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The ternary Goldbach problem in arithmetic progressions

by

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For a large odd integer N and a positive integer r, define $\mathbf{b} = (b_1, b_2, b_3)$ and

$$\mathcal{B}(N, r) = \{\mathbf{b} \in \mathbb{N}^3 : 1 \le b_j \le r, (b_j, r) = 1 \text{ and } b_1 + b_2 + b_3 \equiv N \pmod{r}\}.$$

It is known that

$$\#\mathcal{B}(N, r) = r^2 \prod_{\substack{p \mid r \\ p \mid N}} \frac{(p-1)(p-2)}{p^2} \prod_{\substack{p \mid r \\ p \nmid N}} \frac{p^2 - 3p + 3}{p^2}.$$

Let $\varepsilon > 0$ be arbitrary and $R = N^{1/8-\varepsilon}$. We prove that for all positive integers $r \le R$, with at most $O(R \log^{-A} N)$ exceptions, the Diophantine equation

$$\begin{cases} N = p_1 + p_2 + p_3, \\ p_j \equiv b_j \pmod{r}, \quad j = 1, 2, 3, \end{cases}$$

with prime variables is solvable whenever $\mathbf{b} \in \mathcal{B}(N, r)$, where $A > 0$ is arbitrary.

1. Introduction and statement of results. For given odd integers N we shall be concerned with the solubility of the equation

$$N = p_1 + p_2 + p_3 \tag{1.1}$$

in prime variables p_j; this is known as the *ternary Goldbach problem*. Hardy and Littlewood [HL] proved in 1923 that subject to the generalized Riemann hypothesis (GRH hereafter) the number $J(N)$ of solutions of (1.1) satisfies

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an asymptotic formula

$$J(N)=\sigma(N)\frac{N^2}{2\log^3 N}(1+o(1)). \tag{1.2}$$

Here $\sigma(N)$ is the singular series, and one has $\sigma(N)\gg 1$ for odd N. In 1937 Vinogradov [Vi] obtained for the first time a nontrivial estimate of exponential sums over primes, and managed to establish (1.2) unconditionally.

Since 1923, many authors have considered the corresponding problems with restrictive conditions posed on the three prime variables in (1.1). One of these generalizations was given by Rademacher [R] in 1926. For a positive integer r, define $\mathbf{b}=(b_1,b_2,b_3)$ and

$$\begin{aligned}(1.3)\quad &\mathcal{B}(N,r)\\ &=\{\mathbf{b}\in\mathbb{N}^3: 1\le b_j\le r,\ (b_j,r)=1 \text{ and } b_1+b_2+b_3\equiv N \pmod r\}.\end{aligned}$$

Then, according to Liu and Tsang [LT],

$$\#\mathcal{B}(N,r)=r^2\prod_{\substack{p\mid r\\ p\mid N}}\frac{(p-1)(p-2)}{p^2}\prod_{\substack{p\mid r\\ p\nmid N}}\frac{p^2-3p+3}{p^2}. \tag{1.4}$$

Rademacher [R] showed, subject to GRH, that if r is a fixed positive integer, and $J(N;r,\mathbf{b})$ the number of solutions of the equation

$$\begin{cases} N=p_1+p_2+p_3,\\ p_j\equiv b_j \pmod r, \quad j=1,2,3,\end{cases} \tag{1.5}$$

then we have, for odd N and all $\mathbf{b}\in\mathcal{B}(N,r)$,

$$J(N;r,\mathbf{b})=\sigma(N;r)\frac{N^2}{2\log^3 N}\left(1+o(1)\right), \tag{1.6}$$

and the singular series $\sigma(N;r)$ satisfies

$$\begin{aligned}(1.7)\qquad \sigma(N;r)=\frac{C(r)}{r^2}\prod_{p\mid r}\frac{p^3}{(p-1)^3+1}&\prod_{\substack{p\mid N\\ p\nmid r}}\frac{(p-1)((p-1)^2-1)}{(p-1)^3+1}\\ &\times\prod_{p>2}\left(1+\frac{1}{(p-1)^3}\right)\gg 1,\end{aligned}$$

where $p>2$ throughout, $C(r)=2$ for odd r, and $C(r)=8$ for even r. Following the work of Vinogradov [Vi], several authors established Rademacher's result unconditionally; see for example Ayoub [A] and Zulauf [Zu].

The arguments of [A] and [Zu] with some minor modifications actually give (1.6) for all $r\le\log^A N$, where $A>0$ is arbitrary. A natural problem is whether (1.6) is still true for larger r. The purpose of the present paper is to give a result in this direction. We prove that (1.6) is true for almost all positive moduli $r \le N^{1/8-\varepsilon}$ and all $\mathbf{b} \in \mathcal{B}(N,r)$. Precisely speaking, we have the following

THEOREM 1. *Let N be a fixed large odd integer, $\varepsilon > 0$ arbitrarily small and*

$$R = N^{1/8-\varepsilon}.$$

Let also $A > 0$ be arbitrary. For a positive integer r, define $\mathcal{B}(N,r)$ as in (1.3). *Then for all positive integers $r \le R$, with at most $O(R\log^{-A} N)$ exceptions, the Diophantine equation* (1.5) *with prime variables is solvable whenever $\mathbf{b} \in \mathcal{B}(N,r)$, and the number of solutions is given by* (1.7).

The above result is a consequence of the following mean-value theorem.

THEOREM 2. *Let N be a fixed large odd integer, $\varepsilon > 0$ arbitrarily small and*

$$R = N^{1/8-\varepsilon}.$$

Let also $A > 0$ be arbitrary. For a positive integer r, define $\mathcal{B}(N,r)$ as in (1.3). *Then*

$$\sum_{r\le R} r \max_{\mathbf{b}\in\mathcal{B}(N,r)} \Bigg| \sum_{\substack{N=n_1+n_2+n_3\\ n_j\equiv b_j \ (\mathrm{mod}\, r)}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) - \sigma(N;r)\frac{N^2}{2}\Bigg| \ll N^2\log^{-A} N, \tag{1.8}$$

where $\Lambda(n)$ denotes the von Mangoldt function.

Remark. If the r's in the theorems are restricted to primes, then the exponent $1/8$ can be improved to $3/20$. This improvement is useful in studying the ternary Goldbach problem with the three prime summands restricted to a thin subset of primes. This problem has been investigated in another paper [Li].

Since the derivation of Theorem 1 from Theorem 2 is immediate, we give it here.

Proof of Theorem 1. Let $E(R)$ be the set of positive integers $r \le R$ for which

$$\max_{\mathbf{b}\in\mathcal{B}(N,r)} \Bigg| \sum_{\substack{N=n_1+n_2+n_3\\ n_j\equiv b_j \ (\mathrm{mod}\, r)}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) - r\sigma(N;r)\frac{N^2}{2}\Bigg| > \frac{r}{\varphi^3(r)}\cdot\frac{N^2}{\log N}.$$

Then one deduces from Theorem 2 that

$$\sum_{r\in E(R)} \frac{r^2}{\varphi^3(r)} \le \log^{-A} N$$

for arbitrary $A > 0$, and consequently,

$$\#E(R) = \sum_{r\in E(R)} 1 \le R \sum_{r\in E(R)} \frac{r^2}{\varphi^3(r)} \ll R \log^{-A} N.$$

Since

$$\frac{r}{\varphi^3(r)} \ll \sigma(N;r) \ll \frac{r}{\varphi^3(r)}, \tag{1.9}$$

one sees that (1.6) is true for all $r \notin E(R)$ and all $\mathbf{b} \in \mathcal{B}(N,r)$. This completes the proof of Theorem 1.

Now it remains to establish Theorem 2.

The proof of Theorem 2 is motivated by a paper of Wolke [W], which contains several new ideas to study the problem under consideration and the ternary Goldbach problem with the prime summands restricted to a thin subset of prime numbers. His method actually gave Theorem 2 for almost all *prime* moduli $r = p \le N^{1/11}$.

The basic tool of our proof, as can be expected, is the circle method. On the minor arcs, one needs a nontrivial estimate for exponential sums over primes in arithmetic progressions to every individual and large modulus r. All known results of this kind are, however, nontrivial when the choice of minor arcs is very "thin". Consequently, the major arc is much "larger" than usual. By defining the major and minor arcs in this way, the minor arcs can then be treated easily by a result of Balog and Perelli [BP] on exponential sums over primes in an arithmetic progression (see Lemma 1 below). The main difficulty of the proof comes from the major arcs, where we use the following ideas:

(a) The starting point is Lemma 2 in §2, where we establish a new formula for

$$\sum_{\substack{n\le N\\ n\equiv b \,(\mathrm{mod}\, r)}} \Lambda(n)e(n\alpha)$$

in terms of Dirichlet characters. It plays a similar role as the formula

$$\sum_{n\le N} \Lambda(n)e\bigg(n\bigg(\frac{a}{q}+\lambda\bigg)\bigg) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} G(a,\overline{\chi}) \sum_{n\le N} \Lambda(n)e(n\lambda)$$

does in the treatment of the original ternary Goldbach problem, where $G(a,\chi)$ is the Gaussian sum defined as

$$G(a,\chi) = \sum_{n=1}^{k} \chi(n)e\bigg(\frac{an}{k}\bigg).$$

Consequently, a generalization of the Gaussian sum, namely $G(b,f,m,\chi_g,k)$ defined as in §2, occurs. We need upper estimates for $G(b,f,m,\chi_g,k)$, and these are established in §3.

(b) The treatment of the major arcs eventually reduces to the following form of mean-value estimates for exponential sums over primes:

THEOREM 3. *For any $A > 0$, there exists a constant $E = E(A) > 0$ such that if*

$$1 \le K \le x^{1/3} L^{-E}, \qquad \theta = K^{-3} L^{-E}, \tag{1.10}$$

then

$$\sum_{q \le K} \max_{y \le x} \max_{(a,q)=1} \max_{|\lambda| \le \theta} \left| \sum_{n \le y} \Lambda(n) e\left(n\left(\frac{a}{q} + \lambda\right)\right) - \frac{\mu(q)}{\varphi(q)} \sum_{n \le y} e(n\lambda) \right| \ll x L^{-A}.$$

In §4, a general result (Theorem 4) containing this theorem is established. These mean-value estimates play important roles in the proof of Theorem 2, and the exponent $1/8$ results from them.

