The class number one problem for the non-abelian normal CM-fields of degree 16

by

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1. Introduction. We determine in Theorems 1–3 all the non-abelian normal CM-fields $\mathbf{N}$ of degree 16 with class number one which are composita of two normal octic CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ with the same maximal totally real subfield $\mathbf{K}$: there are exactly seven such composita. According to [LouOka 2], this enables us to complete the determination of all the non-abelian normal CM-fields of degree 16 with class number one: there are twenty five such CM-fields (seventeen of them being dihedral). Indeed, according to [LouOka 2] the determination of all the non-abelian normal CM-fields of degree 16 with relative class number one is reduced to the determination of all the non-abelian normal CM-fields $\mathbf{N}$ of degree 16 with relative class number one which are composita of two normal octic CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ with the same maximal totally real subfield. Note that each extension $\mathbf{N}_i/\mathbb{Q}$ is either quaternionic, dihedral or abelian. Moreover, we proved:

THEOREM 1. *Let* $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ *be a non-abelian normal CM-field of degree* 16 *which is a compositum of two normal octic CM-fields* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *with the same maximal totally real subfield, and assume that one of the* $\mathbf{N}_i$*'s is a quaternion CM-field, say* $\mathbf{N}_2$. *Then the relative class number of* $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ *is equal to one if and only if*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-3}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-(2+\sqrt{2})(3+\sqrt{3})}).$$

Moreover, this number field $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ *has class number one.*

Therefore, we may now assume that none of the $\mathbf{N}_i$'s is a quaternion octic CM-field, and there are two cases left to cope with: both the $\mathbf{N}_i$'s are dihedral octic CM-fields (see Section 2), or one of the $\mathbf{N}_i$'s is a dihedral octic CM-field and the other is an abelian imaginary octic number field (see Section 3). We will prove:

THEOREM 2. *Let* $\mathbf{N} = \mathbf{N}_1 \mathbf{N}_2$ *be a non-abelian normal CM-field of degree* 16 *which is a compositum of two dihedral octic CM-fields* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *with the same maximal totally real subfield. Then the relative class number* $h_{\mathbf{N}}^-$ *of* $\mathbf{N}$ *is equal to one if and only if* $\mathbf{N}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5+\sqrt{17})/2})$ *(a dihedral octic CM-field, cyclic over* $\mathbb{Q}(\sqrt{34})$*) and either*

$$\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{17 + 4\sqrt{17}}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(17+3\sqrt{17})/2})$$

($\mathbf{N}^+$ *is abelian,* $\mathbf{N}_2$ *is a dihedral octic CM-field, cyclic over* $\mathbb{Q}(\sqrt{2})$*,* $h_{\mathbf{N}_2} = 2$ *and* $d_{\mathbf{N}_2} = 2^{12} \cdot 17^6$*), or*

$$\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-5(5+\sqrt{17})/2})$$

($\mathbf{N}^+$ *is abelian,* $\mathbf{N}_2$ *is a dihedral octic CM-field, cyclic over* $\mathbb{Q}(\sqrt{34})$*,* $h_{\mathbf{N}_2} = 2$ *and* $d_{\mathbf{N}_2} = 2^{12} \cdot 5^4 \cdot 17^4$*). Moreover, both these number fields* $\mathbf{N}$ *have class number one.*

THEOREM 3. *Let* $\mathbf{N} = \mathbf{N}_1 \mathbf{N}_2$ *be a non-abelian normal CM-field of degree* 16 *which is a compositum of an abelian octic CM-field* $\mathbf{N}_1$ *and of a dihedral octic CM-field* $\mathbf{N}_2$ *with the same maximal totally real subfield. Then the class number* $h_{\mathbf{N}}$ *of* $\mathbf{N}$ *is equal to one if and only if either*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-3}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-(3+\sqrt{3})})$$

(note that $\mathbf{N}_2$ *is cyclic over* $\mathbb{Q}(\sqrt{2})$*,* $h_{\mathbf{N}_2} = 2$ *and* $d_{\mathbf{N}_2} = 2^{22} \cdot 3^6$*), or*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-11}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{-(15+8\sqrt{3})})$$

(note that $\mathbf{N}_2$ *is cyclic over* $\mathbb{Q}(\sqrt{11})$*,* $h_{\mathbf{N}_2} = 2$ *and* $d_{\mathbf{N}_2} = 2^8 \cdot 3^6 \cdot 11^4$*), or*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-3}, \sqrt{2}, \sqrt{17}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5+\sqrt{17})/2})$$

(note that $\mathbf{N}_2$ *is cyclic over* $\mathbb{Q}(\sqrt{34})$*,* $h_{\mathbf{N}_2} = 1$ *and* $d_{\mathbf{N}_2} = 2^{12} \cdot 17^4$*), or*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{17}, \sqrt{-(13+2\sqrt{13})}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9+\sqrt{13})/2})$$

(note that $\mathbf{N}_2$ *is cyclic over* $\mathbb{Q}(\sqrt{221})$*,* $h_{\mathbf{N}_2} = 1$ *and* $d_{\mathbf{N}_2} = 13^4 \cdot 17^4$*).*

Note that in [LouOka 1] we proved that there are exactly seventeen non-abelian normal CM-fields of degree 8 with class number one, that in [LOO] we proved that there are exactly nine non-abelian normal CM-fields of degree 12 with class number one, and that one can easily see that a non-abelian normal CM-field of degree ≤ 16 must have degree 8, 12 or 16. Therefore, this paper completes the determination of all the non-abelian normal CM-fields of degree ≤ 16 with class number one: there are exactly thirty nine such CM-fields.

Note also that due to A. Odlyzko's [Odl] and J. Hoffstein's [Hof] results, we know that there are only finitely many normal CM-fields with class number one, and their degrees satisfy $2n \leq 436$. Moreover, K. Yamamura [Yam] determined all the abelian CM-fields with class number one: there are exactly 172 such CM-fields. Finally, according to [LOO] a non-abelian normal CM-field of degree $2n < 32$ must have degree 8, 12, 16, 20, 24 or 28, and if it has degree 20 or 28 then it must be a dihedral CM-field. However, according to [Lef] we know that there is only one dihedral CM-field of degree $4p > 12$ with class number one and it has degree 20. Therefore, up to now, we have determined all the non-abelian normal CM-fields of degree $2n < 32$ and $2n \neq 24$ with class number one. Moreover (see [LLO]), we have lately started working on the determination of all the non-abelian normal CM-fields of degree $2n = 24$ with class number one. Note that there are nine non-abelian groups of order 16, twelve non-abelian groups of order 24 and forty four non-abelian groups of order 32. Hence, it seems reasonable to be able to settle soon the class number one problem for the non-abelian normal CM-fields of degree 24. However, settling the same problem for those of degree 32 seems much more difficult, and it is quite clear that prior to solving the class number one problem for all the normal CM-fields, we need a much better upper bound on their degrees than that given by J. Hoffstein.

1.1. *Notation.* For any number field $\mathbf{E}$ we let $d_{\mathbf{E}}$, $\zeta_{\mathbf{E}}$, $h_{\mathbf{E}}$, $W_{\mathbf{E}}$ and $w_{\mathbf{E}}$ denote the absolute value of its discriminant, its Dedekind zeta function, its class number, its group of roots of unity and the order of this finite group, respectively. If $\mathbf{E} = \mathbb{Q}(\sqrt{\alpha})$ for some algebraic number α, we let $P_{\mathbf{E}}(X)$ denote the minimal polynomial of α over $\mathbb{Q}$. Moreover, if $\mathbf{E}$ is a CM-field, we let $h_{\mathbf{E}}^-$, $Q_{\mathbf{E}} \in \{1, 2\}$ and $\mathbf{E}^+$ denote its relative class number, its Hasse unit index and its maximal totally real subfield, respectively. Therefore, $\mathbf{E}/\mathbf{E}^+$ is a quadratic extension and $h_{\mathbf{E}}^- = h_{\mathbf{E}}/h_{\mathbf{E}^+}$.

LEMMA 4. *Let* $\mathbf{E}$ *be a CM-field.*

1 (see [LouOka 1, proof of Proposition 2]). *If* t *prime ideals of* $\mathbf{E}^+$ *ramify in the quadratic extension* $\mathbf{E}/\mathbf{E}^+$ *then* 2^{t-1} *divides* $h_{\mathbf{E}}^-$.

2 (see [Wa, Remark p. 185 and Exercise 10.7] and [Lou 6, Proposition 6]). *If* $Q_{\mathbf{E}} = 2$ *and* $h_{\mathbf{E}}^-$ *is odd then* $h_{\mathbf{E}}$ *is odd and* $\mathbf{E}/\mathbf{E}^+$ *is unramified at all the finite places.*

3 (see [CH, Lemma (13.5) p. 70]). *If a prime ideal of* $\mathbf{E}^+$ *lying above an odd rational prime is ramified in* $\mathbf{E}/\mathbf{E}^+$ *then* $Q_{\mathbf{E}} = 1$.

4 (see [Lou 6, Proposition 13]). *If* $\mathbf{E} = \mathbf{E}_1\mathbf{E}_2$ *is a compositum of two CM-fields* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *with the same maximal totally real subfield* $\mathbf{E}_1^+ = \mathbf{E}_2^+$, *then*

$$(1) \qquad h_{\mathbf{E}}^- = \frac{Q_{\mathbf{E}}}{Q_{\mathbf{E}_1} Q_{\mathbf{E}_2}} \cdot \frac{w_{\mathbf{E}}}{w_{\mathbf{E}_1} w_{\mathbf{E}_2}} h_{\mathbf{E}_1}^- h_{\mathbf{E}_2}^-.$$

We also refer the reader to [Wa] for the various well known results on CM-fields we will be freely using.

If p is a positive odd prime and n a relative integer, we let (n/p) denote the Legendre symbol.

Let p be any prime. A cyclic number field is called *p-primary* if its conductor is a power of the prime p. We let $\mathbf{C}_p$ denote a p-primary cyclic number field. If $q \equiv 1 \pmod 8$ is prime and if we write $q = A^2 + B^2$ with $A \geq 1$, $B \geq 1$ and B even, then

$$\mathbf{C}_q = \mathbb{Q}(\sqrt{q + B\sqrt{q}})$$

is the only q-primary real cyclic quartic field. It is real and has conductor q. If $q \equiv 5 \pmod 8$ is prime and if we write $q = A^2 + B^2$ with $A \geq 1$, $B \geq 1$ and B even, then

$$\mathbf{C}_q = \mathbb{Q}(\sqrt{-(q + B\sqrt{q})})$$

is the only q-primary cyclic quartic field. It is imaginary and has conductor q. Note that in both cases our generator β_q of $\mathbf{C}_q$ is primary in $\mathbb{Q}(\sqrt{q})$, i.e., is equal to some square modulo the principal ideal (4) of $\mathbb{Q}(\sqrt{q})$. In fact, we have $\beta_q \equiv 1 \pmod{(4)}$.

In the same way,

$$\mathbf{C}_2 = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$$

is the only 2-primary real cyclic quartic field. It has conductor 16. Moreover,

$$\mathbf{C}_2 = \mathbb{Q}(\sqrt{-(2 + \sqrt{2})})$$

is the only 2-primary imaginary cyclic quartic field. It has conductor 16.

