \sum_{w\in Q^-} \lambda(\operatorname{tr} w\sigma_r^-) + \sum_{\substack{0\le r\le n-1\\ r \text{ odd}}} |B_r^-\backslash Q^-| \sum_{w\in Q^-} \lambda(\operatorname{tr} \varrho w\sigma_r^-).$$

Here one has to observe that, for each $y\in Q^-$,

$$\sum_{w\in Q^-} \lambda(\operatorname{tr} \varrho w\sigma_r^- y) = \sum_{w\in Q^-} \lambda(\operatorname{tr} y\varrho w\sigma_r^-) = \sum_{w\in Q^-} \lambda(\operatorname{tr} \varrho y' w\sigma_r^-) = \sum_{w\in Q^-} \lambda(\operatorname{tr} \varrho w\sigma_r^-),$$

where $y' = \varrho^{-1} y \varrho \in Q^- = Q^-(2n,q)$ with ϱ as in (3.53).

Glancing through the above argument about $\sum_{w\in O^-(2n,q)} \lambda(\operatorname{tr} w)$, we see that (5.11) equals

$$q^{(n-1)(n+2)/2} \times\Big\{ \sum_{i\in SO^-(2,q)} \lambda(\operatorname{tr} i) \sum_{\substack{0\le r\le n-1\\ r \text{ even}}} |B_r^-\backslash Q^-| q^{r(n-r-3)} b_r(\lambda) K_{GL(n-1-r,q)}(\lambda;1,1)$$

$$+ \sum_{i\in SO^-(2,q)} \lambda(\operatorname{tr} \delta_1 i) \sum_{\substack{0\le r\le n-1\\ r \text{ odd}}} |B_r^-\backslash Q^-| q^{r(n-r-3)} b_r(\lambda) K_{GL(n-1-r,q)}(\lambda;1,1)\Big\},$$

where δ_1 is as in (4.11).

So we get the following result.

THEOREM 5.3. *Let* λ *be a nontrivial additive character of* $\mathbb{F}_q$. *Then the*

Gauss sum over $SO^-(2n,q)$,

$$\sum_{w\in SO^-(2n,q)} \lambda(\operatorname{tr} w),$$

is given by

$$(5.12)\qquad q^{n^2-n-1}$$
$$\times\Big\{\Big(-\frac{1}{q-1}\sum_{j=1}^{q-1} G(\psi^j,\lambda)^2\Big)\sum_{r=0}^{[(n-1)/2]} q^{r(r+3)}\begin{bmatrix} n-1\\ 2r\end{bmatrix}_q\prod_{j=1}^{r}(q^{2j-1}-1)$$
$$\times\sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda;1,1)^{n-2r+1-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)$$
$$-(q+1)\sum_{r=0}^{[(n-2)/2]} q^{r(r+3)+1}\begin{bmatrix} n-1\\ 2r+1\end{bmatrix}_q\prod_{j=1}^{r+1}(q^{2j-1}-1)$$
$$\times\sum_{l=1}^{[(n-2r)/2]} q^l K(\lambda;1,1)^{n-2r-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\},$$

where $G(\psi^j,\lambda)$ *is the usual Gauss sum as in* (2.20) *with* ψ *a multiplicative character of* $\mathbb{F}_q$ *of order* $q-1$, *and* $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.21). *In addition, the first unspecified sum in* (5.12) *is over all integers* $j_1,\dots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\dots\le j_1\le n-2r$ *and the second one is over all integers* $j_1,\dots,j_{l-1}$ *satisfying* $2l-1\le j_{l-1}\le j_{l-2}\le\dots\le j_1\le n-2r-1$.

Remark. We see that the expressions in (5.10) and (5.12) are the same as the corresponding ones in [5] for q odd.

6. $O^+(2n,q)$ **and** $O(2n+1,q)$ **cases.** In this section, we will consider the sum

$$\sum_{w\in G}\lambda(\operatorname{tr} w)$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and $G=O^+(2n,q)$ or $SO^+(2n,q)$ or $O(2n+1,q)$, and find explicit expressions for them by using the decompositions in (3.4), (3.46) and (3.30).

First, we consider the sum

$$\sum_{w\in O(2n+1,q)}\lambda(\operatorname{tr} w). \tag{6.1}$$

With $P=P(2n+1,q),\sigma_r,A_r$ respectively as in (2.19), (3.26), (3.32) and by using the decomposition in (3.30), (6.1) can be written as

$$\sum_{r=0}^{n} |A_r \backslash P| \sum_{w \in P} \lambda(\operatorname{tr} w\sigma_r). \tag{6.2}$$

With $P' = P'(2n, q), \sigma'_r, A'_r$ respectively as in (3.28), (3.29), (3.33), we see that

$$|A_r \backslash P| = |A'_r \backslash P'|$$

(cf. (3.38) and [8], (3.10)), and, for $w \in P$,

$$\operatorname{tr} w\sigma_r = \operatorname{tr}(\iota(w)\sigma'_r) + 1,$$

where ι is the isomorphism in (2.18).