It should be mentioned that Maier and Pomerance [MP], Balog [B] and Mikawa [Mi] studied the distribution of prime twins with one of them in arithmetic progressions. Their methods can deal with the binary Goldbach problem with one of the summands in arithmetic progressions, but we cannot apply them to the problem considered in the present paper.

We use standard notations in number theory. In particular, the letter r in the sequel stands always for *positive integers,* while L for $\log N$ except in §4 where $L = \log x$. The letter δ denotes a sufficiently small positive number, whose value may vary in different occurrences. For example, we can write

$$N^\delta L^5 \ll N^\delta, \qquad N^\delta N^\delta \ll N^\delta.$$

The expression $r \sim R$ means $\frac{1}{2}R < r \le R$. A Dirichlet character $\chi \bmod q$ will be written as χ_q if necessary.

2. Outline of the proof of Theorem 2. Let

$$R \le N^{1/8-\varepsilon}, \tag{2.1}$$

and

$$P = R^2 L^{3C}, \qquad Q = N R^{-2} L^{-4C}; \tag{2.2}$$

the constant C will be specified later. For each positive integer $r \sim R$, the major arc of the circle method is defined as

$$E_1(R) = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Since $2P < Q$, no two major arcs intersect. The minor arc is defined as

$$E_2(R) = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] - E_1(R).$$

Write $\alpha \in [0,1]$ in the form

$$\alpha = a/q + \lambda, \quad 1 \le a \le q, \quad (a,q) = 1. \tag{2.3}$$

It follows from Dirichlet's lemma on rational approximations that

$$E_2(R) = \{\alpha : P < q \le Q,\ |\lambda| \le 1/(qQ)\}.$$

Let $\Lambda(n)$ be the von Mangoldt function, $e(\alpha) = e^{2\pi i\alpha}$ as usual, and

$$S(\alpha; r, b) = \sum_{\substack{n \le N \\ n \equiv b \,(\mathrm{mod}\, r)}} \Lambda(n) e(n\alpha). \tag{2.4}$$

Then the statement of Theorem 2 is equivalent to that, for arbitrary $A > 0$,

$$\sum_{r \sim R} r \max_{\mathbf{b} \in \mathcal{B}(N,r)} \left| \int_0^1 S(\alpha; r, b_1) S(\alpha; r, b_2) S(\alpha; r, b_3) e(-N\alpha)\, d\alpha - \sigma(N; r) \frac{N^2}{2} \right| \ll N^2 L^{-A}.$$

It thus suffices to prove

$$\sum_{r \sim R} r \max_{\mathbf{b} \in \mathcal{B}(N,r)} \left| \int_{E_1(R)} S(\alpha; r, b_1) S(\alpha; r, b_2) S(\alpha; r, b_3) e(-N\alpha)\, d\alpha - \sigma(N; r) \frac{N^2}{2} \right| \ll N^2 L^{-A}, \tag{2.5}$$

and

$$\sum_{r \sim R} r \max_{\mathbf{b} \in \mathcal{B}(N,r)} \left| \int_{E_2(R)} S(\alpha; r, b_1) S(\alpha; r, b_2) S(\alpha; r, b_3) e(-N\alpha)\, d\alpha \right| \ll N^2 L^{-A}. \tag{2.6}$$

The estimate of $S(\alpha; r, b)$ with $(b,r) = 1$ on the minor arcs is given in the following lemma.

Lemma 1. *Let $A > 0$ be arbitrary and $\alpha \in E_2(R)$. If C is sufficiently large, then*

$$S(\alpha; r, b) \ll \frac{N}{r \log^A N}, \tag{2.7}$$

uniformly for $r \sim R$.

Proof. We need the following result of Balog and Perelli [BP]: For $M \le N$ and $h = (r,q)$,

$$\sum_{\substack{n \le M \\ n \equiv b \,(\mathrm{mod}\, r)}} \Lambda(n) e\left(\frac{a}{q} n\right) \ll L^3 \left(\frac{hN}{rq^{1/2}} + \frac{q^{1/2} N^{1/2}}{h^{1/2}} + \frac{N^{4/5}}{r^{2/5}} \right). \tag{2.8}$$

(A similar result was also obtained by Lavrik [La].) Now the desired estimate can be easily derived from (2.8) via partial summation.

We can now give

Proof of (2.6). It follows from Lemma 1 that the integral over $E_2(R)$ is

$$\int\limits_{E_2(R)} S(\alpha;r,b_1)S(\alpha;r,b_2)S(\alpha;r,b_3)e(-N\alpha)\,d\alpha$$
$$\ll \max_{\alpha\in E_2(R)}|S(\alpha;r,b_1)|\Big(\int\limits_0^1|S(\alpha;r,b_2)|^2\,d\alpha\Big)^{1/2}\Big(\int\limits_0^1|S(\alpha;r,b_3)|^2d\alpha\Big)^{1/2}$$
$$\ll \frac{N^2}{r^2L^{A+1}},$$

uniformly for $r\sim R$. Hence the quantity on the left-hand side of (2.6) is $\ll N^2L^{-A}$, which proves (2.6).

Theorem 2 now reduces to (2.5), which will be established in the following four sections.

The starting point of the proof of (2.5) is Lemma 2 below, which transforms the exponential sum $S(\alpha;r,b)$ into character sums. To state the lemma, we need some more notations.

Let d,f,g,k,m be fixed positive integers, and χ_g a Dirichlet character mod g. Define

$$G(d,f,m,\chi_g,k)=\sum_{\substack{n=1\\(n,k)=1\\n\equiv f\ (\mathrm{mod}\ d)}}^{k}\chi(n)e(mn/k). \tag{2.9}$$

Obviously, this is a generalization of the Gaussian sum $G(m,\chi)$.

For positive integers r and q, let

$$h=(r,q). \tag{2.10}$$

Then $r,\ q$ and h can be written as

$$r=p_1^{\alpha_1}\dots p_s^{\alpha_s}r_0,\quad (p_j,r_0)=1,$$
$$q=p_1^{\beta_1}\dots p_s^{\beta_s}q_0,\quad (p_j,q_0)=1,$$
$$h=p_1^{\gamma_1}\dots p_s^{\gamma_s},$$

where α_j,β_j and γ_j are positive integers with $\gamma_j=\min(\alpha_j,\beta_j)$, $j=1,\dots,s$. Define

$$h_1=p_1^{\delta_1}\dots p_s^{\delta_s}, \tag{2.11}$$

where $\delta_j=\alpha_j$ or 0 according as $\alpha_j=\gamma_j$ or not. Then $h_1\mid h$. Write

$$h_2=h/h_1. \tag{2.12}$$

Then

$$h_1h_2 = h, \quad (h_1,h_2) = 1, \quad \left(\frac{r}{h_1},\frac{q}{h_2}\right) = 1. \tag{2.13}$$

LEMMA 2. *Let* a,q,r *be positive integers, and* h, h_1, h_2 *defined as in* (2.10), (2.11) *and* (2.12) *respectively so that* (2.13) *holds. Then*

$$S\left(\frac{a}{q}+\lambda;r,b\right) = \frac{1}{\varphi(r/h_1)\varphi(q/h_2)} \sum_{\xi \bmod r/h_1} \overline{\xi}(b) \sum_{\eta \bmod q/h_2} G(h,b,a,\overline{\eta},q) \times \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda) + O(L^2),$$

where $G(h,b,a,\overline{\eta},q)$ *is defined as in* (2.9).

Proof. It is easily seen that

$$S\left(\frac{a}{q}+\lambda;r,b\right) = \sum_{\substack{c=1\\(c,q)=1}}^{q} e\left(\frac{ac}{q}\right) \sum_{\substack{n\le N\\ n\equiv b \,(\mathrm{mod}\, r)\\ n\equiv c\,(\mathrm{mod}\, q)}} \Lambda(n)e(n\lambda).$$

The inner sum is empty unless $c \equiv b \pmod h$; we can therefore add the restriction $c \equiv b \pmod h$ to the sum over c. On the other hand, under the condition $c \equiv b \pmod h$, the simultaneous congruences

$$n \equiv b \pmod r, \quad n \equiv c \pmod q$$

are equivalent to

$$n \equiv b \pmod{r/h_1}, \quad n \equiv c \pmod{q/h_2}$$

according to (2.13). And consequently,

$$\begin{aligned} S\left(\frac{a}{q}+\lambda;r,b\right) &= \sum_{\substack{c=1\\(c,q)=1\\ c\equiv b\,(\mathrm{mod}\, h)}}^{q} e\left(\frac{ac}{q}\right) \sum_{\substack{n\le N\\ n\equiv b \,(\mathrm{mod}\, r)\\ n\equiv c\,(\mathrm{mod}\, q)}} \Lambda(n)e(n\lambda) \\ &= \sum_{\substack{c=1\\(c,q)=1\\ c\equiv b\,(\mathrm{mod}\, h)}}^{q} e\left(\frac{ac}{q}\right) \sum_{\substack{n\le N\\ n\equiv b \,(\mathrm{mod}\, r/h_1)\\ n\equiv c\,(\mathrm{mod}\, q/h_2)}} \Lambda(n)e(n\lambda). \end{aligned}$$

Introducing the Dirichlet characters $\xi \bmod r/h_1$ and $\eta \bmod q/h_2$, one has

$$S\left(\frac{a}{q}+\lambda;r,b\right) = \frac{1}{\varphi(r/h_1)\varphi(q/h_2)} \sum_{\substack{c=1\\(c,q)=1\\ c\equiv b\,(\mathrm{mod}\, h)}}^{q} e\left(\frac{ac}{q}\right) \sum_{\xi \bmod r/h_1} \overline{\xi}(b)$$

$$\times \sum_{\eta \bmod q/h_2} \overline{\eta}(c) \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda) + O(L^2)$$

$$= \frac{1}{\varphi(r/h_1)\varphi(q/h_2)} \sum_{\xi \bmod r/h_1} \overline{\xi}(b) \sum_{\eta \bmod q/h_2} G(h,b,a,\overline{\eta},q)$$

$$\times \sum_{n\le N} \xi\eta(n)\Lambda(n)e(n\lambda) + O(L^2).$$

This proves the lemma.