1.2. *On Galois groups of extensions.* Let $\mathbf{M}/\mathbf{K}$ and $\mathbf{K}/\mathbf{k}$ be two quadratic extensions; assume that $\mathbf{M} = \mathbf{K}(\sqrt{\alpha})$ where $\alpha \in \mathbf{K}$ is not a square in $\mathbf{K}$. Let $\{1, \sigma\}$ denote the Galois group of the quadratic extension $\mathbf{K}/\mathbf{k}$. It is easily seen that the quartic extension $\mathbf{M}/\mathbf{k}$ is normal if and only if $\alpha^{\sigma-1}$ is a square in $\mathbf{K}$, i.e., if and only if $N_{\mathbf{K}/\mathbf{k}}(\alpha)$ is a square in $\mathbf{K}$. Now, assume that $\alpha^{\sigma-1} = \beta^2$ in $\mathbf{K}$. Then we have $\beta^{\sigma+1} = \pm 1$, and we will use the following result first to check whether a given normal octic field $\mathbf{N}$ is dihedral, and second to determine the only quadratic subfield $\mathbf{L}$ of a dihedral octic field $\mathbf{N}$ such that $\mathbf{N}/\mathbf{L}$ is cyclic quartic:

LEMMA 5 (see also [Lem, Lemma 1]). *The extension $\mathbf{M}/\mathbf{k}$ is cyclic quartic if and only if $\beta^{\sigma+1} = -1$, and the extension $\mathbf{M}/\mathbf{k}$ is bicyclic quartic if and only if $\beta^{\sigma+1} = +1$.*

P r o o f. If $\mathbf{M}/\mathbf{k}$ is cyclic quartic with Galois group of order 4 generated by τ, then $(\tau(\sqrt{\alpha}))^2 = \tau(\alpha) = \sigma(\alpha) = \beta^2 \alpha$, which yields $\tau(\sqrt{\alpha}) = \varepsilon\beta\sqrt{\alpha}$ with $\varepsilon = \pm 1$. Then, as τ^2 is the non-trivial element of the Galois group of the quadratic extension $\mathbf{M}/\mathbf{K}$, we get $-\sqrt{\alpha} = \tau^2(\sqrt{\alpha}) = \tau(\varepsilon\beta\sqrt{\alpha}) = \beta^{\sigma+1}\sqrt{\alpha}$

and $\beta^{\sigma+1} = -1$. In the same way, if $\mathbf{M}/\mathbf{k}$ is bicyclic quartic and if τ in the Galois group of this extension satisfies $\tau_{/\mathbf{K}} = \sigma$, then $\tau^2 = 1$ and $\tau(\sqrt{\alpha}) = \varepsilon\beta\sqrt{\alpha}$ with $\varepsilon = \pm 1$, and we get $\sqrt{\alpha} = \tau^2(\sqrt{\alpha}) = \tau(\varepsilon\beta\sqrt{\alpha}) = \beta^{\sigma+1}\sqrt{\alpha}$ and $\beta^{\sigma+1} = +1$. ■

In particular, let $\mathbf{k} = \mathbb{Q}(\sqrt{D})$ be a quadratic number field, take $\alpha \in \mathbf{k}$ which is not a square in $\mathbf{k}$ and set $D' = N_{\mathbf{k}/\mathbb{Q}}(\alpha)$. Then the quartic field $\mathbf{K} = \mathbf{k}(\sqrt{\alpha})$ is normal if and only if D' is a square in $\mathbf{k}$. Moreover, if D' is not a square in $\mathbf{k}$ then the normal closure $\mathbf{N}$ of $\mathbf{K}$ is a dihedral octic field, $\mathbf{N} = \mathbb{Q}(\sqrt{D}, \sqrt{D'}, \sqrt{\alpha})$ and $\mathbf{N}$ is cyclic over $\mathbf{L} = \mathbb{Q}(\sqrt{DD'})$ (apply Lemma 5). Conversely, as any dihedral octic field is the normal closure of any one of its four non-normal quartic subfields, we have:

LEMMA 6. *Let* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ *be a given real bicyclic quartic field, where* $D_1 > 0$ *and* $D_2 > 0$ *are positive square-free integers. Then* $\mathbf{N}^+$ *is the maximal abelian subfield of some dihedral octic field which is cyclic over* $\mathbf{L} = \mathbb{Q}(\sqrt{D_1 D_2})$ *if and only if the ternary quadratic form* $X^2 - D_1 Y^2 - D_2 Z^2$ *represents zero non-trivially, which amounts to asking*

- $(D_1/p_2) = +1$ *for all odd primes* p_2 *which divide* D_2 *but do not divide* D_1,
- $(D_2/p_1) = +1$ *for all odd primes* p_1 *which divide* D_1 *but do not divide* D_2, *and*
- $\left(-\frac{D_1 D_2}{p^2}/p\right) = +1$ *for all odd primes* p *which divide both* D_1 *and* D_2.

We conclude this subsection by quoting the following lemma which we will be using in Subsection 2.1 and whose proof readily follows from Lemma 5:

LEMMA 7. *Let* $\mathbf{N}_1 = \mathbf{K}(\sqrt{-\alpha_1})$ *and* $\mathbf{N}_2 = \mathbf{K}(\sqrt{-\alpha_2})$ *be two non-abelian normal octic CM-fields with the same maximal totally real subfield* $\mathbf{K}$ *(where* α_1 *and* α_2 *are totally positive elements of* $\mathbf{K}$), *let* $\mathbf{k}$ *be any quadratic subfield of* $\mathbf{K}$, *and set* $\mathbf{N} = \mathbf{N}_1 \mathbf{N}_2$. *Then* $\mathbf{N}$ *is a normal CM-field of degree 16,* $\mathbf{N}^+ = \mathbf{K}(\sqrt{\alpha_1 \alpha_2})$, *and* $\mathbf{N}^+/\mathbf{k}$ *is cyclic (quartic) if and only if exactly one of the two quartic extensions* $\mathbf{N}_1/\mathbf{k}$ *and* $\mathbf{N}_2/\mathbf{k}$ *is cyclic (quartic).*

1.3. *On dihedral octic CM-fields.* Let $\mathbf{M}$ be a dihedral octic CM-field. Let $\mathbf{K}$ denote any of the four non-normal quartic subfields of $\mathbf{M}$. According to [Lou 2], these four $\mathbf{K}$'s are CM-fields, they have Hasse unit index equal to one and

$$(2) \qquad\qquad h_{\mathbf{M}}^- = (Q_{\mathbf{M}}/2)(h_{\mathbf{K}}^-)^2.$$

Therefore, these four $\mathbf{K}$'s have the same relative class number, and $h_{\mathbf{M}}^- = 1$ if and only if $h_{\mathbf{K}}^- = 1$, and $h_{\mathbf{M}}^- = h_{\mathbf{K}}^- = 1$ implies $Q_{\mathbf{M}} = 2$. In the same way, $h_{\mathbf{M}}^-$ is odd if and only if $h_{\mathbf{K}}^-$ is odd, and $h_{\mathbf{M}}^-$ odd implies $Q_{\mathbf{M}} = 2$. Using [LouOka 1, Th. 4] [Lou 5, Th. 10] and point (4) p. 52 of [LouOka 1, Th. 4], we get:

THEOREM 8. *Let* $\mathbf{M}$ *be a dihedral octic CM-field. Then* $h_{\mathbf{M}}$ *is odd if and only if* $h_{\mathbf{M}}^-$ *is odd and* $h_{\mathbf{M}}^-$ *is odd if and only if* $h_{\mathbf{K}}^-$ *is odd. Moreover,* $h_{\mathbf{M}}^-$ *is odd if and only if* $\mathbf{M}$ *is some* $\mathbf{N}_{(p,q)}$. *Here,* p *and* q *denote two distinct primes not equal to* 3 *modulo* 4 *such that* $(p/q) = +1$ (*Kronecker symbol*) *and such that* 4 *does not divide the class number of the real quadratic field* $\mathbf{k}_+ = \mathbb{Q}(\sqrt{pq})$, *and* $\mathbf{N}_{(p,q)}$ *denotes the narrow Hilbert* 2-*class field of* $\mathbf{k}_+$. *We have* $\mathbf{N}_{(p,q)}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$, *and* $\mathbf{N}_{(p,q)}/\mathbb{Q}(\sqrt{p})$ *and* $\mathbf{N}_{(p,q)}/\mathbb{Q}(\sqrt{q})$ *are bicyclic quartic, whereas* $\mathbf{N}/\mathbb{Q}(\sqrt{pq})$ *is cyclic quartic.*

Let $\mathbf{K}_{(p,q)}$ *denote any of the two non-normal quartic subfields of* $\mathbf{N}_{(p,q)}$ *containing* $\mathbb{Q}(\sqrt{p})$, *let* $\mathbf{K}_{(q,p)}$ *denote any of the two non-normal quartic subfields of* $\mathbf{N}_{(p,q)}$ *containing* $\mathbb{Q}(\sqrt{q})$ *and let* h_p *denote the class number of* $\mathbb{Q}(\sqrt{p})$. *Then the diophantine equation* $x^2 - py^2 = 4q^{h_p}$ *has integral solutions coprime with* q, *and for any integral solutions* $x \geq 1$ *and* $y \geq 1$ *coprime with* q *of this equation we have* $\mathbf{K}_{(p,q)} = \mathbb{Q}(\sqrt{-\alpha_{(p,q)}})$ *and* $\mathbf{N}_{(p,q)} = \mathbf{N}_{(q,p)} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-\alpha_{(p,q)}}) = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-\alpha_{(q,p)}})$, *where* $\alpha_{(p,q)} = (x + y\sqrt{p})/2$.

COROLLARY 9. *Let* $\mathbf{M}$ *be a dihedral octic CM-field. Then* $h_{\mathbf{M}} = 2$ *implies* $h_{\mathbf{M}}^- = 2$.

Let us also remind the reader of the following determination:

THEOREM 10 (see [LouOka 1]). *There are exactly* 38 *non-isomorphic non-normal quartic CM-fields with relative class number* 1, *namely the* $\mathbf{K}_{(p,q)}$*'s and* $\mathbf{K}_{(q,p)}$*'s with*

$$
\begin{aligned}
(p,q) \in \{&(2, 17), (2, 73), (2, 89), (2, 233), (2, 281), \\
&(5, 41), (5, 61), (5, 109), (5, 149), (5, 269), (5, 389), \\
&(13, 17), (13, 29), (13, 157), (13, 181), \\
&(17, 137), (17, 257), (29, 53), (73, 97)\}.
\end{aligned}
$$

2. First case: $\mathbf{N}_1$ **and** $\mathbf{N}_2$ **are dihedral.** The aim of this section is to prove Theorem 2. For the remainder of this section we let $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ denote a non-abelian normal CM-field of degree 16 which is a compositum of two dihedral octic CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ with the same maximal totally real subfield. Hence, $\mathbf{N}^+$ is a totally real normal octic number field and $\mathbf{N} = \mathbf{N}^+\mathbf{N}_2$. We let $\mathbf{K}_i$ denote any one of the four non-normal quartic CM-subfields of $\mathbf{N}_i$, we let $\mathbf{k}_i$ denote the quadratic subfield of $\mathbf{K}_i$ (therefore, $\mathbf{k}_i = \mathbf{K}_i^+$) and we let $\mathbf{L}_i$ denote the quadratic subfield of $\mathbf{N}_i$ such that $\mathbf{N}_i/\mathbf{L}_i$ is cyclic quartic.

2.1. *Description of* $\mathbf{N}^+$ *when* $h_{\mathbf{N}}^-$ *is odd.* The following lemma shows that $\mathbf{N}^+$ is abelian, and Proposition 13 will then give a precise description of $\mathbf{N}^+$ when $h_{\mathbf{N}}^-$ is odd.

