ACTA ARITHMETICA
LXXXII.3 (1997)

On the irreducibility of some polynomials in two variables

by

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To the memory of Paul Erdős

Let $f(X)$ and $g(Y)$ be polynomials with integral coefficients in the single independent variables X and Y. The diophantine problem $f(x) = g(y)$ is strongly related to the absolute irreducibility and the genus of $f(X) - g(Y)$ as pointed out by Davenport, Lewis and Schinzel [DLS]:

Theorem A. *Let $f(X)$ be of degree $n > 1$ and $g(Y)$ of degree $m > 1$. Let $D(\lambda) = \operatorname{disc}(f(x) + \lambda)$ and $E(\lambda) = \operatorname{disc}(g(y) + \lambda)$. Suppose there are at least $[n/2]$ distinct roots of $D(\lambda) = 0$ for which $E(\lambda) \neq 0$. Then $f(X) - g(Y)$ is irreducible over the complex field. Further, the genus of the equation $f(x) - g(y) = 0$ is strictly positive except possibly when $m = 2$ or $m = n = 3$. Apart from these possible exceptions, the equation has at most a finite number of integral solutions.*

The purpose of this note is to handle some special cases. For an integer $k > 1$ we set

$$f_k(X) = X(X+1)\dots(X+k-1).$$

For several scattered effective and ineffective results on the equation

$$f_k(x) = f_l(y) \quad \text{in integers } x, y \tag{1}$$

we refer to [BS], [MB], [SS], [SST1], [SST2] and [Sh].

By using an algebraic number-theoretic argument we can guarantee the conditions of Theorem A in certain cases. Let I denote the set of integers k for which $f_k'(X)$ is either irreducible or it has an irreducible factor of degree $k - 2$. Our conjecture, based upon several numerical examples, is that I is the whole set of positive integers, more exactly, either $f_k'(X)$ or $f_k'(X)/(2X + k - 1)$ are irreducible depending on the parity of k. Applying

1991 *Mathematics Subject Classification*: Primary 11D41.

Research supported in part by the Hungarian Academy of Sciences, by Grants 16975, 19479 and 23992 from the Hungarian National Foundation for Scientific Research.

Eisenstein's theorem one can see that the primes belong to I and we have checked by computer that $\{1, 2, 3, \ldots, 30\} \subset I$.

THEOREM 1. *If k and l are elements of I with $2 < k < l$, then the polynomial $f_k(X) - f_l(Y)$ is irreducible (over $\mathbb{C}$) and (1) has only finitely many solutions.*

Moreover, some simple inequalities lead to

THEOREM 2. *Let k and m be integers greater than 2. Then the equation*

$$f_k(x) = \binom{y}{m} \qquad \text{in positive integers } x \text{ and } y$$

has only finitely many solutions.

Remark. Similar (effective) results in the cases $k = 2$, $l > 2$; $k = 2$, $m > 2$ and $m = 2, k > 2$ were obtained in [Y] and [SST2], respectively. These equations can be treated by Baker's method.

Proof of Theorem 1. The discriminant of the polynomial $f_k(X)+\lambda$ is denoted by $D_k(\lambda)$, i.e.

$$D_k(\lambda) = C \prod_{f_k'(x)=0} (f_k(x) + \lambda)$$

(cf. [DLS]) where C is a non-zero absolute constant. To show that $D_k(\lambda)$ and $D_l(\lambda)$ have no common zeros, we take any irrational zeros α_k and β_l of f_k' and f_l', respectively, and put

$$\mathbb{K} = \mathbb{Q}(\alpha_k, \beta_l).$$

The crucial step is that instead of the comparison of $f_k(\alpha_k)$ and $f_l(\beta_l)$ we show that their field norms with respect to $\mathbb{K}$ are not equal. If $f_k'(X)$ is irreducible, then a simple calculation yields

$$N_{\mathbb{K}/\mathbb{Q}}(f_k(\alpha_k)) = \left(\frac{f_k'(0)\ldots f_k'(1-k)}{k^k}\right)^{[\mathbb{K}:\mathbb{Q}(\alpha_k)]};$$

furthermore, if k is even then $f_k'(X)$ is always divisible by the linear factor $2X + k - 1$ and in case $k \in I$, as was pointed out by A. Schinzel, we get

$$N_{\mathbb{K}/\mathbb{Q}}(f_k(\alpha_k)) = \left(\frac{2^k f_k'(0)\ldots f_k'(1-k)}{(-1)^{k/2} k^k (k-1)!!}\right)^{[\mathbb{K}:\mathbb{Q}(\alpha_k)]}.$$

According to these formulae, for an integer $n > 2$, we write

$$a_n = \begin{cases} \left|\dfrac{f_n'(0)\ldots f_n'(1-n)}{n^n}\right|^{1/(n-1)} & \text{if } n \text{ is odd,} \\[2ex] \left|\dfrac{2^n f_n'(0)\ldots f_n'(1-n)}{n^n (n-1)!!}\right|^{1/(n-2)} & \text{if } n \text{ is even.} \end{cases}$$