3. The generalized Gaussian sum $G(d,f,m,\chi_g,k)$**.** Let d, f, g, m, k be fixed positive integers, and $\chi \bmod g$ a Dirichlet character. The purpose of this section is to give upper estimates for the sum $G(d,f,m,\chi_g,k)$ defined as in (2.9).

The main result of this section is the following

LEMMA 3. *Let* $d\,|\,k$, $g\,|\,k$ *and* $(m,k)=(f,k)=1$. *Let also* $\chi \bmod g$ *be induced by the primitive character* $\chi^* \bmod g^*$. *Then*

$$|G(d,f,m,\chi_g,k)| \le {g^*}^{1/2}.$$

In the special case $g=k$, define

$$G(d,f,m,\chi)=G(d,f,m,\chi_k,k)=\sum_{\substack{n=1\\ n\equiv f \,(\mathrm{mod}\, d)}}^{k} \chi(n)e(mn/k). \tag{3.1}$$

Then Lemma 3 is a consequence of the following

LEMMA 4. *Let* $d\,|\,k$ *and* $(m,k)=(f,k)=1$. *Let also* $\chi \bmod k$ *be induced by the primitive character* $\chi^* \bmod k^*$. *Then*

$$|G(d,f,m,\chi)| \le {k^*}^{1/2}.$$

Now we derive Lemma 3 from Lemma 4.

Proof of Lemma 3. Let χ_k^0 be the principal character mod k. Then $\chi_g\chi_k^0$ is a character mod k, and consequently,

$$G(d,f,m,\chi_g,k)=G(d,f,m,\chi_g\chi_k^0).$$

The desired result follows from Lemma 4 on noting that $\chi_g\chi_k^0 \bmod k$ is also induced by the primitive character $\chi^* \bmod g^*$.

It remains to prove Lemma 4. To this end, we investigate $G(d,f,m,\chi)$ for some special characters $\chi \bmod k$ in the following Lemmas 5–7. The proof of Lemma 4 will then be given at the end of this section.

LEMMA 5. *Let $d\,|\,k$, and $\chi \bmod k$ be primitive. Then*

$$G(d,f,m,\chi)=\frac{k}{d}\cdot\frac{1}{\tau(\overline{\chi})}e\Big(\frac{mf}{k}\Big)G\Big(\frac{k}{d},-m,f,\chi\Big), \tag{3.2}$$

and consequently,

$$|G(d,f,m,\chi)|\le k^{1/2}. \tag{3.3}$$

Here and in the sequel $\tau(\chi)$ is defined by

$$\tau(\chi)=\sum_{n=1}^{k}\chi(n)e(n/k).$$

Proof. Making the substitution $n=jd+f$, one sees that

$$\begin{aligned} G(d,f,m,\chi)&=\sum_{j=1}^{k/d}\chi(jd+f)e\Big(\frac{m(jd+f)}{k}\Big) \\ &=e\Big(\frac{mf}{k}\Big)\sum_{j=1}^{k/d}\chi(jd+f)e\Big(\frac{mj}{k/d}\Big). \end{aligned} \tag{3.4}$$

Now we appeal to the identity

$$\chi(a)=\frac{1}{\tau(\overline{\chi})}\sum_{n=1}^{k}\overline{\chi}(n)e\Big(\frac{an}{k}\Big),$$

which holds for the primitive character $\chi \bmod k$. Therefore,

$$\begin{aligned} \sum_{j=1}^{k/d}\chi(jd+f)e\Big(\frac{mj}{k/d}\Big)&=\frac{1}{\tau(\overline{\chi})}\sum_{n=1}^{k}\overline{\chi}(n)e\Big(\frac{fn}{k}\Big)\sum_{j=1}^{k/d}e\Big(\frac{njd}{k}\Big)e\Big(\frac{mj}{k/d}\Big) \\ &=\frac{1}{\tau(\overline{\chi})}\sum_{n=1}^{k}\overline{\chi}(n)e\Big(\frac{fn}{k}\Big)\sum_{j=1}^{k/d}e\Big(\frac{(n+m)j}{k/d}\Big). \end{aligned}$$

The inner sum equals k/d or 0 according as $n+m\equiv 0 \pmod{k/d}$ or not. Hence the right-hand side above is equal to

$$\frac{k}{d}\cdot\frac{1}{\tau(\overline{\chi})}\sum_{\substack{n=1\\ n\equiv -m \ (\mathrm{mod}\, k/d)}}^{k}\overline{\chi}(n)e\Big(\frac{fn}{k}\Big)=\frac{k}{d}\cdot\frac{1}{\tau(\overline{\chi})}G\Big(\frac{k}{d},-m,f,\overline{\chi}\Big).$$

This in combination with (3.4) gives (3.2).

The inequality (3.3) follows from the well-known fact that $|\tau(\chi)|=k^{1/2}$ and the trivial estimate $|G(k/d,-m,f,\overline{\chi})|\le d$. This completes the proof of the lemma.

LEMMA 6. *Let* $d\,|\,k$, $(m,k)=1$ *and* $\chi \bmod k$ *be induced by the primitive character* $\chi^* \bmod k^*$. *If* k^* *satisfies*

$$p\,|\,k \Rightarrow p\,|\,k^*,$$

then

$$|G(d,f,m,\chi)| \le {k^*}^{1/2}. \tag{3.5}$$

Proof. From the assumption of the lemma, one deduce that

$$G(d,f,m,\chi) = \sum_{\substack{n=1\\ n\equiv f \ (\mathrm{mod}\ d)}}^{k} \chi^*(n)e(mn/k).$$

The following argument is divided into 2 cases.

Special case. We start from the simplest case where $k=p^\alpha$ for some prime p and positive integer α. Since $k^*\,|\,k$ and $d\,|\,k$, we can suppose that $k^*=p^\beta$ and $d=p^\gamma$, where β and γ are integers satisfying $1\le\beta\le\alpha$ and $0\le\gamma\le\alpha$. It is obvious that one has either $k^*\,|\,d$ or $d\,|\,k^*$.

If $k^*\,|\,d$, then on setting $n=du+f$ the above sum becomes

$$\begin{aligned} G(d,f,m,\chi) &= \sum_{u=1}^{k/d} \chi^*(ud+f)e\left(\frac{m(ud+f)}{k}\right)\\ &= \chi^*(f)e\left(\frac{mf}{k}\right)\sum_{u=1}^{k/d} e\left(\frac{mu}{k/d}\right). \end{aligned}$$

Since $(m,k)=1$, the last sum vanishes, and consequently,

$$G(d,f,m,\chi)=0. \tag{3.6}$$

If $d\,|\,k^*$, then on making the substitution $n=uk^*+v$ one has

$$G(d,f,m,\chi) = \sum_{u=1}^{k/k^*}\sum_{v=1}^{k^*} \chi^*(uk^*+v)e\left(\frac{m(uk^*+v)}{k}\right),$$

where the double sums over u,v are further restricted by the condition $uk^*+v\equiv f \pmod d$. The restriction $uk^*+v\equiv f \pmod d$ is equivalent to $v\equiv f \pmod d$. Therefore the above quantity can be written as

$$\sum_{u=1}^{k/k^*} e\left(\frac{mu}{k/k^*}\right) \sum_{\substack{v=1\\ v\equiv f\ (\mathrm{mod}\ d)}}^{k^*} \chi^*(v)e\left(\frac{mv}{k}\right).$$

The first sum vanishes unless $k=k^*$, hence for $k^*\ne k$ one has

$$G(d,f,m,\chi)=0. \tag{3.7}$$

While for $k^* = k$ one obtains

$$G(d,f,m,\chi) = \sum_{\substack{v=1 \\ v\equiv f \ (\mathrm{mod}\ d)}}^{k^*} \chi^*(v) e\left(\frac{mv}{k^*}\right) = G(d,f,m,\chi^*),$$

hence by Lemma 4,

$$|G(d,f,m,\chi)| \le k^{*1/2}. \tag{3.8}$$

We therefore conclude from (3.6)–(3.8) that (3.5) holds for $k = p^\alpha$.