LEMMA 11. *The totally real octic number field $\mathbf{N}^+$ is a non-cyclic abelian octic number field. Moreover, $\mathrm{Gal}(\mathbf{N}^+/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ if $\mathbf{N}_1$ and $\mathbf{N}_2$ are cyclic over the same quadratic subfield, and $\mathrm{Gal}(\mathbf{N}^+/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ otherwise.*

P r o o f. Use Lemmas 5 and 7. See also [Lem, Lemma 1]. ∎

Since $w_{\mathbf{N}} = w_{\mathbf{N}_1} = w_{\mathbf{N}_2} = 2$, according to point 3 of Lemma 4 we have

$$(3) \qquad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}}}{2 Q_{\mathbf{N}_1} Q_{\mathbf{N}_2}} h_{\mathbf{N}_1}^- h_{\mathbf{N}_2}^-.$$

Using (2) and (3) yields

$$(4) \qquad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}}}{8} (h_{\mathbf{K}_1}^- h_{\mathbf{K}_2}^-)^2.$$

PROPOSITION 12. 1. *We may, and will, choose notation so that $h_{\mathbf{N}}^-$ is odd if and only if $h_{\mathbf{K}_1}^-$ is odd and $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$. In that situation $Q_{\mathbf{N}} = 2$, $h_{\mathbf{N}}$ is odd, $\mathbf{N}/\mathbf{N}^+$ is unramified at all the finite places, $\mathbf{N}_1^+ = \mathbf{N}_2^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ with p and q two distinct primes not equal to 3 modulo 4, and $\mathbf{N}_1/\mathbb{Q}(\sqrt{pq})$ is cyclic quartic.*
2. *$h_{\mathbf{N}}^- = 1$ if and only if $h_{\mathbf{K}_1}^- = 1$ and $h_{\mathbf{K}_2}^- = 2$.*

P r o o f. Assume that $h_{\mathbf{N}}^-$ is odd. Using (4) gives $Q_{\mathbf{N}} = 2$, which according to point 2 of Lemma 4 yields $h_{\mathbf{N}}$ odd and $\mathbf{N}/\mathbf{N}^+$ is unramified at all the finite places, and Theorem 8 yields the desired description of $\mathbf{N}_1^+$. ∎

PROPOSITION 13. *Assume that $h_{\mathbf{N}}^-$ is odd. Then the abelian octic field $\mathbf{N}^+$ is its own narrow genus field and one of the following two assertions holds*:

1. *$\mathbf{N}^+ = \mathbf{C}_p \mathbf{C}_q$ is the compositum of two primary real cyclic fields, one of them being quadratic and the other quartic. We choose notation so that $\mathbf{C}_p$ is quadratic and $\mathbf{C}_q$ is cyclic quartic (which implies $q = 2$ or $q \equiv 1 \pmod 8$). Then $\mathbf{N}_1/\mathbb{Q}(\sqrt{pq})$ and $\mathbf{N}_2/\mathbb{Q}(\sqrt{p})$ are cyclic quartic. Therefore, we may assume that both $\mathbf{K}_1$ and $\mathbf{K}_2$ are quadratic extensions of $\mathbb{Q}(\sqrt{q})$, i.e., we may assume that $\mathbf{K}_1^+ = \mathbf{K}_2^+ = \mathbb{Q}(\sqrt{q})$. Therefore, $\mathbf{K}_1 = \mathbf{K}_{(q,p)} = \mathbb{Q}(\sqrt{-\alpha_{(q,p)}})$ with $\alpha_{(q,p)} \in \mathbf{K}_1^+ = \mathbb{Q}(\sqrt{q})$ as in Theorem 8, and if $\mathbf{C}_q = \mathbb{Q}(\sqrt{\beta_q})$ with $\beta_q = q + B\sqrt{q}$ as in Subsection 1.1 then $\mathbf{K}_2 = \mathbb{Q}(\sqrt{-\alpha_{(q,p)}\beta_q})$.*
2. *$\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l})$ for some prime l not equal to 3 modulo 4. Then $\mathbf{N}_1/\mathbb{Q}(\sqrt{pq})$ and $\mathbf{N}_2/\mathbb{Q}(\sqrt{pq})$ are both cyclic quartic, and l is inert in both $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$. Therefore, we may assume that both $\mathbf{K}_1$ and $\mathbf{K}_2$ are quadratic extensions of $\mathbb{Q}(\sqrt{p})$, i.e., we may assume that $\mathbf{K}_1^+ = \mathbf{K}_2^+ = \mathbb{Q}(\sqrt{p})$. Therefore, $\mathbf{K}_1 = \mathbf{K}_{(p,q)} = \mathbb{Q}(\sqrt{-\alpha_{(p,q)}})$ with $\alpha_{(p,q)} \in \mathbf{K}_1^+ = \mathbb{Q}(\sqrt{p})$ as in Theorem 8, and $\mathbf{K}_2 = \mathbb{Q}(\sqrt{-l\alpha_{(p,q)}})$.*

Proof. Let $\mathbf{G}$ denote the narrow genus field of $\mathbf{N}^+$, i.e., $\mathbf{G}$ is the maximal abelian number field containing $\mathbf{N}^+$ and such that the extension $\mathbf{G}/\mathbf{N}^+$ is unramified at all the finite places. Since the degree of $\mathbf{N}^+$ is a power of two, so is the degree of $\mathbf{G}$. Since the class number of $\mathbf{N}$ is odd (first point of Proposition 12), $\mathbf{G}$ is included in $\mathbf{N}$. Indeed, $\mathbf{GN}/\mathbf{N}$ being an abelian extension of 2-power degree unramified at all places, we have $\mathbf{GN} \subseteq \mathbf{N}$ and $\mathbf{G} \subseteq \mathbf{N}$. Since $\mathbf{N}^+ \subseteq \mathbf{G} \subseteq \mathbf{N}$ and since $\mathbf{N}$ is non-abelian, we get $\mathbf{G} = \mathbf{N}^+$. Finally, using $\mathbf{N}_2^+ = \mathbf{N}_1^+ \subseteq \mathbf{N}^+$ and the description of $\mathbf{N}_1^+ = \mathbf{N}_2^+$ in the first point of Proposition 12 we get the desired results on the description of $\mathbf{N}^+$. Finally, as $\mathbf{N}_1/\mathbb{Q}(\sqrt{pq})$ is cyclic quartic (first point of Proposition 12), using Lemma 7 we can easily determine which quadratic field $\mathbf{L}$ out of the three quadratic subfields of $\mathbf{N}_2^+ = \mathbf{N}_1^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ is the one such that $\mathbf{N}_2/\mathbf{L}$ is cyclic quartic. It now only remains to prove that l is inert both in $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$ when $\mathbf{N}^+$ is as in point 2 of this Proposition 13. Clearly, it suffices to prove that l is inert in $\mathbb{Q}(\sqrt{p})$.

Hence, we assume that

$$\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l}).$$

Since $\mathbf{N}/\mathbf{N}^+$ is unramified at all the finite places (first point of Proposition 12), the index of ramification of any prime which is ramified in $\mathbf{N}/\mathbb{Q}$ is equal to 2. In particular, $\mathbf{N}_i^+/\mathbf{L}_i$ is unramified. Therefore, $\mathbf{L}_1 = \mathbf{L}_2 = \mathbb{Q}(\sqrt{pq})$ and we may choose $\mathbf{K}_1$ and $\mathbf{K}_2$ such that $\mathbf{k}_1 = \mathbf{k}_2 = \mathbb{Q}(\sqrt{p})$. Now, let $\mathcal{L}$ denote any prime ideal of $\mathbf{N}_2$ lying above l. Since l is not ramified in $\mathbf{N}_2^+/\mathbb{Q}$, $\mathbf{N}_2^+$ is the inertia field of $\mathcal{L}$. Therefore, for any subfield $\mathbf{M}$ of $\mathbf{N}_2$, $\mathcal{L}$ is ramified in $\mathbf{N}_2/\mathbf{M}$ if and only if $\mathbf{M} \subseteq \mathbf{N}_2^+$. Hence, all the prime ideals of $\mathbf{k}_2$ lying above l are ramified in $\mathbf{K}_2/\mathbf{k}_2$. Moreover, as 2^{t_2-1} divides $h_{\mathbf{K}_2}^-$ where t_2 denotes the number of prime ideals of $\mathbf{k}_2$ which are ramified in the quadratic extension $\mathbf{K}_2/\mathbf{k}_2$, it follows that $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$ implies $t_2 \leq 2$ (point 1 of Lemma 4). Since one of the prime ideals of $\mathbf{k}_2$ lying above q (which splits in $\mathbf{k}_2/\mathbb{Q}$ and is ramified in $\mathbf{N}^+/\mathbb{Q}$) is ramified in $\mathbf{K}_2/\mathbf{k}_2$, the congruence $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$ implies that l is inert in $\mathbf{k}_2$. ∎

Therefore, to prove Theorem 2, for each possible $\mathbf{K}_1$ with $h_{\mathbf{K}_1}^- = 1$ (given in Theorem 10), we determine all the possible $\mathbf{K}_2$'s with $h_{\mathbf{K}_2}^- = 2$, and we finally compute $h_{\mathbf{N}^+}$. According to Proposition 13, the determination of all the possible $\mathbf{K}_2$'s falls naturally into two subcases:

1. $\mathbf{N}^+ = \mathbf{C}_p\mathbf{C}_q$, in which case $\mathbf{K}_2$ is well determined by $\mathbf{K}_1$. Subsection 2.2 is devoted to settling this case.

2. $\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l})$, in which case $\mathbf{K}_2$ is well determined by $\mathbf{K}_1$ and l, so that we will need an upper bound on l when $h_{\mathbf{K}_2}^- = 2$ to end up with a finite list of $\mathbf{K}_2$'s. Subsection 2.3 is devoted to settling this case.

2.2. *Determination of all* **N***'s with* $h_{\mathbf{N}}^- = 1$ *and* $\mathbf{N}^+$ *as in Proposition* 13, *point* 1. Since $h_{\mathbf{K}_1}^- = 1$ (second point of Proposition 12) and since we have assumed that both $\mathbf{K}_1$ and $\mathbf{K}_2$ are quadratic extensions of $\mathbb{Q}(\sqrt{q})$ (first point of Proposition 13), we have $\mathbf{K}_1 = \mathbf{K}_{(q,p)} = \mathbb{Q}(\sqrt{-\alpha_{(q,p)}})$, with $\alpha_{(q,p)} \in \mathbb{Q}(\sqrt{q})$ as in Theorem 8, where (p,q) or (q,p) must appear in Theorem 10 and where we must have $q = 2$ or $q \equiv 1 \pmod 8$ (first point of Proposition 13). Therefore, there are only 18 possible values for the pair (p,q):

$$(p,q) \in \{(2,17),(2,73),(2,89),(2,233),(2,281),$$
$$(17,2),(73,2),(89,2),(233,2),(281,2),$$
$$(5,41),(13,17),(17,137),(17,257),$$
$$(73,97),(97,73),(137,17),(257,17)\}.$$

Moreover, $\mathbf{N}^+ = \mathbf{C}_p \mathbf{C}_q$ with $\mathbf{C}_p$ quadratic and $\mathbf{C}_q = \mathbb{Q}(\sqrt{\beta_q})$ cyclic quartic (with $\beta_q = q + B\sqrt{q}$ as in Subsection 1.1), and $\mathbf{K}_2 = \mathbb{Q}(\sqrt{-\alpha_{(q,p)}\beta_q})$ (see Proposition 13). Now, using [Lou 4], we can easily compute the relative class numbers of all these possible $\mathbf{K}_2$'s and we get Table 1 of Section 4 according to which $h_{\mathbf{N}}^- = 1$ if and only if $(p,q) = (2,17)$. In that case, we have

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-\alpha_{(17,2)}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5+\sqrt{17})/2}),$$

$$\mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-\alpha_{(17,2)}\beta_{17}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(153+37\sqrt{17})/2})$$

$$= \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(17+3\sqrt{17})/2})$$

for we have $153 + 37\sqrt{17} = (4+\sqrt{17})^2(17-3\sqrt{17})$, and

$$\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{\beta_{17}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17+4\sqrt{17}}),$$

$d_{\mathbf{N}^+} = 2^{12} \cdot 17^6$, and $h_{\mathbf{N}^+} = 1$.

2.3. *Determination of all* **N***'s with* $h_{\mathbf{N}}^- = 1$ *and* $\mathbf{N}^+$ *as in Proposition* 13, *point* 2. In this case, since the requirement $h_{\mathbf{K}_1}^- = 1$ only determines p and q (use Theorem 10), we need an upper bound on l to get a finite list of possible fields $\mathbf{K}_2$. We will get it thanks to lower bounds on relative class numbers of non-normal quartic CM-fields. The lower bounds of the next subsection are much better than the ones we got in [Lou 2].