So (6.2) can be rewritten as

$$\begin{aligned}\lambda(1) \sum_{r=0}^{n} |A'_r \backslash P'| \sum_{w \in P} \lambda(\operatorname{tr} \iota(w)\sigma'_r) &= \lambda(1) \sum_{r=0}^{n} |A'_r \backslash P'| \sum_{w \in P'} \lambda(\operatorname{tr} w\sigma'_r) \\ &= \lambda(1) \sum_{w \in Sp(2n,q)} \lambda(\operatorname{tr} w),\end{aligned}$$

in view of the decomposition in (3.31) and the fact that $\iota(P) = P'$.

An explicit expression for $\sum_{w \in Sp(2n,q)} \lambda(\operatorname{tr} w)$, for q a power of any prime, was obtained in Theorem 5.4 of [8].

Theorem 6.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then the Gauss sum over $O(2n+1, q)$,*

$$\sum_{w \in O(2n+1,q)} \lambda(\operatorname{tr} w),$$

equals

$$\lambda(1) \sum_{w \in Sp(2n,q)} \lambda(\operatorname{tr} w),$$

so that it is $\lambda(1)$ times

$$\begin{aligned}q^{n^2-1} &\sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r} (q^{2j-1} - 1) \\ &\times \sum_{l=1}^{[(n-2r+2)/2]} q^l K(\lambda; 1, 1)^{n-2r+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),\end{aligned}$$

where $K(\lambda; 1, 1)$ is the usual Kloosterman sum as in (2.21) and the innermost sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n - 2r + 1$.

Remark. The Gauss sum (6.1) has the same expression as the sum $\sum_{w\in SO(2n+1,q)} \lambda(\operatorname{tr} w)$ for q odd (cf. [4], Theorem 5.1). On the other hand, the sum

$$\sum_{w\in O(2n+1,q)} \lambda(\operatorname{tr} w)$$

for q odd is given by

$$(\lambda(1)+\lambda(-1)) \sum_{w\in Sp(2n,q)} \lambda(\operatorname{tr} w)$$

(cf. [4], Theorem 6.1).

Next, we consider the sum

$$\sum_{w\in O^+(2n,q)} \lambda(\operatorname{tr} w). \tag{6.3}$$

In view of the decomposition in (3.4), (6.3) can be written as

$$\sum_{r=0}^{n} |A_r^+\backslash P^+| \sum_{w\in P^+} \lambda(\operatorname{tr} w\sigma_r^+). \tag{6.4}$$

By proceeding just as in the odd q case (cf. [9]), we see that (6.4) equals

$$q^{\binom{n}{2}} \sum_{r=0}^{n} |A_r^+\backslash P^+| q^{r(n-r)} s_r K_{GL(n-r,q)}(\lambda;1,1),$$

where s_r denotes the number of all $r\times r$ nonsingular symmetric matrices over $\mathbb{F}_q$ ($s_r=1$, for $r=0$), and $K_{GL(n-r,q)}(\lambda;1,1)$ is as in (5.7).

On the other hand, the sum

$$\sum_{w\in SO^+(2n,q)} \lambda(\operatorname{tr} w)$$

is given by

$$q^{\binom{n}{2}} \sum_{\substack{0\le r\le n\\ r\ \text{even}}} |A_r^+\backslash P^+| q^{r(n-r)} s_r K_{GL(n-r,q)}(\lambda;1,1),$$

in view of (3.46).

Note that $|A_r^+\backslash P^+|$ and s_r as well as $K_{GL(n-r,q)}(\lambda;1,1)$ are the same as the corresponding formulas for q odd (cf. (3.9) and (4.5); [9], (3.13) and (4.7)). So we should get the same results as for the odd q case.

THEOREM 6.2. *Let λ be a nontrivial additive character of $\mathbb{F}_q$. Then the Gauss sum over $SO^+(2n,q)$,*

$$\sum_{w\in SO^+(2n,q)} \lambda(\operatorname{tr} w),$$

is given by

$$q^{n^2-n-1}\sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r}(q^{2j-1}-1)$$
$$\times \sum_{l=1}^{[(n-2r+2)/2]} q^l K(\lambda;1,1)^{n-2r+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.21) *and the innermost sum is over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r+1$.

THEOREM 6.3. *Let* λ *be a nontrivial additive character of* $\mathbb{F}_q$. *Then the Gauss sum over* $O^+(2n,q)$,

$$\sum_{w\in O^+(2n,q)} \lambda(\operatorname{tr} w),$$

is given by

$$q^{n^2-n-1}\Big\{\sum_{r=0}^{[n/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r \end{bmatrix}_q \prod_{j=1}^{r}(q^{2j-1}-1)$$
$$\times \sum_{l=1}^{[(n-2r+2)/2]} q^l K(\lambda;1,1)^{n-2r+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)$$
$$+ \sum_{r=0}^{[(n-1)/2]} q^{r(r+1)} \begin{bmatrix} n \\ 2r+1 \end{bmatrix}_q \prod_{j=1}^{r+1}(q^{2j-1}-1)$$
$$\times \sum_{l=1}^{[(n-2r+1)/2]} q^l K(\lambda;1,1)^{n-2r+1-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\},$$

where $K(\lambda;1,1)$ *is the usual Kloosterman sum as in* (2.21), *and the first and second unspecified sums are respectively over all integers* $j_1,\ldots,j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r+1$ *and over the same set of integers satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le n-2r$.

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