General case. We now turn to general k. To this end, we first prove that $G(d,f,m,\chi)$ is multiplicative with respect to k. Let $k = k_1k_2$ with $(k_1,k_2) = 1$. Then for $\chi \bmod k$ there exist a unique couple of characters $\chi_1 \bmod k_1$ and $\chi_2 \bmod k_2$ such that $\chi = \chi_1\chi_2$. Therefore, on making the substitution $n = k_2n_1 + k_1n_2$, one has

$$G(d,f,m,\chi) = \sum_{n_1=1}^{k_1} \sum_{n_2=1}^{k_2} \chi_1\chi_2(k_2n_1 + k_1n_2) e\left(\frac{m(k_2n_1 + k_1n_2)}{k_1k_2}\right), \tag{3.9}$$

where the double sums are further restricted by

$$k_2n_1 + k_1n_2 \equiv f \pmod{d}. \tag{3.10}$$

On noting that $d \mid k$, we set $d = d_1d_2$ with $d_1 \mid k_1$ and $d_2 \mid k_2$. It follows from $(k_1,k_2) = 1$ that $(d_1,d_2) = 1$, hence (3.10) is equivalent to

$$n_1 \equiv f\overline{k}_2 \pmod{d_1}, \qquad n_2 \equiv f\overline{k}_1 \pmod{d_2}, \tag{3.11}$$

where $\overline{k}_1$ and $\overline{k}_2$ are defined by $k_1\overline{k}_1 \equiv 1 \pmod{d_2}$ and $k_2\overline{k}_2 \equiv 1 \pmod{d_1}$. Now (3.9) becomes

$$\begin{aligned} G(d,f,m,\chi) &\\ = \sum_{\substack{n_1=1 \\ n_1\equiv f\overline{k}_2 \ (\mathrm{mod}\ d_1)}}^{k_1} &\chi_1(k_2n_1) e\left(\frac{mn_1}{k_1}\right) \sum_{\substack{n_2=1 \\ n_2\equiv f\overline{k}_1 \ (\mathrm{mod}\ d_2)}}^{k_2} \chi_2(k_1n_2) e\left(\frac{mn_2}{k_2}\right) \\ = \chi_1(k_2)&\chi_2(k_1) G(d_1, f\overline{k}_2, m, \chi_1) G(d_2, f\overline{k}_1, m, \chi_2). \end{aligned} \tag{3.12}$$

Now let

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$$

be the canonical decomposition of k, where p_j stands for primes, and α_j positive integers. Accordingly, k^* and d can be written as

$$k^* = p_1^{\beta_1} p_2^{\beta_2} \dots p_s^{\beta_s} \quad \text{and} \quad d = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_s^{\gamma_s},$$

where β_j and γ_j are integers satisfying $1 \le \beta_j \le \alpha_j$ and $0 \le \gamma_j \le \alpha_j$. It follows that there are primitive characters $\chi_j^* \bmod p_j^{\beta_j}$, $j = 1, \dots, s$, such that $\chi^* = \chi_1^*\chi_2^* \dots \chi_s^*$, and each $\chi_j^* \bmod p_j^{\beta_j}$ induces $\chi_j \bmod p_j^{\alpha_j}$.

Making the substitution $n = n_1K_1 + n_2K_2 + \ldots + n_sK_s$, where K_j is defined by $p_j^{\alpha_j}K_j = k$, one sees that

$$G(d,f,m,\chi) = \prod_{j=1}^{s} \chi_j(K_j) G(p_j^{\gamma_j}, f\overline{K}_j, m, \chi_j),$$

where $\overline{K}_j$ satisfies

$$\overline{K}_j K_j \equiv 1 \pmod{p_j^{\gamma_j}}, \qquad j = 1, \ldots, s.$$

It follows that

$$|G(d,f,m,\chi)| \le \prod_{j=1}^{s} |G(p_j^{\gamma_j}, f\overline{K}_j, m, \chi_j)| = \prod_{j=1}^{s} p_j^{\beta_j/2} = k^{*\,1/2}.$$

This completes the proof of the lemma.

Lemma 7. *Let $d \mid k$ and $(m,k) = (f,k) = 1$. Let also $\chi^0 \bmod k$ be the principal character. Then for $(d, k/d) > 1$,*

$$G(d,f,m,\chi^0) = 0;$$

and for $(d,k/d) = 1$,

$$G(d,f,m,\chi^0) = \mu\left(\frac{k}{d}\right) e\left(\frac{fmt}{d}\right),$$

where t is defined by $tk/d \equiv 1 \pmod{d}$.

This is Hilfssatz 2 of Rademacher [R] or Theorem 2.2 of Ayoub [A].

We can now give

Proof of Lemma 4. Let

$$k = k_1k_2 \quad \text{with} \quad (k_1,k_2) = 1, \quad k^* \mid k_1, \quad \text{and} \quad p \mid k_1 \Rightarrow p \mid k^*. \tag{3.13}$$

Then for $\chi \bmod k$ there exist a unique couple of characters $\chi_1 \bmod k_1$ and $\chi_2^0 \bmod k_2$ such that $\chi = \chi_1\chi_2^0$, where $\chi_2^0 \bmod k_2$ is the principal character. On noting that $d \mid k$, we set $d = d_1d_2$ with $d_1 \mid k_1$ and $d_2 \mid k_2$. It therefore follows from (3.12) that

$$G(d,f,m,\chi) = \chi_1(k_2)\chi_2^0(k_1) G(d_1, f\overline{k}_2, m, \chi_1) G(d_2, f\overline{k}_1, m, \chi_2^0).$$

The statement of the lemma now follows from Lemmas 6 and 7.

4. A mean-value estimate for exponential sums over primes. Wolke [W] was the first to study the mean-value estimate as in Theorem 3. He proved that Theorem 3 is true for

$$1 \le K = x^{1/4}, \qquad \theta = \min(K^{-4}, L^{-E}).$$

Actually, Theorem 3 was recently given by the authors in another joint paper [ZL] as an improvement of Wolke's result. Unfortunately, however, there is a gap in the proof of [ZL]: the statement "$h''(\beta) > 0$ for $1/2 \le \beta \le 1$" on p. 365 of [ZL] is not always true. The proof therefore needs corrections.

In this section we prove the following general result, which contains the assertion of Theorem 3. One can see from the proof of Theorem 2 that this general theorem is necessary.

THEOREM 4. *Let $z \ge 1$ be arbitrary. For any $A > 0$, there exists a constant $E = E(A) > 0$ such that if*

$$1 \le K \le z^{2/3} x^{1/3} L^{-E}, \quad \theta = z^2 K^{-3} L^{-E}, \tag{4.1}$$

then

$$\sum_{q \le K} \max_{y \le x} \max_{(a,q)=1} \max_{|\lambda| \le \theta} \left| \sum_{n \le y} \Lambda(n) e\left(n\left(\frac{a}{q} + \lambda\right)\right) - \frac{\mu(q)}{\varphi(q)} \sum_{n \le y} e(n\lambda) \right| \ll z x L^{-A}.$$

We need some lemmas to establish this result.

LEMMA 8. *Suppose that $F(u)$ and $G(u)$ are real functions defined on $[a,b]$, and $G(u)$ and $1/F'(u)$ are monotonic.*

(i) *If $|F'(u)| \gg m$ and $|G(u)| \ll M$, then*

$$\int_a^b G(u) e(F(u))\, du \ll M/m.$$

(ii) *If $|F''(u)| \gg r$ and $|G(u)| \ll M$, then*

$$\int_a^b G(u) e(F(u))\, du \ll M/\sqrt{r}.$$

For the proof of these results, see Lemmas 3.3 and 3.4 in Titchmarsh [T].

LEMMA 9. *Let $N(\sigma, T, \chi)$ be the number of zeros $\varrho = \beta + i\gamma$ of the Dirichlet L-function $L(s,\chi)$ in the rectangle $\sigma \le \beta \le 1, -T \le \gamma \le T$. Suppose $q \le 1$ and $T \le 2$. Then, for $1/2 \le \sigma \le 1$, we have*

$$\sum_{\chi \bmod q} N(\sigma, T, \chi) \ll (qT)^{3(1-\sigma)/(2-\sigma)} (\log qT)^9.$$

This is Theorem 12.1 in Montgomery [Mo].

LEMMA 10. *Let $a_n, n = 1, 2, \ldots,$ be complex numbers and $\chi \bmod q$ a character. Then*

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{T_0}^{T} \Big| \sum_{n \le N} a_n \chi(n) n^{it} \Big|^2 dt \ll (Q^2 T + N) \sum_{n \le N} |a_n|^2$$

for arbitrary Q, T_0, and T.

For this, see Theorem 7.1 in Montgomery [Mo].

LEMMA 11. *Let $\zeta(s)$ be the Riemann zeta-function, and*

$$F(s)=\sum_{n\le U}\Lambda(n)/n^s,\qquad G(s)=\sum_{n\le U}\mu(n)/n^s.$$

Then

$$\Big(-\frac{\zeta'}{\zeta}(s)-F(s)\Big)=G(s)(-\zeta'(s))-F(s)G(s)\zeta(s)\\ -(\zeta(s)G(s)-1)\Big(-\frac{\zeta'}{\zeta}(s)-F(s)\Big).$$

This is Vaughan's identity; for the proof, see [Va].

Now we can, using the idea due to Zhan [Zh], give the proof of Theorem 4.

Proof of Theorem 4. Introducing the Dirichlet characters, the exponential sum under consideration becomes

$$\sum_{n\le y}\Lambda(n)e\Big(n\Big(\frac{a}{q}+\lambda\Big)\Big)\\ =\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\sum_{n\le y}\Lambda(n)\chi(n)e(n\lambda)\sum_{h=1}^{q}\overline{\chi}(h)e\Big(\frac{ah}{q}\Big)+O(L^2),$$

and consequently,

$$\sum_{q\le Q}\max_{y\le x}\max_{|\lambda|\le\theta}\max_{(a,q)=1}\Big|\sum_{n\le y}\Lambda(n)e\Big(n\Big(\frac{a}{q}+\lambda\Big)\Big)-\frac{\mu(q)}{\varphi(q)}\sum_{n\le y}e(n\lambda)\Big| \tag{4.2}$$

$$\ll\sum_{q\le K}\max_{y\le x}\max_{|\lambda|\le\theta}\max_{(a,q)=1}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\Big|G(a,\chi)\sum_{n\le y}\Lambda(n;\chi)\chi(n)e(n\lambda)\Big|+KL^2,$$

where $G(a,\chi)$ is defined as in §3, and

$$\Lambda(n;\chi)=\begin{cases}\Lambda(n) & \text{for } \chi\ne\chi^0,\\ \Lambda(n)-1 & \text{for } \chi=\chi^0.\end{cases}$$

To estimate the sums on the right-hand side of (4.2), one notes that if the primitive character $\chi \bmod q$ induces the character $\eta \bmod k$, then $q\,|\,k$, and $|G(a,q)|\le q^{1/2}$ for $(a,q)=1$. We now combine all contributions made by an individual primitive character, so that the first term on the right-hand side of (4.2) is

$$\ll\sum_{q\le K}\max_{y\le x}\max_{|\lambda|\le\theta}\max_{(a,q)=1}\sum_{\substack{k\le K\\ q|k}}\frac{q^{1/2}}{\varphi(k)}\sideset{}{^*}\sum_{\chi \bmod q}\Big|\sum_{n\le y}\Lambda(n;\chi)\chi(n)e(n\lambda)\Big|$$

$$\ll L\sum_{q\le K}\max_{y\le x}\max_{|\lambda|\le\theta}\frac{q^{1/2}}{\varphi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\Big|\sum_{n\le y}\Lambda(n;\chi)\chi(n)e(n\lambda)\Big|.$$

Hence the assertion of the theorem reduces to

$$S := \sum_{q\sim D} \max_{y\le x} \max_{|\lambda|\le\theta} \sideset{}{^*}\sum_{\chi \bmod q} \Big| \sum_{n\sim y} \Lambda(n;\chi)\chi(n)e(n\lambda)\Big| \ll zxD^{1/2}L^{-A-3}, \tag{4.3}$$

with $1 \le D \le K$ and K,θ satisfying (4.1).