2.3.1. *Lower bounds on relative class numbers of some octic and quartic CM-fields*

THEOREM 14. *Let* **M** *be a dihedral octic CM-field. Then* $d_{\mathbf{M}} \geq 7 \cdot 10^{14}$ *implies*

$$(5) \qquad h_{\mathbf{M}}^- \geq \frac{Q_{\mathbf{M}}\sqrt{d_{\mathbf{M}}/d_{\mathbf{M}^+}}}{9(\log d_{\mathbf{M}^+} + 0.14)^3 \log d_{\mathbf{M}}}.$$

P r o o f. First, according to [Lou 3, Proposition A], $\beta \in [1-(2/\log d_{\mathbf{M}}), 1[$ and $\zeta_{\mathbf{M}}(\beta) \leq 0$ imply

$$(6) \qquad\qquad \mathrm{Res}_{s=1}(\zeta_{\mathbf{M}}) \geq \varepsilon_{\mathbf{M}} \frac{1-\beta}{e},$$

where $\varepsilon_{\mathbf{M}} = 1 - (8\pi e^{1/4}/d_{\mathbf{M}}^{1/8})$ is very close to 1 when $d_{\mathbf{M}}$ is large. Second, set $c = (2 + \gamma - \log(4\pi))/2 = 0.0230957\ldots$ where $\gamma = 0.577215\ldots$ denotes Euler's constant. Then, according to [Lou 1], for any even, primitive Dirichlet character modulo $f > 1$ we have

$$(7) \qquad\qquad L(1, \chi) \leq \frac{1}{2} \log f + c.$$

Third, according to [Lou 7], for any even, primitive Dirichlet character modulo $f > 1$, $\beta \in \,]1/2, 1[$ and $L(\beta, \chi) = 0$ imply

$$(8) \qquad\qquad L(1, \chi) \leq \frac{1-\beta}{8} \log^2 f.$$

Fourth, we have

$$(9) \qquad\qquad h_{\mathbf{M}}^- = \frac{Q_{\mathbf{M}} w_{\mathbf{M}}}{(2\pi)^4} \sqrt{\frac{d_{\mathbf{M}}}{d_{\mathbf{M}^+}}} \cdot \frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}^+})}.$$

Here, $Q_{\mathbf{M}} \in \{1, 2\}$ denotes the Hasse unit index of $\mathbf{M}$ and $w_{\mathbf{M}} \geq 2$ denotes the number of roots of unity in $\mathbf{M}$. In fact, we have $w_{\mathbf{M}} = 2$.

Now, assume that $\zeta_{\mathbf{M}^+}(1 - (2/\log d_{\mathbf{M}})) \leq 0$. Since $\zeta_{\mathbf{M}}/\zeta_{\mathbf{M}^+} = (\zeta_{\mathbf{K}}/\zeta_{\mathbf{k}})^2$ where $\mathbf{k}$ is the quadratic subfield of any one of the four non-normal quartic subfields $\mathbf{K}$ of $\mathbf{M}$ (see [Lou 2]), we have $\zeta_{\mathbf{M}}(s) \leq 0$ and using (6) we get

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}}) \geq \varepsilon_{\mathbf{M}} \frac{2}{e \log d_{\mathbf{M}}}.$$

On the other hand, let f_1, f_2 and f_3 denote the conductors of the three real quadratic subfields of $\mathbf{M}^+$. Using (7) and $d_{\mathbf{M}^+} = f_1 f_2 f_3$ yields

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}^+}) \leq \frac{1}{8}(\log f_1 + 2c)(\log f_2 + 2c)(\log f_3 + 2c)$$

$$\leq \frac{1}{8}\left(\frac{\log f_1 f_2 f_3 + 6c}{3}\right)^3 = \frac{1}{216}(\log d_{\mathbf{M}^+} + 6c)^3.$$

Finally, using (9) yields

$$(10) \qquad\qquad h_{\mathbf{M}}^- \geq \varepsilon_{\mathbf{M}} \frac{54}{e\pi^4} \cdot \frac{Q_{\mathbf{M}}\sqrt{d_{\mathbf{M}}/d_{\mathbf{M}^+}}}{(\log d_{\mathbf{M}^+} + 6c)^3 \log d_{\mathbf{M}}}.$$

On the contrary, now assume that $\zeta_{\mathbf{M}^+}(1 - (2/\log d_{\mathbf{M}})) > 0$. Then there exists $\beta \in \,]1-(2/\log d_{\mathbf{M}}), 1[$ such that $\zeta_{\mathbf{M}^+}(\beta) = 0$, which implies $\zeta_{\mathbf{M}}(\beta) = 0$ and $\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}}) \geq \varepsilon_{\mathbf{M}}(1 - \beta)/e$. Moreover, there exists $i \in \{1, 2, 3\}$ such

that $L(\beta, \chi_i) = 0$. Using (7), (8) and $f_1 \leq d_{\mathbf{M}}^4$ yields

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{M}^+}) \leq \frac{1-\beta}{32} \log^2 f_1 (\log f_2 + 2c)(\log f_3 + 2c)$$

$$\leq \frac{1-\beta}{32} \log f_1 \left(\frac{\log f_1 f_2 f_3 + 4c}{3}\right)^3$$

$$\leq \frac{1-\beta}{2^7 \cdot 3^3} (\log d_{\mathbf{M}^+} + (4c/3))^3 \log d_{\mathbf{M}}.$$

Finally, using (9) yields

$$(11) \qquad h_{\mathbf{M}} \geq \varepsilon_{\mathbf{M}} \frac{432}{e\pi^4} \cdot \frac{Q_{\mathbf{M}}\sqrt{d_{\mathbf{M}}/d_{\mathbf{M}^+}}}{(\log d_{\mathbf{M}^+} + (4c/3))^3 \log d_{\mathbf{M}}}.$$

Since (11) is better than (10), the latter always holds. Noticing that we have $54\varepsilon_{\mathbf{M}}/(e\pi^4) > 1/9$ for $d_{\mathbf{M}} \geq 7 \cdot 10^{14}$, we get the desired result. ∎

COROLLARY 15. *Let* $\mathbf{K}$ *be a non-normal quartic CM-field. Then* $d_{\mathbf{K}} \geq 3 \cdot 10^7$ *yields*

$$(12) \qquad h_{\mathbf{K}}^- \geq \frac{\sqrt{d_{\mathbf{K}}/d_{\mathbf{K}^+}}}{3(\log(d_{\mathbf{K}}/d_{\mathbf{K}^+}) + 0.104)^2}.$$

Therefore, $d_{\mathbf{K}}/d_{\mathbf{K}^+} \geq 3 \cdot 10^5$ *implies* $h_{\mathbf{K}}^- > 1$*, and* $d_{\mathbf{K}}/d_{\mathbf{K}^+} \geq 2 \cdot 10^6$ *implies* $h_{\mathbf{K}}^- > 2$.

Proof. Let $\mathbf{M}$ be the normal closure of $\mathbf{K}$. According to [Lou 2] we have $h_{\mathbf{K}}^- = \sqrt{2h_{\mathbf{M}}^-/Q_{\mathbf{M}}}$ and $d_{\mathbf{M}}/d_{\mathbf{M}^+} = (d_{\mathbf{K}}/d_{\mathbf{K}^+})^2$. Now, noticing that

$$(\log d_{\mathbf{M}^+} + 6c) \log d_{\mathbf{M}} \leq 2(\log(d_{\mathbf{M}}/d_{\mathbf{M}^+}) + 3c)^2$$

and

$$(\log d_{\mathbf{M}^+} + 6c)^2 \leq (\log(d_{\mathbf{M}}/d_{\mathbf{M}^+}) + 6c)^2,$$

and using (5) and $9c/2 < 0.104$ we obtain the desired result. ∎

2.3.2. *The required computations.* For each of the 19 possible (p, q) with $2 \leq p < q$ for which $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is the maximal totally real subfield of a dihedral octic CM-field $\mathbf{N}_1$ with relative class number one (see Theorem 10), we take $\mathbf{K}_1 = \mathbf{K}_{(p,q)} = \mathbb{Q}(\sqrt{-\alpha_{(p,q)}})$ with $\alpha_{(p,q)} \in \mathbb{Q}(\sqrt{q})$ as in Theorem 8. Therefore, $\mathbf{K}_2 = \mathbb{Q}(\sqrt{-l\alpha_{(p,q)}})$. Then we use (12) to get a bound $B_{p,q}$ on l inert in $\mathbb{Q}(\sqrt{p})$ and in $\mathbb{Q}(\sqrt{q})$ such that $l > B_{p,q}$ implies $h_{\mathbf{K}_2}^- > 2$ (note that if $l \neq 2$ then $d_{\mathbf{K}_2}/d_{\mathbf{K}_2^+} = d_{\mathbf{k}}l^2$ where $\mathbf{k} = \mathbb{Q}(\sqrt{pq})$). For example, if $(p, q) = (2, 17)$, then Corollary 15 gives $h_{\mathbf{K}_2}^- > 2$ if $d_{\mathbf{K}_2}/d_{\mathbf{K}_2^+} = d_{\mathbf{k}}l^2 = 136l^2 > 2 \cdot 10^6$, hence if $l > 121 = B_{2,17}$ and there are only five primes l not equal to 3 modulo 4 such that $l \leq 121 = B_{2,17}$ and $(2/l) = (17/l) = -1$,

namely $l \in \{5, 29, 37, 61, 109\}$. For the remaining possible values of (p, q, l) we use [Lou 4] to compute $h_{\mathbf{K}_2}^-$. For example, if $(p, q) = (2, 17)$, we get

(p, q, l)	$P_{\mathbf{K}_2}(X)$	$d_{\mathbf{K}_2}$	$h_{\mathbf{K}_2}^-$
$(2, 17, 5)$	$X^4 + 50X^2 + 425$	$2^6 \cdot 5^2 \cdot 17$	2
$(2, 17, 29)$	$X^4 + 290X^2 + 14297$	$2^6 \cdot 29^2 \cdot 17$	10
$(2, 17, 37)$	$X^4 + 370X^2 + 23273$	$2^6 \cdot 37^2 \cdot 17$	14
$(2, 17, 61)$	$X^4 + 610X^2 + 63257$	$2^6 \cdot 61^2 \cdot 17$	22
$(2, 17, 109)$	$X^4 + 1090X^2 + 201977$	$2^6 \cdot 109^2 \cdot 17$	42

Table 2 in Section 4 provides the reader with some of our computations: for each possible (p, q) it gives the value of the relative class number of $\mathbf{K}_2$ for the smallest possible $l > 1$. Therefore, we have $h_{\mathbf{N}}^- = 1$ if and only if $(p, q, l) = (2, 17, 5)$. In that case, we have

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-\alpha_{(2,17)}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5 + 2\sqrt{2})})$$

$$= \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-\alpha_{(17,2)}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5 + \sqrt{17})/2})$$

and

$$\mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-5\alpha_{(2,17)}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-5(5 + 2\sqrt{2})})$$

$$= \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-5\alpha_{(17,2)}}) = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-5(5 + \sqrt{17})/2})$$

and

$$\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17}),$$

$d_{\mathbf{N}^+} = 2^{12} \cdot 5^4 \cdot 17^4$, and $h_{\mathbf{N}^+} = 1$. Note that since

$$\alpha_{(2,17)}\alpha_{(17,2)} = (5 + \sqrt{2})\frac{5 + \sqrt{17}}{2} = \left(\frac{5 + 2\sqrt{2} + \sqrt{17}}{2}\right)^2,$$

we do not encounter any contradiction.

According to Section 2.1 and this Subsection 2.3.2, Theorem 2 is proved.