For convenience, set $b_1 = b_2 = 1$ and

$$b_k = |f'_k(0) \ldots f'_k(1-k)| \quad (k > 2).$$

Since

$$f'_{k+1}(i) = (i+k)f'_k(i), \quad i = 0, -1, \ldots, 1-k, \quad |f'_{k+1}(-k)| = k!,$$

we have the recursion $b_{k+1} = b_k(k!)^2$, and therefore

$$b_k = (2! \ldots (k-1)!)^2 \quad (k > 2).$$

To prove that the sequence a_n, $n = 3, 4, \ldots$, is strictly increasing we have two cases to distinguish depending on the parity of the indices. To illustrate the tendency, $a_3, \ldots, a_{14}$ are listed below up to several digits:

$$a_3 = 0.38\ldots, \quad a_4 = 1.7\ldots, \quad a_5 = 2.2\ldots, \quad a_6 = 18.1\ldots,$$
$$a_7 = 30.1\ldots, \quad a_8 = 362.9\ldots, \quad a_9 = 711.9\ldots, \quad a_{10} = 11756.1\ldots,$$
$$a_{11} = 26250.9\ldots, \quad a_{12} = 244460.0\ldots, \quad a_{13} = 1.39\cdot 10^6, \quad a_{14} = 1.65\cdot 10^7.$$

If k is even then $a_k < a_{k+1}$ $(k > 2)$ is equivalent to

$$b_k^{2/(k(k-2))} < (k!)^{2/k} k^{k/(k-2)} ((k-1)!!)^{1/(k-2)} (k+1)^{-(k+1)/k} 2^{-k/(k-2)}$$

and in the sequel, we may assume that $k \geq 14$. By using induction we obtain

$$b_k^{2/(k(k-2))} < \frac{k^2}{8} \quad (k \geq 9). \tag{2}$$

Indeed, supposing (2) and the recursion for b_{k+1} we have to show

$$\left(\frac{k^2}{8}\right)^{k(k-2)} (k!)^4 < \left(\frac{(k+1)^2}{8}\right)^{k^2-1}. \tag{3}$$

Assuming (3) not true and applying $k! < \left(\frac{k+1}{2}\right)^k$ $(k > 2)$ we obtain

$$\frac{8^{2k-1} k^{2k^2-4k}}{2^{4k}} \geq (k+1)^{2k^2-4k-2}$$

and

$$2.6^{2k-4} > \frac{(k+1)^2}{8} 2^{2k} \geq \left(1 + \frac{1}{k}\right)^{2k^2-4k},$$

which is false for $k \geq 14$. Therefore (3), and hence (2), is proved for $k \geq 14$.

On the other hand,

$$\left(\frac{k}{e}\right)^2 < (k!)^{2/k}, \quad \left(\frac{k}{e}\right)^{1/2} < ((k-1)!!)^{1/(k-2)},$$

and

$$k^{k/(k-2)} (k+1)^{-(k+1)/k} > \left(\frac{k}{k+1}\right)^{k/(k-2)} \geq \left(\frac{14}{15}\right)^{k/(k-2)} > \left(\frac{14}{15}\right)^{7/6} > 0.92$$

imply

$$k^2/8 < \frac{k^{2.5}}{29.5} < \left(\frac{k}{e}\right)^{1/2} \cdot 0.92 \cdot 2^{-7/6}\left(\frac{k}{e}\right)^2 \qquad (k \geq 14),$$

hence $a_k < a_{k+1}$ is proved if k is even. The remaining case (k is odd) is simple. We get

$$a_k = \left(\frac{b_k}{k^k}\right)^{1/(k-1)},$$

$$a_{k+1} = \left(2^{k+1}(k!)^2 b_k (k+1)^{-(k+1)}(k!!)^{-1}\right)^{1/(k-1)}.$$

One can observe that $k! > k!!$ and

$$k^{-k} < \frac{(k+1)e}{(k+1)^{k+1}} < \frac{2^{k+1}k!}{(k+1)^{k+1}},$$

and thus, Theorem 1 is proved.

Proof of Theorem 2. The exceptional case $(k,m) = (3,3)$ is covered by a rather general result of [S1] (cf. [ST, p. 122]).

Set

$$A(k) = \begin{cases} \dfrac{(1 \cdot 3 \cdot \ldots \cdot (k-1))^2}{2^k} & \text{if } k \text{ is even,} \\[2ex] \dfrac{k \cdot (1 \cdot 3 \cdot \ldots \cdot (k-2))^2}{2^k} & \text{if } k \text{ is odd.} \end{cases}$$

As a matter of fact we prove a little more. Namely, the equation

$$af_k(x) = bf_m(y) \quad \text{in positive integers } x \text{ and } y$$

with $aA(k) > b(m-1)!$ has only finitely many solutions. To guarantee the conditions of Theorem A it is enough to show that

$$a \min_{f'_k(x)=0} |f_k(x)| > b \max_{f'_m(y)=0} |f_m(y)|. \tag{4}$$

Obviously,

$$b(m-1)! > b \max_{f'_m(y)=0} |f_m(y)|.$$

Since all the zeros of $f_k(x)$ are real, also all zeros of $f'_k(x)$ are real and, by Rolle's theorem, they alternate with the zeros of $f_k(x)$. Elementary calculus yields

$$a \min_{f'_k(x)=0} |f_k(x)| > a \min\left(\left|f_k\left(-\frac{1}{2}\right)\right|, \left|f_k\left(-\frac{3}{2}\right)\right|, \ldots, \left|f_k\left(-\frac{2k-3}{2}\right)\right|\right).$$

Since

$$a\cdot\left|f_k\left(-\frac{2j-1}{2}\right)\right| = \frac{(2(k-j)-1)!!(2j-1)!!}{2^k}$$
$$= \frac{a\cdot 1\cdot 3\cdot\ldots\cdot(2j-1)\cdot 1\cdot 3\cdot 5\cdot\ldots\cdot(2k-(2j+1))}{2^k}$$
$$\geq a\cdot A(k)\qquad (j=1,\ldots,k-1),$$

(4) is proved.

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Received on 31.12.1996
and in revised form on 9.6.1997 (3108)