The argument leading to (4.3) falls naturally into two cases according as D is small or large. For $D \le L^F$, where F is some positive constant, one uses the classical zero-density estimate and zero-free region for the Dirichlet L-functions. While for $L^F < D \le K$, one appeals to contour integration, the large sieve inequality and Vaughan's identity.

Case 1. $D \le L^F$, where F is a positive constant to be specified later in terms of A. In this case, it suffices to prove that

$$\Sigma := \sum_{n\sim y} \Lambda(n;\chi)\chi(n)e(n\lambda) \ll zxL^{-2F-A-3} \tag{4.4}$$

for $y \le x$, $|\lambda| \le \theta$ and any primitive character χ mod d.

To estimate Σ, one appeals to the Siegel–Walfisz theorem ([D], §19):

$$\sum_{n\sim u} \Lambda(n;\chi)\chi(n)e(n\lambda) = -\sum_{|\gamma|\le T} \frac{u^\varrho}{\varrho} + b(\chi) + O\bigg(\frac{u(\log uqT)^2}{T}\bigg)$$

where $\varrho = \beta + i\gamma$ denotes nontrivial zeros of $L(s,\chi)$, $b(\chi)$ is a constant depending on χ, and $T \ge 2$ is a parameter. Applying partial summation, we have

$$\begin{aligned}\Sigma &= \int_{y/2}^{y} e(\lambda u)\, d\Big(\sum_{n\le u} \Lambda(n;\chi)\chi(n)\Big) \\ &= \sum_{|\gamma|\le T} \int_{y/2}^{y} u^{\varrho-1}e(\lambda u)\,du + O\bigg((1+|\lambda|x)\frac{xL^2}{T}\bigg).\end{aligned} \tag{4.5}$$

Take $T = x^2$, so that the O-term is acceptable in (4.4). Since, for $u \sim y$,

$$\frac{d}{du}\bigg(\lambda u + \frac{\gamma}{2\pi}\log u\bigg) = \lambda + \frac{\gamma}{2\pi u} \gg \frac{\min_{u\sim y}|\gamma + 2\pi\lambda u|}{y},$$

and

$$\frac{d^2}{du^2}\bigg(\lambda u + \frac{\gamma}{2\pi}\log u\bigg) = -\frac{\gamma}{2\pi u^2} \gg \frac{|\gamma|}{y^2},$$

we deduce from Lemma 8 that the integral on the right-hand side of (4.5) is

$$\int_{y/2}^{y} u^{\beta-1}e\bigg(\lambda u + \frac{\gamma}{2\pi}\log u\bigg)\,du \ll \min\bigg(\frac{y^\beta}{\sqrt{|\gamma|+1}}, \frac{y^\beta}{\min_{u\sim y}|\gamma+2\pi\lambda u|}\bigg).$$

Let $T_0 = 4\pi\theta x$, so that for $T_0 < |\gamma| \le x^2$ and $u \sim y$,

$$|\gamma + 2\pi\lambda u| \ge |\gamma| - 2\pi\theta u \ge |\gamma|/2.$$

Then (4.5) becomes

$$\begin{aligned}(4.6)\qquad \varSigma &\ll \sum_{|\gamma|\le x^2} \min\left(\frac{y^\beta}{\sqrt{|\gamma|+1}}, \frac{y^\beta}{\min_{u\sim y}|\gamma+2\pi\lambda u|}\right) + O(xL^{-2F-A-3}) \\ &\ll \sum_{|\gamma|\le T_0} \frac{x^\beta}{\sqrt{|\gamma|+1}} + \sum_{T_0<|\gamma|\le x^2} \frac{x^\beta}{|\gamma|} + O(xL^{-2F-A-3}).\end{aligned}$$

It is well known that for any χ mod q there is a constant $c_1 > 0$ such that $L(s,\chi)$ has no zero in the region

$$\sigma \ge 1 - \frac{c_1}{\log q + \log^{4/5}(|t|+2)},$$

except the possible Siegel zero. But the Siegel zero does not exist in the present situation, since $q \le L^F$. Therefore, one has

$$(4.7)\qquad x^{\beta-1} \ll \exp\left\{-\frac{c_1 \log x}{\log q + \log^{4/5} T}\right\} \ll \exp(-c_2 L^{1/5}),$$

for some constant $c_2 > 0$. Hence the second sum on the right-hand side of (4.6) is acceptable.

To deal with the first term, one notes that

$$\sum_{|\gamma|\le T_0} \frac{x^\beta}{\sqrt{|\gamma|+1}} \ll xL \max_{T_1\sim T_0} T_1^{-1/2} \sum_{|\gamma|\le T_1} x^{\beta-1},$$

which is, on applying Lemma 9,

$$\begin{aligned}&\ll xL \max_{T_1\le T_0} T_1^{-1/2}(\log qT_1)^9 \max_{1/2\le\sigma\le1}(qT_1)^{(3-3\sigma)/(2-\sigma)}x^{\sigma-1} \\ &\ll xL^{F+11} \max_{T_1\le T_0}\max_{1/2\le\sigma\le1} \exp\left\{-(1-\sigma)L + \left(\frac{3-3\sigma}{2-\sigma} - \frac{1}{2}\right)\log T_1\right\} \\ &=: xL^{F+11} \max_{T_1\le T_0}\max_{1/2\le\sigma\le1} f(T_1,\sigma),\end{aligned}$$

say. Therefore, in view of (4.6), the estimate (4.4) reduces to

$$(4.8)\qquad \max_{T_1\le T_0}\max_{1/2\le\sigma\le1} f(T_1,\sigma) \ll zL^{-3F-A-20}.$$

Suppose first $4/5 \le \sigma \le 1$, so that

$$\frac{3-3\sigma}{2-\sigma} \le \frac{1}{2}.$$

It follows from (4.7) that

$$(4.9)\qquad \max_{T_1\le T_0}\max_{4/5\le\sigma\le1} f(T_1,\sigma) \ll \max_{4/5\le\sigma\le1}\exp\{-(1-\sigma)L\} \ll \exp\{-c_2L^{1/5}\},$$

which is acceptable in (4.8). Now we turn to $3/5 \le \sigma \le 4/5$, which ensures that

$$\frac{3-3\sigma}{2-\sigma} \ge \frac{1}{2}.$$

On noting that $\log T_1 \le L + O(1)$, and

$$\max_{3/5\le\sigma\le4/5}\left\{-\frac{\sigma(\sigma-1/2)}{2-\sigma}\right\} = -\frac{3}{70},$$

one deduces that

$$\begin{aligned}(4.10)\qquad &\max_{T_1\le T_0}\max_{3/5\le\sigma\le4/5} f(T_1,\sigma)\\ &\ll \max_{3/5\le\sigma\le4/5}\exp\left\{-(1-\sigma)L + \left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)L\right\}\\ &= \max_{3/5\le\sigma\le4/5}\exp\left\{-\frac{\sigma(\sigma-1/2)}{2-\sigma}L\right\}\\ &\ll x^{-3/70},\end{aligned}$$

and this is also acceptable in (4.8). Finally, we consider $1/2 \le \sigma \le 3/5$. Now we have

$$\frac{6}{7} \le \frac{3-3\sigma}{2-\sigma} - \frac{1}{2},$$

and consequently,

$$\begin{aligned}&\max_{T_1\le T_0}\max_{1/2\le\sigma\le3/5} f(T_1,\sigma)\\ &\ll \max_{T_1\le T_0}\max_{1/2\le\sigma\le3/5}\exp\left\{-(1-\sigma)L + \left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)\log x\right\}\\ &\quad\times\exp\left\{-\left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)\log\frac{x}{T_1}\right\}.\end{aligned}$$

Since $T_1 \ll T_0 \ll \theta x \ll xL^{-E}$, the above quantity is

$$\begin{aligned}(4.11)\qquad &\ll \max_{1/2\le\sigma\le3/5}\exp\left\{-\frac{\sigma(\sigma-1/2)}{2-\sigma}L\right\}\exp\left\{-\frac{6}{7}E\log\log x\right\}\\ &\ll L^{-6E/7},\end{aligned}$$

which is acceptable in (4.8) if $E \ge 6F + 2A + 28$.

Combining (4.9)–(4.11) we get (4.8), hence (4.4). This proves (4.3) in Case 1.

Case 2. $L^F < D \le K$, where F is a constant to be specified in the following argument. In this case, we use Vaughan's identity to establish (4.3).