3. Second case: $\mathbf{N}_1$ is abelian and $\mathbf{N}_2$ is dihedral. The aim of this section is to prove Theorem 3. For the remainder of this section, we let $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ be a non-abelian normal CM-field of degree 16 which is a compositum of two normal octic CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ with the same maximal real subfield, $\mathbf{N}_1$ being an abelian imaginary octic field and $\mathbf{N}_2$ being a dihedral octic CM-field. Since $w_{\mathbf{N}} = w_{\mathbf{N}_1}$ (for $\mathbf{N}_1$ is the maximal abelian subfield of $\mathbf{N}$) and $w_{\mathbf{N}_2} = 2$, (3) still holds. Using (2) and (3) yields

$$(13) \qquad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}}}{4Q_{\mathbf{N}_1}}h_{\mathbf{N}_1}^-(h_{\mathbf{K}_2}^-)^2.$$

3.1. *Description of* $\mathbf{N}_1$ *when* $h_{\mathbf{N}} = 1$

PROPOSITION 16. *We have* $h_{\mathbf{N}}^- = 1$ *if and only if one of the following three assertions holds*:

1. $h_{\mathbf{N}_1}^- = 4$ *and* $h_{\mathbf{K}_2}^- = 1$ *(which implies* $Q_{\mathbf{N}} = Q_{\mathbf{N}_1}$*).*
2. $h_{\mathbf{N}_1}^- = 2$ *and* $h_{\mathbf{K}_2}^- = 1$ *(which implies* $Q_{\mathbf{N}} = 2$ *and* $Q_{\mathbf{N}_1} = 1$*).*
3. $h_{\mathbf{N}_1}^- = 1$ *and* $h_{\mathbf{K}_2}^- = 2$ *(which implies* $Q_{\mathbf{N}} = Q_{\mathbf{N}_1}$*).*

P r o o f. Use (13) and the fact that since $W_{\mathbf{N}} = W_{\mathbf{N}_1}$, it follows that $Q_{\mathbf{N}_1} = 2$ implies $Q_{\mathbf{N}} = 2$. ∎

Here, in contrast with Proposition 12, we do not always have $Q_{\mathbf{N}} = 2$. This prevents us from readily getting: $h_{\mathbf{N}}^-$ odd implies $h_{\mathbf{N}}$ odd, hence prevents us from readily getting: $h_{\mathbf{N}}^-$ odd implies $\mathbf{N}_1$ is its own narrow genus field. Hence, in contrast with Theorem 2, here we only solve the class number one problem.

PROPOSITION 17. *Assume that* $h_{\mathbf{N}}$ *is odd. Then* $\mathbf{N}_1$ *is equal to its own narrow genus field. Hence, one of the following two assertions holds*:

1. *There exist relative integers* p, q *and* l *either equal to* -1, *or prime and not equal to* 3 *modulo* 4 *such that* $\mathbf{N}_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l})$. *Moreover, if* $h_{\mathbf{N}} = 1$ *and* $h_{\mathbf{N}_1}^- = 2$ *(i.e., if we are in case* 2 *of Proposition* 16*) then we may, and will, choose notation such that* p *and* q *are positive.*
2. *There exist two positive primes* p *and* q *such that* $\mathbf{N}_1 = \mathbf{C}_p \mathbf{C}_q$ *is a compositum of a* p-*primary real quadratic field* $\mathbf{C}_p$ *and of a* q-*primary imaginary cyclic quartic field.*

P r o o f. Let $\mathbf{G}$ denote the narrow genus field of $\mathbf{N}_1$. As in the proof of Proposition 13, $\mathbf{G}$ is included in $\mathbf{N}$. Since $\mathbf{N}_1$ is the maximal abelian subfield of $\mathbf{N}$, $\mathbf{G}$ is included in $\mathbf{N}_1$, hence is equal to $\mathbf{N}_1$. Now, as $\mathbf{N}_1^+ = \mathbf{N}_2^+$ is bicyclic quartic, $\mathbf{N}_1$ is not cyclic. Therefore, if $\mathbf{N}_1$ is not elementary, then $\mathbf{N}_1 = \mathbf{C}_p \mathbf{C}_q$ is a compositum of a p-primary quadratic number field $\mathbf{C}_p$ associated with a quadratic character χ_p and of a q-primary cyclic quartic field $\mathbf{C}_p$ associated with a quartic character χ_q. If both $\mathbf{C}_p$ and $\mathbf{C}_q$ were imaginary then $\mathbf{N}_1^+$ would be a cyclic quartic field (associated with the quartic character $\chi_p \chi_q$), a contradiction. In the same way, if $\mathbf{C}_p$ were imaginary and $\mathbf{C}_q$ were real then $\mathbf{N}_1^+ = \mathbf{C}_q$ would be a cyclic quartic field, a contradiction.

If we are in case 2 of Proposition 16 and if $h_{\mathbf{N}} = 1$, then $h_{\mathbf{K}_2}^- = 1$. Hence, there exist two primes p and q not equal to 3 modulo 4 such that $\mathbf{N}_1^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ (see Theorem 8). We could also say that we must have $Q_{\mathbf{N}_1} = 1$. Therefore, if we assume that $\mathbf{N}_1$ is elementary and equal to its narrow genus field, then according to [Uch, Proposition 3] we conclude that either $\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{2}, \sqrt{q})$ for some positive prime q equal to 1 modulo 4,

or $\mathbf{N}_1$ is the compositum of three primary quadratic fields, exactly one of them being imaginary. ∎

PROPOSITION 18. *We have $h_\mathbf{N} = 1$ if and only if one of the following two assertions holds*:

1. $h_{\mathbf{N}_1}^- = 1$, $h_{\mathbf{N}_2} = 2$ *and* $h_{\mathbf{N}^+} = 1$. *In that case*, $h_{\mathbf{K}_2}^- = 2$ *and* $\mathbf{N}$ *is the Hilbert class field of* $\mathbf{N}_2$.

2. $h_{\mathbf{N}_1}^- = 2$, $h_{\mathbf{N}_2} = 1$ *and* $h_{\mathbf{N}^+} = 1$. *In that case*, $h_{\mathbf{K}_2}^- = 1$, $h_{\mathbf{N}_1} = 1$ *and* $\mathbf{N}$ *is the Hilbert class field of* $\mathbf{N}_1$.

P r o o f. Assume we are in case 1. Since $h_{\mathbf{N}_2} = 2$, we have $h_{\mathbf{N}_2}^- = 2$ (Corollary 9), and (2) yields

$$2 = h_{\mathbf{N}_2}^- = (Q_{\mathbf{N}_2}/2)(h_{\mathbf{K}_2}^-)^2,$$

which implies $h_{\mathbf{K}_2}^- = 2$ (and $Q_{\mathbf{N}_2} = 1$). Hence, according to the third point of Proposition 16 we have $h_\mathbf{N}^- = 1$, and $h_{\mathbf{N}^+} = 1$ gives $h_\mathbf{N} = 1$. In the same way, assume we are in case 2. Then $\mathbf{N}_1^+ = \mathbf{N}_2^+$ and $h_{\mathbf{N}_2} = 1$ yield $h_{\mathbf{N}_1^+} = h_{\mathbf{N}_2^+} = 1$. Therefore, we get $h_{\mathbf{N}_1} = 2$. Moreover, $h_{\mathbf{N}_2} = 1$ gives $h_{\mathbf{N}_2}^- = 1$, which implies $h_{\mathbf{K}_2}^- = 1$ (and $Q_{\mathbf{N}_2} = 2$). Hence, according to the second point of Proposition 16 we have $h_\mathbf{N}^- = 1$, and $h_{\mathbf{N}^+} = 1$ gives $h_\mathbf{N} = 1$.

Conversely, assume $h_\mathbf{N} = 1$. Then $h_{\mathbf{N}^+} = 1$ and using the Hilbert class fields of $\mathbf{N}_1$ and $\mathbf{N}_2$ we easily get $h_{\mathbf{N}_1} \in \{1, 2\}$ and $h_{\mathbf{N}_2} \in \{1, 2\}$. Hence, $h_{\mathbf{N}_1}^- \in \{1, 2\}$ and the first case of Proposition 16 needs not be considered. If we are in the second case of Proposition 16, then $h_{\mathbf{N}_1}^- = 2$, $h_{\mathbf{K}_2}^- = 1$ and (2) gives $h_{\mathbf{N}_2}^- = 1$ (and $Q_{\mathbf{N}_2} = 2$) and $h_{\mathbf{N}_2}$ is odd (Theorem 8). Therefore, $h_{\mathbf{N}_2} = 1$ and the second assertion of the proposition holds. In the same way, if we are in the third case of Proposition 16, then $h_{\mathbf{N}_1}^- = 1$ and $h_{\mathbf{K}_2}^- = 2$. Therefore, $h_{\mathbf{N}_2}^- = 2Q_{\mathbf{N}_2}$ is even. Since we have $h_{\mathbf{N}_2} \in \{1, 2\}$, we get $h_{\mathbf{N}_2} = h_{\mathbf{N}_2}^- = 2$ and the first assertion holds. ∎

Therefore, the proof of Theorem 3 will be divided into three steps.

First, we determine in Section 3.2 all the imaginary abelian octic fields $\mathbf{N}_1$ as in Proposition 17 which have relative class number 1 (there are twenty three such $\mathbf{N}_1$'s) or 2 (there are five such $\mathbf{N}_1$'s).

Second, for each of these five possible $\mathbf{N}_1$ with $h_{\mathbf{N}_1}^- = 2$, using Theorem 10 we will easily find all the possible $\mathbf{N}_2$'s such that $h_{\mathbf{N}_2} = 1$ and $\mathbf{N}_2^+ = \mathbf{N}_1^+$. Finally, for each compositum $\mathbf{N} = \mathbf{N}_1 \mathbf{N}_2$ we will only have to compute $h_{\mathbf{N}^+}$.

Third, for each of these twenty three possible $\mathbf{N}_1$ with $h_{\mathbf{N}_1}^- = 1$, we do not want to use the determination in [YPJK] of all the dihedral octic CM-fields with class number two, for we do not in fact need this difficult determination. Instead, using some of the ideas developed in [YPJK] we show in Subsection 3.3 that for each of these twenty three possible $\mathbf{N}_1$, we

can determine whether there exists at least one dihedral octic CM-field $\mathbf{N}_2$ such that $\mathbf{N}_2^+ = \mathbf{N}_1^+$ and $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$. Then, whenever there exists at least one such $\mathbf{N}_2$, we prove that there exists exactly one such $\mathbf{N}_2$, we find a generator for $\mathbf{N}_2$ and compute $h_{\mathbf{K}_2}^-$. For each case where $h_{\mathbf{K}_2}^- = 2$, we finally compute $h_{\mathbf{N}_2}$, and when it is equal to 2, we compute $h_{\mathbf{N}^+}$.

3.2. *Determination of certain imaginary abelian octic number fields with relative class numbers equal to 1 or 2*

PROPOSITION 19. 1. *Let $\mathbf{N}_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l})$ be an elementary octic imaginary number field. Assume that $\mathbf{N}_1$ is its own narrow genus field. Then $h_{\mathbf{N}_1}^- = 1$ if and only if*

$$(p, q, l) \in \{(-1, -2, -3), (-1, -2, -11), (-1, -3, -11), (-1, -3, -19),$$
$$(-1, -3, -7), (-1, -7, -19), (-2, -3, -7), (-3, -11, -19),$$
$$(-1, -2, 5), (-1, -3, 5), (-1, -7, 5), (-1, -7, 13),$$
$$(-2, -3, 5), (-2, -7, 5),$$
$$(-3, -7, 5), (-3, -11, 2), (-3, -11, 17)\}.$$

2. *Let $\mathbf{N}_1 = \mathbf{C}_p \mathbf{C}_q$ be a compositum of a p-primary real quadratic number field $\mathbf{C}_p$ and of a q-primary imaginary cyclic quartic field. Then $h_{\mathbf{N}_1}^- = 1$ if and only if*

$$(p, q) \in \{(5, 2), (2, 5), (2, 13), (13, 5), (17, 5), (5, 13)\}.$$

3. *Let $\mathbf{N}_1 = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{l})$ be an elementary octic imaginary number field. Assume that $\mathbf{N}_1^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ where $2 \leq p < q$ are two distinct primes and that $\mathbf{N}_1$ is its own narrow genus field. Then $h_{\mathbf{N}_1}^- = 2$ if and only if*

$$(p, q, l) \in \{(2, 5, -3), (2, 17, -3)\}.$$

4. *Let $\mathbf{N}_1 = \mathbf{C}_p \mathbf{C}_q$ be a compositum of a p-primary real quadratic number field $\mathbf{C}_p$ and of a q-primary imaginary cyclic quartic field. Then $h_{\mathbf{N}_1}^- = 2$ if and only if*

$$(p, q) \in \{(5, 29), (29, 5), (17, 13)\}.$$

P r o o f. To begin with, let us recall that if $\mathbf{E}$ is an imaginary abelian field of 2-power degree, then

$$h_{\mathbf{E}}^- = Q_{\mathbf{E}} w_{\mathbf{E}} \prod_{\mathbf{F}} (h_{\mathbf{F}}/w_{\mathbf{F}}),$$

where this product ranges over all the imaginary cyclic subfields $\mathbf{F}$ of $\mathbf{E}$.

Let us first prove point 1. By Table 2 pp. 126–127 of [Lou 5] we only have to prove that the only imaginary octic fields $\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{q})$ with relative class number one, where $q \equiv 1 \pmod 4$ is an odd prime, are obtained when $q \in \{5, -3, -11\}$.

Let us first assume that $q > 1$. Then $Q_{\mathbf{N}_1} = 1$ and $h_{\mathbf{N}_1}^- = h_{-q}h_{-2q}/4$ where h_d denote the class number of the quadratic number field $\mathbb{Q}(\sqrt{d})$. According to genus theory, h_{-q} and h_{-2q} are both even. Hence, $h_{\mathbf{N}_1}^- = 1$ if and only if $h_{-q} = 2$ and $h_{-2q} = 2$, hence if and only if $q = 5$. In the same way, $h_{\mathbf{N}_1}^- = 2$ if and only if $h_{-q} = 2$ and $h_{-2q} = 4$, or $h_{-q} = 4$ and $h_{-2q} = 2$. Therefore, we never have $h_{\mathbf{N}_1}^- = 2$.

Let us now assume that $q < -1$. Then $Q_{\mathbf{N}_1} = 2$ and $h_{\mathbf{N}_1}^- = h_q h_{-2q}/2$. According to genus theory, h_q is odd. Hence, $h_{\mathbf{N}_1}^- = 1$ if and only if $h_q = 1$ and $h_{-2q} = 2$, hence if and only if $q = -3$ or $q = -11$. In the same way, $h_{\mathbf{N}_1}^- = 2$ if and only if $h_q = 1$ and $h_{-2q} = 4$, hence if and only if $q = -7$.

Let us now prove point 3. We have $l < 0$ and if we let w_d and h_d denote the number of roots of unity and class number of an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, we have

$$h_{\mathbf{N}_1}^- = w_{\mathbf{N}_1}(h_l/w_l)(h_{pl}/w_{pl})(h_{ql}/w_{ql})(h_{pql}/w_{pql}).$$

First, assume that either $2 < p < q$ and $l \leq -1$, or $2 = p < q$ and $l < -2$. Then $w_{\mathbf{N}_1} = w_l$ and we get $h_{\mathbf{N}_1}^- = 2h_l(h_{pl}/2)(h_{ql}/2)(h_{pql}/4)$. Therefore, $h_{\mathbf{N}_1}^-$ is always even and $h_{\mathbf{N}_1}^- = 2$ if and only if $(h_l, h_{pl}, h_{ql}, h_{pql}) = (1, 2, 2, 4)$. Since the class number one and two problems have been solved for the imaginary quadratic number fields (see Table 1 in [Lou 5]), we know the possible values of l, p and q. Namely, $(h_l, h_{pl}, h_{ql}) = (1, 2, 2)$ if and only if:

$$l = -1, \quad p, q \in \{5, 13, 37\},$$
$$l = -2, \quad p, q \in \{5, 29\},$$
$$l = -3, \quad p, q \in \{2, 5, 17, 41, 89\},$$
$$l = -7, \quad p, q \in \{5, 13, 61\},$$
$$l = -11, \quad p, q \in \{2, 17\}.$$

Then we have $h_{\mathbf{N}_1}^- = h_{pql}/4$ and the following table provides us with the values of the class numbers h_{pql} for these possible values of l, p and q:

l	p	q	pql	h_{pql}	l	p	q	pql	h_{pql}
-1	5	13	-65	8	-3	5	41	-615	20
-1	5	37	-185	16	-3	5	89	-1335	28
-1	13	37	-481	16	-3	17	41	-2091	12
-2	5	29	-290	20	-3	17	89	-4539	20
-3	2	5	-30	4	-3	41	89	-10947	28
-3	2	17	-102	4	-7	5	13	-455	20
-3	2	41	-246	12	-7	5	61	-2135	44
-3	2	89	-534	20	-7	13	61	-5551	52
-3	5	17	-255	12					

Second, assume that $l \in \{-1, -2\}$ and $2 = p < q$. Therefore, $\mathbf{N}_1 =$

$\mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{q})$ and $q \equiv 1 \pmod 4$ is a positive prime. According to the previous proof of point 1, we have $h_{\mathbf{N}_1}^- = (h_{-q}/2)(h_{-2q}/2)$ and $h_{\mathbf{N}_1}^- = 2$ if and only if $(h_{-q}, h_{-2q}) \in \{(2,4),(4,2)\}$. However, we have the following table of class numbers:

q	5	13	29	37
h_{-q}	2	2	6	2
h_{-2q}	2	6	2	10
$h_{\mathbf{N}_1}^-$	1	3	3	5

Therefore, we never have $h_{\mathbf{N}_1}^- = 2$.

Let us finally prove points 2 and 4. Let $\mathbf{N}_1 = \mathbf{C}_p \mathbf{C}_q$ be a compositum of a p-primary real quadratic number field $\mathbf{C}_p$ of conductor f_p and of a q-primary imaginary cyclic quartic $\mathbf{C}_q$ of conductor f_q. According to [Uch] we have $Q_{\mathbf{N}_1} = 1$ and

$$h_{\mathbf{N}_1}^- = h_{\mathbf{C}_q}^- (h_{\mathbf{F}_{(p,q)}}^- / 2)$$

where $\mathbf{F}_{(p,q)}$ denotes the imaginary cyclic quartic subfield of $\mathbf{N}_1$ associated with the odd quartic Dirichlet character $\chi_p \chi_q$ of conductor $f_{(p,q)} = f_p f_q$. Moreover, according to [Wa, Th. 10.4(b)], the class number $h_{\mathbf{C}_q}$ is odd. Hence, $h_{\mathbf{C}_q}^-$ is odd. Therefore, $h_{\mathbf{N}_1}^- = 1$ if and only if $(h_{\mathbf{C}_q}^-, h_{\mathbf{F}_{(p,q)}}^-) = (1,2)$, and $h_{\mathbf{N}_1}^- = 2$ if and only if $(h_{\mathbf{C}_q}^-, h_{\mathbf{F}_{(p,q)}}^-) = (1,4)$. Now, in [Lou 8] we solved the relative class number one and two problems for the imaginary cyclic quartic fields. Using the techniques developed in [Lou 8], we can also easily solve the relative class number four problem for the imaginary cyclic quartic fields $\mathbf{F}_{(p,q)}$. Indeed, according to [Lou 8, Th. 4] or [Lou 3, proof of Corollary a], $h_{\mathbf{F}_{(p,q)}}^- \le 4$ implies $f_{(p,q)} \le 2 \cdot 10^4$. Now, up to this upper bound, we compute the relative class numbers of $\mathbf{C}_q$ and $\mathbf{F}_{(p,q)}$ thanks to the following two formulas:

$$h_{\mathbf{C}_q}^- = \frac{w_q}{4 f_q^2} \left| \sum_{k=1}^{f_q - 1} k \chi_q(k) \right|^2$$

with

$$w_q = \begin{cases} 2 & \text{if } q \neq 5, \\ 10 & \text{if } q = 5, \end{cases}$$

and

$$h_{\mathbf{F}_{(p,q)}}^- = \frac{1}{2 f_{(p,q)}^2} \left| \sum_{k=1}^{f_{(p,q)} - 1} k \chi_p(k) \chi_q(k) \right|^2.$$

Note also that if $q = 2$, then $f_q = 16$ and we may take χ_q defined by the following table:

k	1	3	5	7	9	11	13	15
$\chi_q(k)$	1	i	i	1	-1	$-i$	$-i$	-1

We get

$$h^-_{\mathbf{C}_q} = 1 \Leftrightarrow q \in \{2, 5, 13, 29, 37, 53, 61\},$$

$$h^-_{\mathbf{F}_{(p,q)}} = 2 \Leftrightarrow (p, q) \in \{(5, 13), (13, 5), (17, 5), (2, 5), (2, 13), (5, 2)\},$$

$$h^-_{\mathbf{F}_{(p,q)}} = 4 \Leftrightarrow (p, q) \in \{(5, 29), (29, 5), (17, 13)\}.$$

The desired results easily follow. ∎

3.3. *Determination of* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *when* $h^-_{\mathbf{N}_1} = 1$, $h^-_{\mathbf{K}_2} = 2$ *and* $h_{\mathbf{N}} = 1$. The aim of this subsection is:

• first, to show that we can get rid of some of the $23 = 17 + 6$ imaginary abelian fields $\mathbf{N}_1$ with relative class number one which appear in points 1 and 2 of Proposition 19 for their maximal totally real subfields $\mathbf{N}_1^+$ cannot be the maximal totally real subfield of any dihedral octic CM-field $\mathbf{N}_2$ (to this end, we will use Lemma 6);

• second, to show that for some of the remaining $\mathbf{N}_1$'s, their maximal totally real subfields $\mathbf{N}_1^+$ cannot be the maximal totally real subfield of any dihedral octic CM-field $\mathbf{N}_2$ with relative class number two (to this end, we will use Lemma 20 below); and

• third, to show that for each of the few $\mathbf{N}_1$'s remaining we can determine $\mathbf{N}_2$ provided that $h^-_{\mathbf{K}_2} = 2$ (to this end, we will use Proposition 23 below).

LEMMA 20. *Let* $\mathbf{N}$ *be a dihedral octic CM-field. Let* $\mathbf{N}^+$ *denote its maximal totally real subfield. Hence,* $\mathbf{N}^+/\mathbb{Q}$ *is bicyclic quartic. Let* $\mathbf{L}$ *denote the only quadratic subfield of* $\mathbf{N}^+$ *such that* $\mathbf{N}/\mathbf{L}$ *is cyclic quartic, and let* $\mathbf{k}_1$ *and* $\mathbf{k}_2$ *denote the two other quadratic subfields of* $\mathbf{N}^+$. *Let* $\mathbf{K}_i$ *and* $\mathbf{K}'_i$ *denote the two non-normal quartic subfields of* $\mathbf{N}$ *which contain* $\mathbf{k}_i$. *Hence,* $\mathbf{K}_i$ *and* $\mathbf{K}'_i$ *are isomorphic non-normal quartic CM-field and* $h^-_{\mathbf{K}_1} = h^-_{\mathbf{K}_2}$. *If at least three distinct primes ramify in* $\mathbf{N}^+/\mathbf{L}$ *and* $\mathbf{N}^+/\mathbf{k}_2$ *then* 4 *divides* $h^-_{\mathbf{K}_2}$.

P r o o f. We use the two following points. First, if a prime ideal $\mathcal{P}_+$ of $\mathbf{N}^+$ is ramified in $\mathbf{N}^+/\mathbf{L}$ then it is ramified in $\mathbf{N}/\mathbf{N}^+$, for the extension $\mathbf{N}/\mathbf{L}$ is cyclic quartic. Second, let $\mathcal{P}_2$ be the prime ideal of $\mathbf{k}_2$ lying below a prime ideal $\mathcal{P}_+$ of $\mathbf{N}^+$ ramified in the quadratic extension $\mathbf{N}/\mathbf{N}^+$. If $\mathcal{P}_2$ were unramified in both $\mathbf{K}_2/\mathbf{k}_2$ and $\mathbf{K}'_2/\mathbf{k}_2$ then it would be unramified in $\mathbf{K}'_2\mathbf{K}_2/\mathbf{k}_2 = \mathbf{N}/\mathbf{k}_2$, hence unramified in $\mathbf{N}/\mathbf{N}^+$, a contradiction. Therefore, $\mathcal{P}_2$ is either ramified in $\mathbf{K}_2/\mathbf{k}_2$ or ramified in $\mathbf{K}'_2/\mathbf{k}_2$, and since $\mathbf{K}_2$ and $\mathbf{K}'_2$ are isomorphic and $\mathbf{k}_2$ is normal, one of the ideals of $\mathbf{k}_2$ conjugate to $\mathcal{P}_2$ is ramified in $\mathbf{K}_2/\mathbf{k}_2$. ∎

PROPOSITION 21. *Let* p, q *and* l *be positive primes,* $\mathbf{N}$ *be a dihedral octic CM-field with maximal totally real subfield* $\mathbf{N}^+$, $\mathbf{L}$ *be the only quadratic subfield of* $\mathbf{N}^+$ *such that* $\mathbf{N}/\mathbf{L}$ *is cyclic quartic, and* $\mathbf{K}$ *be any of the four non-normal quartic CM-subfields of* $\mathbf{N}$.

1. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{pq}, \sqrt{l})$ *with* $p \equiv q \equiv 3 \pmod 4$, $l \not\equiv 3 \pmod 4$ *and* $(p/l) = -1$ *or* $(q/l) = -1$, *then* $\mathbf{L} \neq \mathbb{Q}(\sqrt{pql})$ *and* 4 *divides* $h_{\mathbf{K}}^-$.

2. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2p}, \sqrt{q})$ *with* $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 4$ *and* $(p/q) = -1$, *then* $\mathbf{L} \neq \mathbb{Q}(\sqrt{2pq})$ *and* 4 *divides* $h_{\mathbf{K}}^-$.

3. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ *with* $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 4$ *and* $(p/q) = -1$, *then* $\mathbf{L} \neq \mathbb{Q}(\sqrt{pq})$ *and* 4 *divides* $h_{\mathbf{K}}^-$.

4. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ *with* $p \equiv q \equiv 3 \pmod 4$, *then* $\mathbf{L} = \mathbb{Q}(\sqrt{p})$ *if* $(q/p) = +1$, *and* $\mathbf{L} = \mathbb{Q}(\sqrt{q})$ *if* $(p/q) = +1$. *Moreover, if* $p \not\equiv q \pmod 8$ *then* 4 *divides* $h_{\mathbf{K}}^-$.

5. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ *with* $p \equiv q \equiv 1 \pmod 4$, *then* $(p/q) = +1$.

6. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ *with* $p \equiv 1 \pmod 4$, *then* $p \equiv 1 \pmod 8$.

7. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ *with* $p \equiv 3 \pmod 8$, *then* $\mathbf{L} = \mathbb{Q}(\sqrt{2})$.

8. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{pq}, \sqrt{pl})$ *with* $p \equiv q \equiv l \equiv 3 \pmod 4$ *then at least one of the three symbols* (pq/l), (pl/q) *and* (ql/p) *is equal to* $+1$.

9. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2p}, \sqrt{2q})$ *with* $p \equiv q \equiv 3 \pmod 4$ *then* $(2p/q) = +1$ *or* $(2q/p) = +1$.

P r o o f. It is easily checked that all our results on $\mathbf{L}$ and on the values of the Legendre symbols follow from Lemma 6. Moreover, using Lemma 20 easily yields that 4 divides $h_{\mathbf{K}}^-$ in cases 1, 2 and 3. Only the same result in case 4 is difficult. We may assume that $(q/p) = +1$, which yields $\mathbf{L} = \mathbb{Q}(\sqrt{p})$, and we set $\mathbf{k}_1 = \mathbb{Q}(\sqrt{pq})$, which gives $\mathbf{k}_2 = \mathbb{Q}(\sqrt{q})$.

First, we note that $(q) = \mathcal{Q}_2^2$ is ramified in $\mathbf{k}_2$. Since q is inert in $\mathbf{L}$, $(\mathcal{Q}_2) = \mathcal{Q}_+$ is inert in $\mathbf{N}^+/\mathbf{k}_2$ and since $\mathcal{Q}_+$ is ramified in $\mathbf{N}^+/\mathbf{L}$ and $\mathbf{N}/\mathbf{L}$ is cyclic quartic, it follows that $(\mathcal{P}_+) = \mathcal{P}_{\mathbf{N}}^2$ is ramified in $\mathbf{N}/\mathbf{N}^+$ and $\mathbf{N}^+$ is the inertia field of $\mathcal{Q}_{\mathbf{N}}$ in $\mathbf{N}/\mathbf{k}_2$. Therefore, $\mathcal{Q}_2$ is ramified in both $\mathbf{K}_2/\mathbf{k}_2$ and $\mathbf{K}_2'/\mathbf{k}_2$.

Second, we note that p splits in $\mathbf{k}_2$, say $(p) = \mathcal{P}_2 \mathcal{P}_2'$, and both $\mathcal{P}_2$ and $\mathcal{P}_2'$ are ramified in $\mathbf{N}^+/\mathbf{k}_2$. Since $\mathbf{K}_2$ and $\mathbf{K}_2'$ are isomorphic and $\mathbf{k}_2$ is normal, as in the proof of the second point of Lemma 20, we may choose notation such that $\mathcal{P}_2$ is ramified in $\mathbf{K}_2/\mathbf{k}_2$. Hence, we have already got two distinct prime ideals $\mathcal{P}_2$ and $\mathcal{Q}_2$ of $\mathbf{k}_2$ which are ramified in $\mathbf{K}_2/\mathbf{k}_2$.

Third, if $p \not\equiv q \pmod 8$ then $pq \equiv 5 \pmod 8$ and 2 is inert in $\mathbf{k}_1$, the prime ramified ideal $\mathcal{L}_2$ of $\mathbf{k}_2$ lying above 2 is inert in $\mathbf{N}^+/\mathbf{k}_2$, say $(\mathcal{L}_2) = \mathcal{L}_+$ and $\mathcal{L}_+$ is inert in $\mathbf{N}^+/\mathbf{L}$. If $\mathcal{L}_+$ is not ramified in $\mathbf{N}/\mathbf{N}^+$ then $(\mathcal{L}_+) = \mathcal{L}_{\mathbf{N}}$ is inert in $\mathbf{N}/\mathbf{N}^+$ and inert in $\mathbf{N}/\mathbf{k}_2$. A contradiction since $\mathbf{N}/\mathbf{k}_2$ is bicyclic quartic. Therefore, $\mathcal{L}_+$ is ramified in $\mathbf{N}/\mathbf{N}^+$ and $\mathbf{N}^+$ is the inertia field of $\mathcal{L}_{\mathbf{N}}$ in $\mathbf{N}/\mathbf{k}_2$. Therefore, $\mathcal{L}_2$ is ramified in both $\mathbf{K}_2/\mathbf{k}_2$ and $\mathbf{K}_2'/\mathbf{k}_2$. Therefore, as we have found a third prime ideal of $\mathbf{k}_2$ ramified in $\mathbf{K}_2/\mathbf{k}_2$ we see that $4 = 2^{3-1}$ divides $h_{\mathbf{K}_2}^-$. ∎

COROLLARY 22. *Let* $\mathbf{N}_1$ *be one of the* $23 = 17 + 6$ *imaginary abelian number fields with relative class number one given in points* 1 *and* 2 *of*

Proposition 19. *If* $\mathbf{N}_1^+$ *is the maximal totally real subfield of some dihedral octic CM-field* $\mathbf{N}_2$ *such that* $h_{\mathbf{K}_2}^- = 2$, *then* $\mathbf{N}_1$ *is one of the following four fields*:

$$(p, q, l) \in \{(-1, -2, -3), (-1, -2, -11), (-1, -3, -11), (-1, -3, -19)\}.$$

P r o o f. For example, according to points 5 and 6 of Proposition 21, if $\mathbf{N}_1$ is one of the six fields which appear in point 2 of Proposition 19, then $\mathbf{N}_1^+$ cannot be the maximal real subfield of any dihedral octic CM-field $\mathbf{N}_2$. ∎

PROPOSITION 23. *Let the notation be as in Lemma* 20.

1. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ *with* $p \equiv 3 \pmod 8$ *then* $\mathbf{L} = \mathbb{Q}(\sqrt{2})$ *and we may assume that* $\mathbf{k}_2 = \mathbb{Q}(\sqrt{p})$. *In both cases the prime ideal* $\mathcal{L}_2$ *of* $\mathbf{k}_2$ *lying above* 2 *and the prime ideal* $\mathcal{P}_2$ *of* $\mathbf{k}_2$ *lying above* p *are ramified in both* $\mathbf{K}_2/\mathbf{k}_2$ *and* $\mathbf{K}_2'/\mathbf{k}_2$. *In particular, if* $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$ *then* $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-(x + y\sqrt{p})\sqrt{p}})$ *or* $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\varepsilon(x + y\sqrt{p})\sqrt{p}})$ *where* $x \geq 1$ *and* $y \geq 1$ *are integral solutions of* $x^2 - py^2 = -2$ *and where* $\varepsilon > 1$ *denotes the fundamental unit of* $\mathbf{k}_2$. *Finally,* $\mathbf{k}_2(\sqrt{-(x + y\sqrt{p})\sqrt{p}})$ *and* $\mathbf{k}_2(\sqrt{-\varepsilon(x + y\sqrt{p})\sqrt{p}})$ *being isomorphic, we may assume that* $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-(x + y\sqrt{p})\sqrt{p}})$.

2. *If* $\mathbf{N}^+ = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ *with* $p \equiv q \equiv 3 \pmod 4$ *then we may assume that* $(q/p) = +1$, $\mathbf{L} = \mathbb{Q}(\sqrt{p})$ *and* $\mathbf{k}_2 = \mathbb{Q}(\sqrt{q})$. *The prime ideal* $\mathcal{Q}_2$ *of* $\mathbf{k}_2$ *lying above* q *is ramified in both* $\mathbf{K}_2/\mathbf{k}_2$ *and* $\mathbf{K}_2'/\mathbf{k}_2$, *and one of the two prime ideals of* $\mathbf{k}_2$ *lying above* p, *say* $\mathcal{P}_2$, *is ramified in* $\mathbf{K}_2/\mathbf{k}_2$, *the other one being ramified in* $\mathbf{K}_2'/\mathbf{k}_2$. *In particular, if* $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$ *then* $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-(x + y\sqrt{p})\sqrt{q}})$ *or* $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\varepsilon(x + y\sqrt{q})\sqrt{q}})$ *where* $x \geq 1$ *and* $y \geq 1$ *are integral solutions of* $x^2 - qy^2 = -p^h$ *and where* $\varepsilon > 1$ *and* h *denote the fundamental unit and class number of* $\mathbf{k}_2$, *respectively.*

P r o o f. 1. Let us prove the first part of this point. We first note that 2 is totally ramified in $\mathbf{N}^+$, hence totally ramified in $\mathbf{N}$ (for $\mathbf{N}/\mathbf{L}$ is cyclic quartic) and indeed the prime ideal $\mathcal{L}_2$ of $\mathbf{k}_2$ lying above 2 is ramified in both $\mathbf{K}_2/\mathbf{k}_2$ and $\mathbf{K}_2'/\mathbf{k}_2$. Second, $(p) = \mathcal{P}_{\mathbf{L}}$ is inert in $\mathbf{L}/\mathbb{Q}$, $(\mathcal{P}_{\mathbf{L}}) = \mathcal{P}_+^2$ is ramified in $\mathbf{N}^+/\mathbf{L}$, hence totally ramified in $\mathbf{N}/\mathbf{L}$, $(p) = \mathcal{P}_2^2$ is ramified in $\mathbf{k}_2/\mathbb{Q}$ and $\mathcal{P}_2$ is inert in $\mathbf{N}^+/\mathbf{k}_2$. Therefore, $\mathbf{N}^+$ is the inertia field of $\mathcal{P}_{\mathbf{N}}$ in $\mathbf{N}/\mathbf{k}_2$ and $\mathcal{P}_2$ is ramified in both $\mathbf{K}_2/\mathbf{k}_2$ and $\mathbf{K}_2'/\mathbf{k}_2$.