Estimation of the sum of type I. We first show that

$$\Sigma' := \sum_{d\sim D} \max_{y\le x} \max_{|\lambda|\le\theta} \sideset{}{^*}\sum_{\chi \bmod d} \Big| \sum_{\substack{mn\sim y\\ m\sim M\\ n\sim N}} a(m)b(n)\chi(mn)e(mn\lambda)\Big| \ll zxD^{1/2}L^{-A-7} \tag{4.12}$$

holds for $a(m) \ll d(m)$ and $b(n) \ll d(n)$ with $m\sim M$, $n\sim N$ and

$$x \ll MN \ll x, \quad M,N \le xD^{-1}L^{-2A-20}. \tag{4.13}$$

Let

$$f_1(s,\chi) = \sum_{m\sim M} \frac{a(m)\chi(m)}{m^s} \quad \text{and} \quad f_2(s,\chi) = \sum_{n\sim N} \frac{b(n)\chi(n)}{n^s},$$

where $s=\sigma+it$ is a complex variable. Then one sees that

$$f_1(s,\chi)f_2(s,\chi) \ll M^{1-\sigma}N^{1-\sigma} \ll x^{1-\sigma} \tag{4.14}$$

uniformly for $-2\le\sigma\le 2$. Applying Perron's summation formula (see e.g. Lemma 3.12 in [T]) and then shifting the contour to the left, one gets

$$\begin{aligned}\sum_{\substack{mn\le u\\ m\sim M\\ n\sim N}} & a(m)b(n)\chi(m)\chi(n) \\ &= \frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} f_1(s,\chi)f_2(s,\chi)\frac{u^s}{s}\,ds + O(L) \\ &= \frac{1}{2\pi i}\Big\{\int_{1+\varepsilon-ix^2}^{1/2-ix^2} + \int_{1/2-ix^2}^{1/2+ix^2} + \int_{1/2+ix^2}^{1+\varepsilon+ix^2}\Big\} f_1(s,\chi)f_2(s,\chi)\frac{u^s}{s}\,ds + O(L).\end{aligned}$$

By (4.14), the integrals on the horizontal parts are clearly $O(L)$. Therefore,

$$\begin{aligned}\sum_{\substack{mn\le u\\ m\sim M\\ n\sim N}} & a(m)b(n)\chi(m)\chi(n) \\ &= \frac{1}{2\pi}\int_{-x^2}^{x^2} f_1\Big(\frac12+it,\chi\Big)f_2\Big(\frac12+it,\chi\Big)\frac{u^{1/2+it}}{1/2+it}\,dt + O(L).\end{aligned}$$

Now, by partial summation, the inner sum of Σ' is

$$\begin{aligned}\int_{y/2}^{y} & e(\lambda u)\, d\Big\{\sum_{\substack{mn\le u\\ m\sim M\\ n\sim N}} a(m)b(n)\chi(m)\chi(n)\Big\} \\ &= \frac{1}{2\pi}\int_{-x^2}^{x^2} f_1\Big(\frac12+it,\chi\Big)f_2\Big(\frac12+it,\chi\Big)\frac{1}{1/2+it}\int_{y/2}^{y} u^{1/2+it}e(\lambda u)\,du\,dt \\ &\qquad + O((1+|\lambda|x)L)\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-x^2}^{x^2} f_1\left(\frac{1}{2}+it,\chi\right) f_2\left(\frac{1}{2}+it,\chi\right)$$
$$\times \frac{1}{1/2+it} \int_{y/2}^{y} u^{-1/2} e\left(\lambda + \frac{t}{2\pi}\log u\right) du\, dt + O(\theta x L),$$

which, by the argument leading to (4.6), is estimated as

$$\ll x^{1/2} \int_{|t|\le T_0} \left| f_1\left(\frac{1}{2}+it,\chi\right) f_2\left(\frac{1}{2}+it,\chi\right)\right| \frac{dt}{\sqrt{|t|+1}}$$
$$+ x^{1/2} \int_{T_0<|t|\le x^2} \left| f_1\left(\frac{1}{2}+it,\chi\right) f_2\left(\frac{1}{2}+it,\chi\right)\right| \frac{dt}{|t|} + O(\theta x L).$$

It therefore suffices to show

$$\sum_{d\sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2}^{T_2} \left| f_1\left(\frac{1}{2}+it,\chi\right) f_2\left(\frac{1}{2}+it,\chi\right)\right| dt \ll z x^{1/2} D^{1/2} T_2^{1/2} L^{-A-8} \tag{4.15}$$

for $1 \le T_2 \le T_0$, and

$$\sum_{d\sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_3/2}^{T_3} \left| f_1\left(\frac{1}{2}+it,\chi\right) f_2\left(\frac{1}{2}+it,\chi\right)\right| dt \ll z x^{1/2} D^{1/2} T_3 L^{-A-8} \tag{4.16}$$

for $T_0 \le T_3 \le x^2$.

The left-hand side of (4.15) is, by Cauchy's inequality and Lemma 10,

$$\ll \left\{ \sum_{d\sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2}^{T_2} \left| f_1\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \right\}^{1/2} \tag{4.17}$$
$$\times \left\{ \sum_{d\sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2}^{T_2} \left| f_2\left(\frac{1}{2}+it,\chi\right)\right|^2 dt \right\}^{1/2}$$
$$\ll (D^2T_2+M)^{1/2}(D^2T_2+N)^{1/2} L$$
$$\ll \{D^2T_2 + DT_2^{1/2}(M^{1/2}+N^{1/2}) + M^{1/2}N^{1/2}\} L$$
$$\ll z x^{1/2} D^{1/2} T_2^{1/2} L^{-A-8}$$

if $F \ge 2A+20$ and $E \ge 2A+20$. This yields (4.15).

A similar argument gives (4.16). This proves (4.12) subject to (4.13).

Estimation of the sum of type II. Next we prove that

$$\Sigma'' := \sum_{d\sim D} \max_{y\le x} \max_{|\lambda|\le\theta} \sideset{}{^*}\sum_{\chi \bmod d} \Big| \sum_{\substack{mn\sim y\\ m\sim M\\ n\sim N}} b(n)\chi(mn)e(mn\lambda)\Big| \ll zxD^{1/2}L^{-A-7} \tag{4.18}$$

holds for $b(n) \ll d(n)$ with $n\sim N$ and M,N satisfying

$$x \ll MN \ll x, \qquad M \ge DL^{2A+20}. \tag{4.19}$$

Arguing as before, one sees that it suffices to show (4.15) and (4.16) subject to (4.19). Here $f_1(s,\chi), f_2(s,\chi)$ are the same as before except that $a(m)=1$ in the definition of $f_1(s,\chi)$. Since now M is large according to (4.19), the above approach to attack the mean value of $f_1(s,\chi)$ does not work any more; one therefore needs to treat $f_1(s,\chi)$ differently.

Let $w=u+iv$ be a complex variable. Then, applying Perron's formula and then shifting the line of integration as before, one gets

$$\begin{aligned} f_1\Big(\frac12+it,\chi\Big) &= \frac{1}{2\pi i}\int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} L\Big(\frac12+it+w,\chi\Big)\frac{M^w-(M/2)^w}{w}\,dw + O(L)\\ &= \frac{1}{2\pi}\int_{-x^2}^{x^2} L\Big(\frac12+it+iv,\chi\Big)\frac{M^{iv}-(M/2)^{iv}}{iv}\,dv + O(L)\\ &\ll \int_{-x^2}^{x^2}\frac{1}{|v|+1}\Big|L\Big(\frac12+it+iv,\chi\Big)\Big|\,dv + O(L). \end{aligned}$$

Consequently, by Cauchy's inequality,

$$\begin{aligned} &\Big|f_1\Big(\frac12+it,\chi\Big)\Big|^2\\ &\quad\ll \Big\{\int_{-x^2}^{x^2}\frac{1}{|v|+1}\,dv\Big\}\Big\{\int_{-x^2}^{x^2}\frac{1}{|v|+1}\Big|L\Big(\frac12+it+iv,\chi\Big)\Big|^2\,dv\Big\} + L^2\\ &\quad\ll L\int_{-x^2}^{x^2}\frac{1}{|v|+1}\Big|L\Big(\frac12+it+iv,\chi\Big)\Big|^2\,dv + L^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{d\sim D}\sideset{}{^*}\sum_{\chi \bmod d}\int_{T_2/2}^{T_2}\Big|f_1\Big(\frac12+it,\chi\Big)\Big|^2\,dt\\ &\quad\ll L\max_{T_4\le x^2}\frac{1}{T_4}\sum_{d\sim D}\sideset{}{^*}\sum_{\chi \bmod d}\int_{T_2/2}^{T_2}\int_{T_4/2}^{T_4}\Big|L\Big(\frac12+it+iv,\chi\Big)\Big|^2\,dv\,dt + D^2T_2L^2 \end{aligned}$$

$$\ll L \max_{T_2 < T_4 \le x^2} \frac{1}{T_4} \int_{T_2/2}^{T_2} \left\{ \sum_{d \sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_4/2+t}^{T_4+t} \left| L\left(\frac{1}{2}+i\tau, \chi\right)\right|^2 d\tau \right\} dt$$

$$+ L \max_{T_4 \le T_2} \frac{1}{T_4} \int_{T_4/2}^{T_4} \left\{ \sum_{d \sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2+v}^{T_2+v} \left| L\left(\frac{1}{2}+i\tau, \chi\right)\right|^2 d\tau \right\} dv$$

$$+ D^2 T_2 L^2.$$

Applying the classical estimate

$$\sideset{}{^*}\sum_{\chi \bmod q} \int_0^T \left| L\left(\frac{1}{2}+it, \chi\right)\right|^2 dt \ll qT(\log qT)^2,$$

the quantity above is

$$\ll D^2 T_2 L^3 + D^2 T_2 L^3 + D^2 T_2 L^2 \ll D^2 T_2 L^3.$$

Hence by the argument leading to (4.17), one has

$$\sum_{\substack{mn \le u \\ m \sim M \\ n \sim N}} a(m) b(n) \chi(m) \chi(n) \ll \left\{ \sum_{d \sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2}^{T_2} \left| f_1\left(\frac{1}{2}+it, \chi\right)\right|^2 dt \right\}^{1/2}$$

$$\times \left\{ \sum_{d \sim D} \sideset{}{^*}\sum_{\chi \bmod d} \int_{T_2/2}^{T_2} \left| f_2\left(\frac{1}{2}+it, \chi\right)\right|^2 dt \right\}^{1/2}$$

$$\ll (D^2 T_2 L^3)^{1/2} (D^2 T_2 + N)^{1/2} L^{1/2}$$

$$\ll \{D^2 T_2 + D T_2^{1/2} N^{1/2}\} L^2 \ll D^{1/2} x^{1/2} T_2^{1/2} L^{-A-8}$$

if $E \ge 2A + 20$. This proves (4.15) under the condition of (4.19).

A similar argument gives (4.16). This completes the proof of (4.18) subject to (4.19).

Application of Vaughan's identity. By Lemma 11, one sees that the inner sum of S in (4.3) is equal to

$$\sum_{n \sim y} \Lambda(n) \chi(mn) e(mn\lambda) = S_1 - S_2 - S_3,$$

where

$$S_1 = \sum_{\substack{mn \sim y \\ m \le U}} \mu(m)(\log n) \chi(mn) e(mn\lambda),$$

$$S_2 = \sum_{\substack{mn \sim y \\ m \le U^2}} a(m) \chi(mn) e(mn\lambda),$$

$$S_3 = \sum_{\substack{mn \sim y \\ m > U \\ n > U}} a(m)(\log n)\chi(mn)e(mn\lambda),$$

and $a(m) \le d(m)$. Therefore,

$$(4.20) \qquad S = \sum_{q \sim D} \max_{y \le x} \max_{|\lambda \bmod q|} \sum_{\chi \bmod q}{}^{*} |S_1| + \sum_{q \sim D} \max_{y \le x} \max_{|\lambda \bmod q|} \sum_{\chi \bmod q}{}^{*} |S_2| + \sum_{q \sim D} \max_{y \le x} \max_{|\lambda \bmod q|} \sum_{\chi \bmod q}{}^{*} |S_3|.$$

Taking $U = DL^{2A+20}$ in (4.20) and $E \ge 2A + 20$ in (4.1), we have

$$U^2 = D^2 L^{4A+40} \le xD^{-1}L^{-2A-20}.$$

Hence each of the three terms on the right-hand side of (4.20) can be divided into $O(L^4)$ sums of the form Σ' or Σ''. Now, in view of the choice of E, (4.12) and (4.18) are both valid, from which the desired result (4.3) for Case 2 follows in the standard way. This completes the proof of the theorem.

5. Preparation for the major arcs. Let q, r be positive integers and

$$(5.1) \qquad (q, r) = h.$$

For $(a, q) = 1$ and $(b, r) = 1$, define

$$(5.2) \qquad f(r, q, a, b) = \begin{cases} \dfrac{\mu(q/h)}{\varphi(rq/h)} e\left(\dfrac{abt}{h}\right) & \text{if } (q/h, h) = 1,\ tq/h \equiv 1 \pmod{h}, \\ 0 & \text{if } (q/h, h) > 1. \end{cases}$$

And for $S(\alpha; r, b)$ defined by (2.4), let

$$(5.3) \qquad E(r, q, a, b, \lambda) = S\left(\frac{a}{q} + \lambda; r, b\right) - f(r, q, a, b) \sum_{n \le N} e(n\lambda),$$

$$(5.4) \qquad E^*(r, q) = \max_{(a,q)=1} \max_{(b,r)=1} \max_{|\lambda| \le 1/(qQ)} |E(r, q, a, b, \lambda)|.$$

The purpose of this section is to establish the following mean-value estimate, which plays an important role in proving (2.5), hence Theorem 2.

LEMMA 12. *Let R, P and Q be defined as in* (2.1) *and* (2.2), *while f, E and E^* as in* (5.2), (5.3) *and* (5.4). *Then for any $A > 0$, there exists a constant $C > 0$ such that*

$$\sum_{r \sim R} \sum_{q \le P} E^*(r, q) \ll NL^{-A}.$$

This estimate depends on Lemma 13 below, Lemma 3 of §3, and (4.3) of §4 which implies Theorem 4.

LEMMA 13. *Let r and q be positive integers, and h, h_1, h_2 be defined as in* (2.10), (2.11) *and* (2.12) *respectively so that* (2.13) *holds. Then for fixed positive integers r^*, q^*, one has*

$$\sum_{\substack{r\le N_1\\ r^*|r/h_1}}\sum_{\substack{q\le N_2\\ q^*|q/h_2}}\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\ll\frac{d(r^*)}{r^*q^*}\log^3 N_1\log^2 N_2. \tag{5.5}$$

Proof. Since $n\ll\varphi(n)\log n$, one has

$$\sum_{\substack{r\le N_1\\ r^*|r/h_1}}\sum_{\substack{q\le N_2\\ q^*|q/h_2}}\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\ll\log N_1\log N_2\sum_{\substack{r\le N_1\\ r^*|r/h_1}}\sum_{\substack{q\le N_2\\ q^*|q/h_2}}\frac{h_1}{r}\cdot\frac{h_2}{q}.$$

For a fixed pair r,h_1, we set $j_1=r/h_1$. To estimate the sums on the right-hand side, one needs the number of pairs q,h_2 such that the quotients q/h_2 assume the same value j_2. Since h_2 of these pairs must satisfy $h_2|r/h_1$, the required number is obviously $\le d(r/h_1)$, where $d(n)$ is the divisor function. Hence the double sum under consideration is

$$\sum_{\substack{r\le N_1\\ r^*|r/h_1}}\sum_{\substack{q\le N_2\\ q^*|q/h_2}}\frac{h_1h_2}{rq}\le\sum_{\substack{j_1\le N_1\\ r^*|j_1}}\sum_{\substack{j_2\le N_2\\ (j_2,j_1)=1\\ q^*|j_2}}\frac{d(j_1)}{j_1j_2}\le\frac{d(r^*)}{r^*q^*}\sum_{j_1\le N_1}\frac{d(j_1)}{j_1}\sum_{j_2\le N_2}\frac{1}{j_2}$$
$$\ll\frac{d(r^*)}{r^*q^*}\log^2 N_1\log N_2.$$

This proves the lemma.

We can now establish the main result of this section.

Proof of Lemma 12. By Lemma 2 we have

$$\begin{aligned}S\left(\frac{a}{q}+\lambda;r,b\right)&\\ &=\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\sum_{\xi\bmod r/h_1}\overline{\xi}(b)\sum_{\eta\bmod q/h_2}G(h,b,a,\overline{\eta},q)\\ &\quad\times\sum_{n\le N}\xi\eta(n)\Lambda(n)e(n\lambda)+O(L^2)\\ &=I+J+K+O(L^2),\end{aligned}\tag{5.6}$$

say, where I,J and K are the sums corresponding to

(i) $\xi=\xi^0 \bmod r/h_1$, $\eta=\eta^0 \bmod q/h_2$,
(j) $\xi=\xi^0 \bmod r/h_1$, $\eta\ne\eta^0 \bmod q/h_2$,
(k) $\xi\ne\xi^0 \bmod r/h_1$

respectively.

It is easily seen that

$$I=\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\sum_{\eta=\eta^0 \bmod q/h_2} G(h,b,a,\overline{\eta},q)\sum_{\substack{n\le N\\ \chi=\chi^0 \bmod rq/h}}\chi(n)\Lambda(n)e(n\lambda)$$

$$=\frac{1}{\varphi(rq/h)}G(h,b,a,\eta^0_{q/h_2})$$

$$\times\Big\{\sum_{n\le N}e(n\lambda)+\sum_{\substack{n\le N\\ \chi=\chi^0 \bmod rq/h}}\chi(n)(\Lambda(n)-1)e(n\lambda)+O\Big(\frac{L^2}{\varphi(rq/h)}\Big)\Big\}$$

$$=f(r,q,a,b)\sum_{n\le N}e(n\lambda)+O\Big(\frac{1}{\varphi(rq/h)}\sum_{n\le N}(\Lambda(n)-1)e(n\lambda)\Big)+O(L^2),$$

where we have used Lemma 5 and (5.2). Taking maxima ever λ, b and a, and then summing over q and r, one gets

$$\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}\Big|I-f(r,q,a,b)\sum_{n\le N}e(n\lambda)\Big| \tag{5.7}$$

$$\ll\sum_{r\sim R}\frac{1}{\varphi(r)}\sum_{q\le P}\max_{|\lambda|\le 1/(qQ)}\Big|\sum_{n\le N}(\Lambda(n)-1)e(n\lambda)\Big|+RPL^2$$

$$\ll L\sum_{k\le P}\max_{|\lambda|\le 1/(kQ)}\Big|\sum_{n\le N}(\Lambda(n)-1)e(n\lambda)\Big|+RPL^2.$$

We proceed to estimate J. One sees that

$$J=\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\sum_{\eta \bmod q/h_2}G(h,b,a,\overline{\eta},q)\sum_{n\le N}\xi^0\eta(n)\Lambda(n)e(n\lambda)$$

$$=\frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\sum_{\eta \bmod q/h_2}G(h,b,a,\overline{\eta},q)\sum_{n\le N}\eta(n)\Lambda(n)e(n\lambda)+O(L^2).$$

Consequently, one has

$$\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|J|$$

$$\ll\sum_{r\sim R}\frac{1}{\varphi(r/h_1)}\sum_{q\le P}\frac{1}{\varphi(q/h_2)}$$

$$\times\max_{|\lambda|\le 1/(qQ)}\sum_{\eta \bmod q/h_2}|G(h,b,a,\overline{\eta},q)|\Big|\sum_{n\le N}\eta(n)\Lambda(n,\eta)e(n\lambda)\Big|+P^{3/2}L^3.$$

To estimate the sums on the right-hand side above, one appeals to Lemma 3, which ensures that if a primitive character $\chi \bmod k$ induces a character $\overline{\eta} \bmod q/h_2$, then $k\,|\,q/h_2$ and $|G(h,b,a,\overline{\eta},q)|\le k^{1/2}$. We now combine all contributions made by an individual primitive character, which gives

$$\begin{aligned}(5.8)\quad &\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|J|\\
&\ll \sum_{k\le P}\Big\{\sum_{r\sim R}\frac{1}{\varphi(r/h_1)}\sum_{\substack{q\le P\\ k|q/h_2}}\frac{k^{1/2}}{\varphi(q/h_2)}\Big\}\\
&\quad\times\max_{|\lambda|\le 1/(kQ)}\sideset{}{^*}\sum_{\chi \bmod k}\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big| + P^{3/2}L^3\\
&\ll L^5\sum_{k\le P}\frac{1}{k^{1/2}}\max_{|\lambda|\le 1/(kQ)}\sideset{}{^*}\sum_{\chi \bmod k}\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big| + P^{3/2}L^3,\end{aligned}$$

where we have used Lemma 13 to estimate the sums in braces.