Let us prove the second part of this point. If $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$ then only $\mathcal{L}_2$ and $\mathcal{P}_2$ must be ramified in $\mathbf{K}_2/\mathbf{k}_2$. Therefore, $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\alpha})$ where $\alpha \in \mathbf{k}_2$ is a totally positive algebraic element such that $(\alpha) = \mathcal{P}_2\mathbf{I}^2$ or $(\alpha) = \mathcal{L}_2\mathcal{P}_2\mathbf{I}^2$ for some integral ideal $\mathbf{I}$ of $\mathbf{k}_2$. Since the class number h of $\mathbf{k}_2$ is odd, since $\mathcal{L}_2$ and $\mathcal{P}_2$ are principal in $\mathbf{k}_2$, and since $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\alpha}) = \mathbf{k}_2(\sqrt{-\alpha^h})$, we may assume that $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\beta})$ where $\beta \in \mathbf{k}_2$ is a totally positive algebraic element such that $(\alpha) = \mathcal{P}_2$ or $(\alpha) = \mathcal{L}_2\mathcal{P}_2$. Finally, as the fundamental unit ε of $\mathbf{k}_2$ has norm $+1$ and as $\mathcal{L}_2$ and $\mathcal{P}_2$ are not principal in

the strict sense, we must have $(\beta) = \mathcal{L}_2\mathcal{P}_2$, hence $\beta = \lambda_2\sqrt{p}$ or $\beta = \varepsilon\lambda_2\sqrt{p}$, where $\lambda_2 = x + y\sqrt{p}$ with $x \geq 1$ and $y \geq 1$ is a given algebraic integer such that $(\lambda_2) = \mathcal{L}_2$, i.e., such that $x^2 - py^2 = -2$.

Finally, as 2 is totally ramified in $\mathbf{N}^+$, we have $(\lambda_2) = (\sqrt{2})$ in $\mathbf{N}^+$. Taking norms down to $\mathbf{k}_2$ we see that $(\lambda_2^2) = (2)$ and $-\lambda_2/\lambda_2' = \lambda_2^2/2 = \varepsilon^m$ for some relative integer m. Moreover, as $\sqrt{2}$ is not in $\mathbf{k}_2$ we get m odd. Now, we conclude that $(\lambda_2\sqrt{p})' = -\lambda_2'\sqrt{p} = \varepsilon^{-m}\lambda_2\sqrt{p}$ and $\mathbf{k}_2(\sqrt{-(\lambda_2\sqrt{p})'}) = \mathbf{k}_2(\sqrt{-\varepsilon\lambda_2\sqrt{p}})$ is isomorphic to $\mathbf{k}_2(\sqrt{-\lambda_2\sqrt{p}})$.

2. The first part of this point was proved during the proof of point 4 of Proposition 21. The proof of its second part is similar to that of the second part of the first point. ∎

According to Proposition 23, if $\mathbf{N}_1$ ranges over the four abelian fields listed in Corollary 22 and if $h_{\mathbf{K}_2}^- \equiv 2 \pmod 4$, then we are in one of the following six cases: $\mathbf{K}_2 = \mathbf{k}_2(\sqrt{-\alpha})$ with $\alpha \in \mathbf{k}_2$ and

α	$P_{\mathbf{K}_2}(X)$	$d_{\mathbf{K}_2}$	$h_{\mathbf{K}_2}^-$
$3 + \sqrt{3}$	$X^4 + 6X^2 + 6$	$2^9 \cdot 3^3$	2
$11 + 3\sqrt{11}$	$X^4 + 22X^2 + 22$	$2^9 \cdot 11^3$	10
$6 + \sqrt{3}$	$X^4 + 12X^2 + 33$	$2^8 \cdot 3^3 \cdot 11$	4
$15 + 8\sqrt{3}$	$X^4 + 30X^2 + 33$	$2^4 \cdot 3^3 \cdot 11$	2
$19 + 4\sqrt{19}$	$X^4 + 38X^2 + 57$	$2^4 \cdot 3 \cdot 19^3$	10
$6194 + 1421\sqrt{19}$	$X^4 + 12388X^2 + 57$	$2^8 \cdot 3 \cdot 19^3$	12

According to this table and to point 3 of Lemma 4 which gives $Q_{\mathbf{N}_2} = 1$ and $h_{\mathbf{N}_2}^- = \frac{1}{2}(h_{\mathbf{K}_2}^-)^2 = 2$ for the two fields $\mathbf{K}_2$ with $h_{\mathbf{K}_2}^- = 2$ which appear in this table, we readily get:

COROLLARY 24. *Let $\mathbf{N}_1$ be any one of the four abelian imaginary fields with relative class number one which appear in Corollary* 22. *Then $\mathbf{N}_1^+ \subseteq \mathbf{N}_2$ with $h_{\mathbf{K}_2}^- = 2$ if and only if either*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-3}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-(3 + \sqrt{3})}),$$

in which case $h_{\mathbf{N}_2} = 2$, $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3 + \sqrt{3}})$ is a dihedral octic field cyclic over $\mathbb{Q}(\sqrt{2})$ such that $d_{\mathbf{N}^+} = 2^{22} \cdot 3^6$ and $h_{\mathbf{N}^+} = 1$, or

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-11}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{-(15 + 8\sqrt{3})}),$$

in which case $h_{\mathbf{N}_2} = 2$, $\mathbf{N}^+ = \mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{15 + 8\sqrt{3}})$ is a dihedral octic field cyclic over $\mathbb{Q}(\sqrt{11})$ such that $d_{\mathbf{N}^+} = 2^8 \cdot 3^6 \cdot 11^4$ and $h_{\mathbf{N}^+} = 1$.

3.4. *Determination of* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *with* $h_{\mathbf{N}_1}^- = 2$, $h_{\mathbf{K}_2}^- = 1$ *and* $h_{\mathbf{N}} = 1$. The following Corollary 25 is easily proved by using Theorem 10.

COROLLARY 25. *Let $\mathbf{N}_1$ be any one of the five abelian imaginary fields with relative class number two which appear in points 3 and 4 of Proposition 19. Then $\mathbf{N}_1^+ \subseteq \mathbf{N}_2$ with $h_{\mathbf{K}_2}^- = 1$ if and only if either*

$$\mathbf{N}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-3}) \quad and \quad \mathbf{N}_2 = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5+\sqrt{17})/2}),$$

in which case $h_{\mathbf{N}_2} = 1$, $\mathbf{N}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{3(5+\sqrt{17})/2})$ is a dihedral octic field cyclic over $\mathbb{Q}(\sqrt{34})$ such that $d_{\mathbf{N}^+} = 2^{12} \cdot 3^4 \cdot 17^4$ and $h_{\mathbf{N}^+} = 1$, or

$$\mathbf{N}_1 = \mathbf{C}_{17}\mathbf{C}_{13} = \mathbb{Q}(\sqrt{17}, \sqrt{-(13+2\sqrt{13})})$$

and

$$\mathbf{N}_2 = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9+\sqrt{13})/2}),$$

in which case $h_{\mathbf{N}_2} = 1$, $\mathbf{N}^+ = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{(143+31\sqrt{13})/2})$ is a dihedral octic field cyclic over $\mathbb{Q}(\sqrt{17})$ such that $d_{\mathbf{N}^+} = 13^6 \cdot 17^4$ and $h_{\mathbf{N}^+} = 1$.

3.5. *Determination of all $\mathbf{N}$'s with $h_{\mathbf{N}} = 1$.* Proposition 18 and Corollaries 24 and 25 readily yield Theorem 3.

4. Tables

Table 1

p	$q = A^2 + B^2$	$\alpha_{(q,p)}$	$d_{\mathbf{K}_2}$	$h_{\mathbf{K}_2}^-$
2	$17 = 1^2 + 4^2$	$(5+\sqrt{17})/2$	$2^3 \cdot 17^3$	2
2	$73 = 3^2 + 8^2$	$(9+\sqrt{73})/2$	$2^3 \cdot 73^3$	8
2	$89 = 5^2 + 8^2$	$(217+23\sqrt{89})/2$	$2^3 \cdot 89^3$	8
2	$233 = 13^2 + 8^2$	$(6121+401\sqrt{233})/2$	$2^3 \cdot 233^3$	32
2	$281 = 5^2 + 16^2$	$(17+\sqrt{281})/2$	$2^3 \cdot 281^3$	52
17	$2 = 1^2 + 1^2$	$5+2\sqrt{2}$	$2^{11} \cdot 17$	42
73	$2 = 1^2 + 1^2$	$9+2\sqrt{2}$	$2^{11} \cdot 73$	6
89	$2 = 1^2 + 1^2$	$11+4\sqrt{2}$	$2^{11} \cdot 89$	10
233	$2 = 1^2 + 1^2$	$19+8\sqrt{2}$	$2^{11} \cdot 233$	22
281	$2 = 1^2 + 1^2$	$17+2\sqrt{2}$	$2^{11} \cdot 281$	18
5	$41 = 5^2 + 4^2$	$13+2\sqrt{41}$	$5 \cdot 41^3$	6
13	$17 = 1^2 + 4^2$	$9+2\sqrt{17}$	$13 \cdot 17^3$	4
17	$137 = 11^2 + 4^2$	$47+4\sqrt{41}$	$17 \cdot 137^3$	142
17	$257 = 1^2 + 16^2$	$95+4\sqrt{257}$	$17 \cdot 257^3$	32
73	$97 = 9^2 + 4^2$	$1891+192\sqrt{97}$	$73 \cdot 97^3$	380
97	$73 = 3^2 + 8^2$	$103+12\sqrt{73}$	$97 \cdot 73^3$	42
137	$17 = 1^2 + 4^2$	$35+8\sqrt{17}$	$137 \cdot 17^3$	48
257	$17 = 1^2 + 4^2$	$23+4\sqrt{17}$	$257 \cdot 17^3$	58

Table 2

(p,q,l)	$\alpha_{(p,q)}$	$d_{\mathbf{K}_2}$	$h^-_{\mathbf{K}_2}$
$(2,17,5)$	$5+2\sqrt{2}$	$17\cdot 2^6\cdot 5^2$	2
$(2,73,5)$	$9+2\sqrt{2}$	$73\cdot 2^6\cdot 5^2$	14
$(2,89,13)$	$11+4\sqrt{2}$	$89\cdot 2^6\cdot 13^2$	38
$(2,233,5)$	$19+8\sqrt{2}$	$233\cdot 2^6\cdot 5^2$	30
$(2,281,13)$	$17+2\sqrt{2}$	$281\cdot 2^6\cdot 13^2$	78
$(5,41,13)$	$(13+\sqrt{5})/2$	$41\cdot 5^2\cdot 13^2$	10
$(5,61,2)$	$(17+3\sqrt{5})/2$	$61\cdot 5^2\cdot 2^6$	6
$(5,109,2)$	$(21+\sqrt{5})/2$	$109\cdot 5^2\cdot 2^6$	14
$(5,149,2)$	$13+2\sqrt{5}$	$149\cdot 5^2\cdot 2^6$	10
$(5,269,2)$	$17+2\sqrt{5}$	$269\cdot 5^2\cdot 2^6$	22
$(5,389,2)$	$(41+5\sqrt{5})/2$	$389\cdot 5^2\cdot 2^6$	26
$(13,17,5)$	$(9+\sqrt{13})/2$	$17\cdot 13^2\cdot 5^2$	6
$(13,29,2)$	$9+2\sqrt{13}$	$29\cdot 13^2\cdot 2^6$	10
$(13,157,2)$	$(41+9\sqrt{13})/2$	$157\cdot 13^2\cdot 2^6$	34
$(13,181,2)$	$(29+3\sqrt{13})/2$	$181\cdot 13^2\cdot 2^6$	54
$(17,137,5)$	$35+8\sqrt{17}$	$137\cdot 17^2\cdot 5^2$	42
$(17,257,5)$	$23+4\sqrt{17}$	$257\cdot 17^2\cdot 5^2$	74
$(29,53,2)$	$13+2\sqrt{29}$	$53\cdot 29^2\cdot 2^6$	14
$(73,97,5)$	$103+12\sqrt{73}$	$97\cdot 73^2\cdot 5^2$	182

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