We now turn to K. One has by the definition of K,

$$\begin{aligned}K &= \frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\sum_{\substack{\xi \bmod r\\ \xi\ne\xi^0}}\sum_{\eta \bmod q/h} G(h,b,a,\overline{\eta},q)\sum_{n\le N}\xi\eta(n)\Lambda(n)e(n\lambda)\\
&\ll \frac{1}{\varphi(r/h_1)\varphi(q/h_2)}\\
&\quad\times\sum_{\substack{\xi \bmod r/h_1\\ \xi\ne\xi^0}}\sum_{\eta \bmod q/h_2}|G(h,b,a,\overline{\eta},q)|\Big|\sum_{n\le N}\xi\eta(n)\Lambda(n)e(n\lambda)\Big|.\end{aligned}$$

Working analogously to the argument above, one sees that

$$\begin{aligned}&\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|K|\\
&\qquad\ll \sum_{k_1\le 2R}\sum_{\substack{k_2\le P\\ (k_1,k_2)=1}}\Big\{\sum_{\substack{r\sim R\\ k_1|r/h_1}}\frac{1}{\varphi(r/h_1)}\sum_{\substack{q\le P\\ k_2|q/h_2}}\frac{k_2^{1/2}}{\varphi(q/h_2)}\Big\}\\
&\qquad\quad\times\max_{|\lambda|\le 1/(k_2Q)}\sideset{}{^*}\sum_{\substack{\chi_1 \bmod k_1\\ \chi_1\ne\chi_1^0}}\sideset{}{^*}\sum_{\chi_2 \bmod k_2}\Big|\sum_{n\le N}\chi_1\chi_2(n)\Lambda(n)e(n\lambda)\Big|.\end{aligned}$$

By Lemma 13, the quantity in braces is

$$\ll \frac{d(k_1)k_2^{1/2}}{k_1k_2}L^5 \ll R^\delta.$$

Hence

$$(5.9)\quad \sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|K| \ll R^{\delta}\sum_{k_1\le 2R}\sum_{\substack{k_2\le P\\(k_1,k_2)=1}}\frac{1}{k_1k_2^{1/2}} \times \max_{|\lambda|\le 1/(k_2Q)}\sideset{}{^*}\sum_{\substack{\chi_1 \bmod k_1\\ \chi_1\ne\chi_1^0}}\sideset{}{^*}\sum_{\chi_2 \bmod k_2}\Big|\sum_{n\le N}\chi_1\chi_2(n)\Lambda(n)e(n\lambda)\Big|.$$

One thus concludes from (5.6)–(5.9) that

$$(5.10)\quad \sum_{r\sim R}\sum_{q\le P}E^*(r,q)$$
$$= \sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}\Big|E(r,q,a,b)-f(r,q,a,b)\sum_{n\le N}e(n\lambda)\Big|$$
$$\ll \sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}\Big|I-f(r,q,a,b)\sum_{n\le N}e(n\lambda)\Big|$$
$$+\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|J|$$
$$+\sum_{r\sim R}\sum_{q\le P}\max_{(a,q)=1}\max_{(b,r)=1}\max_{|\lambda|\le 1/(qQ)}|K| + RPL^2+P^{3/2}L^3$$
$$\ll \frac{L^9}{R^{1/2}}\sum_{k\sim U}\max_{|\lambda|\le 1/(UQ)}\sideset{}{^*}\sum_{\chi \bmod k}\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big|$$
$$+\frac{R^{\delta}}{RU^{1/2}}\sum_{k\sim UR}\max_{|\lambda|\le 1/(UQ)}\sideset{}{^*}\sum_{\chi \bmod k}\Big|\sum_{n\le N}\chi(n)\Lambda(n;\chi)e(n\lambda)\Big|$$
$$+RPL^2+P^{3/2}L^3,$$

where

$$(5.11)\qquad U\le P=R^2L^C.$$

By (4.3) with $z=1$, the first term on the right-hand side of (5.10) is admissible if

$$(5.12)\qquad U\le N^{1/3}L^{-D},\qquad \frac{1}{UQ}\le U^{-3}L^{-D}.$$

Taking $z=R^{1/2-\delta}$ in (4.3), we see that the second term on the right-hand side of (5.10) is admissible if

$$RU \le R^{1/3-\delta} N^{1/3} L^{-D}, \quad \frac{1}{UQ} \le R^{1-\delta} (RU)^{-3} L^{-D}. \tag{5.13}$$

In view of the definitions of Q and U (see (2.2) and (5.11)), the optimal choice of R satisfying (5.12) and (5.13) is

$$R \le N^{1/8-\varepsilon}$$

as stated in (2.1). This proves the lemma.

6. The major arcs. In this section we give

Proof of (2.5). In the course of the proof, the following elementary estimate will be used: If $A_j = B + C$, $j = 1, 2, 3$, then

$$A_1 A_2 A_3 = B^3 + C(A_1^2 + B^2 + A_1 B) = B^3 + O(|C| \cdot |A_1|^2 + |C| \cdot |B|^2). \tag{6.1}$$

If $\alpha \in E_1(R)$, then for $j = 1, 2, 3$,

$$S(\alpha; r, b_j) = f(r, q, a, b_j) \sum_{n \le N} e(n\lambda) + O(E^*(r, q)). \tag{6.2}$$

Applying (6.1), one has

$$\begin{aligned} I_1(r) &:= \int\limits_{E_1(R)} S(\alpha; r, b_1) S(\alpha; r, b_2) S(\alpha; r, b_3) e(-N\alpha)\, d\alpha \\ &= \sum_{q \le P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \bigg\{ f(r, q, a, b_1) f(r, q, a, b_2) f(r, q, a, b_3) e\Big(-\frac{aN}{q}\Big) \\ &\quad \times \int\limits_{|\lambda| \le 1/(qQ)} \Big(\sum_{n \le N} e(n\lambda) \Big)^3 e(-N\lambda)\, d\lambda \\ &\quad + O\bigg(E^*(r, q) \int\limits_{|\lambda| \le 1/(qQ)} \Big| S\Big(\frac{a}{q} + \lambda; r, b_1\Big) \Big|^2 d\lambda \bigg) \\ &\quad + O\bigg(\frac{E^*(r, q)}{\varphi^2(rq/h)} \int\limits_{|\lambda| \le 1/(qQ)} \Big| \sum_{n \le N} e(n\lambda) \Big|^2 d\lambda \bigg) \bigg\}. \end{aligned}$$

The third integral on the right-hand side above is trivially $\ll N$. While the second integral, when summed over a, can be estimated as

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int\limits_{|\lambda| \le 1/(qQ)} \Big| S\Big(\frac{a}{q} + \lambda; r, b_1\Big) \Big|^2 d\lambda \le \int_0^1 \Big| S\Big(\frac{a}{q} + \lambda; r, b_1\Big) \Big|^2 d\lambda \le \frac{N}{r}.$$

On using the estimate

$$\sum_{n\le N} e(n\lambda) \ll \min(N, 1/\|\lambda\|),$$

one sees that the first integral is

$$\int_{-1/2}^{1/2} \Big(\sum_{n\le N} e(n\lambda)\Big)^3 e(-N\lambda)\,d\lambda + O\Big(\int_{1/(qQ)}^{1/2} \lambda^{-3}\,d\lambda\Big)$$
$$= \sum_{\substack{n_1+n_2+n_3=N\\ 1\le n_j\le N}} 1 + O((qQ)^2) = \frac{1}{2}N^2 + O(N^2L^{-C}).$$

We thus have

$$I_1(r) = \Big(\frac{1}{2}N^2 + O(N^2L^{-C})\Big)$$
$$\times \sum_{q\le P} \sum_{\substack{a=1\\ (a,q)=1}}^{q} f(r,q,a,b_1)f(r,q,a,b_2)f(r,q,a,b_3)e\Big(-\frac{aN}{q}\Big)$$
$$+ O\Big(\frac{N}{r}\sum_{q\le P} E^*(r,q)\Big).$$

We now consider the singular series

$$\sum_{q=1}^{\infty} \sum_{\substack{a=1\\ (a,q)=1}}^{q} f(r,q,a,b_1)f(r,q,a,b_2)f(r,q,a,b_3)e\Big(-\frac{aN}{q}\Big).$$

For $(q/h,h)>1$, one has $f(r,q,a,b_j)=0$, $j=1,2,3$, by (6.2), hence the series converges absolutely to 0. For $(q/h,h)=1$, the series reduces to

$$\frac{1}{\varphi^3(r)} \sum_{\substack{q=1\\ (q/h,h)=1}}^{\infty} \frac{\mu(q/h)}{\varphi^3(q/h)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \Big(\frac{a(b_1+b_2+b_3)t}{h} - \frac{aN}{q}\Big).$$

It was proved by Rademacher [R] that if N is odd and $\mathbf{b}\in\mathcal{B}(N,r)$, then the above series converges absolutely and equals $\sigma(N;r)$ defined as in (1.7). One therefore has

$$I_1(r) - \sigma(N;r)\frac{N^2}{2} \ll \frac{N^2}{\varphi^2(r)L^C} + \frac{N}{r}\sum_{q\le P} E^*(r,q),$$

and consequently,

$$\sum_{r\sim R} r \max_{(b_j,r)=1} \Big| I_1(r) - \sigma(N;r)\frac{N^2}{2}\Big| \ll N^2L^{-A}, \tag{6.3}$$

if C is sufficiently large. This proves (2.5), hence Theorem 2.

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