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On decimal and continued fraction expansions of a real number

by

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0. Introduction. Let x be an irrational number. We deal with the problem of finding from the decimal expansion of x, the first k (where k is a given integer) partial quotients of the regular continued fraction expansion of x. More precisely, for each $n \ge 1$, denote by x_n, y_n with $x_n < x < y_n$ the two consecutive nth decimal approximations of x. We assume that the integer n is such that the numbers x_n and y_n have finite continued fraction expansions which coincide up to order k, i.e., $x_n = [\alpha_0; \alpha_1, \ldots, \alpha_k, \ldots]$ and $y_n = [\alpha_0; \alpha_1, \ldots, \alpha_k, \ldots]$ for some integers α_i. Since the set of numbers which have a continued fraction which begins with $\alpha_0, \ldots, \alpha_k$ is an interval, it follows that $x = [\alpha_0; \alpha_1, \ldots, \alpha_k, \ldots]$, in other words $\alpha_0, \alpha_1, \ldots, \alpha_k$ are precisely the first k partial quotients of x. Writing the two rationals x_n, y_n as a quotient p/q of two integers, i.e., writing

$$x_n = \frac{[10^n x]}{10^n} \quad \text{and} \quad y_n = x_n + \frac{1}{10^n},$$

where $[y]$ denotes the largest integer $\le y$ for each real number y, their continued fraction expansion may be computed exactly. In fact, for a rational number p/q, the continued fraction algorithm shows that we only have to perform operations on integers. This gives a practical method to compute the first k partial quotients of an irrational number if we know as above the n digits of its decimal expansion.

We can believe that for most irrational numbers x, the integer n must be very large compared to k. Denote precisely by $k_n = k_n(x)$ the largest integer $k \ge 0$ such that we can write $x_n = [\alpha_0; \alpha_1, \ldots, \alpha_k, \ldots]$ and $y_n = [\alpha_0; \alpha_1, \ldots, \alpha_k, \ldots]$ for some integers α_i with $\alpha_0 = [x]$. Note that such a representation is always possible. In fact, $[x_n] = [x] = \alpha_0$ and $[y_n] = \alpha_0$ or

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$y_n = \alpha_0 + 1$ and in this last case we can write $y_n = [\alpha_0; 1]$. Hence, x_n, y_n will give k_n partial quotients of x.

In [2] Lochs has proved the following beautiful and surprising result.

THEOREM (Lochs). *For almost all irrationals x, with respect to Lebesgue measure, we have*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{6\log 10 \log 2}{\pi^2} \simeq 0.9702.$$

Since the constant $0.9702\ldots$ of the above theorem is rather close to 1, one can almost say that for large n, the n decimals determine the n first partial quotients.

Consider two examples. For $x = \sqrt[3]{2} = 1.259921\ldots$, we have

$$x_5 = 1.25992 \quad \text{and} \quad y_5 = 1.25993.$$

A computation shows that

$$x_5 = [1; 3, 1, 5, 1, 1, 4, 2, 5, 1, 3] \quad \text{and} \quad y_5 = [1; 3, 1, 5, 1, 1, 5, 5, 1, 2, 1, 4, 3].$$

Therefore $k_5(x) = 5$ and $x = [1; 3, 1, 5, 1, 1, \ldots]$. Thus we obtain from the five decimals of x the first five partial quotients. As another example, the first 1000 decimals of π give exactly 968 partial quotients (see [3]).

In this paper we improve the above theorem of Lochs.

Denote by z_0 the constant $(6\log 10 \log 2)/\pi^2$. As probability measure on [0,1] we will consider the Lebesgue measure denoted by P in this paper. We prove the following theorem.

THEOREM 1 (main theorem). *For all $\varepsilon > 0$, the probability of the set of x for which the distance of $k_n(x)/n$ to z_0 is greater than or equal to ε decreases geometrically to 0, i.e., there exist positive constants C, λ (depending on ε) with $0 < \lambda < 1$ such that*

$$P\left(\left|\frac{k_n}{n} - z_0\right| \geq \varepsilon\right) \leq C\lambda^n$$

for all integers $n \geq 1$.

The above theorem yields immediately that $\sum P(|k_n/n - z_0| \geq \varepsilon) < \infty$ for all $\varepsilon > 0$. Then with the Borel–Cantelli lemma, we deduce easily as a corollary the theorem of Lochs.

The proof of the main theorem will show more precisely that

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{k_n}{n} \leq z_0 - \varepsilon\right) \leq \theta_1(\varepsilon) \qquad (0 < \varepsilon < z_0),$$
$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{k_n}{n} \geq z_0 + \varepsilon\right) \leq \theta_2(\varepsilon),$$

with

$$\theta_1(\varepsilon) = \inf_{0<t<1/2} \frac{1}{t+1}(-t\log 10 + (z_0 - \varepsilon)\log\lambda(2-2t)) < 0$$

and

$$\theta_2(\varepsilon) = \inf_{\alpha>0}(\alpha\log 10 + (z_0+\varepsilon)\log\lambda(2+2\alpha)) < 0.$$

In the above formulas $\lambda(2-2t)$ and $\lambda(2+2\alpha)$ are the dominant eigenvalues of some operators L_s, $s>1$ (transfer operators) defined in Section 2.

The formulas giving θ_1 and θ_2 are interesting. If it is possible to extract further information about the location of the eigenvalues of the operators L_s then we will have more precise estimates of θ_1 and θ_2.

We will also prove a result on approximation. For some irrationals x it may happen that some decimals x_n are better approximations of x than p_n/q_n, i.e., $x - x_n < |x - p_n/q_n|$. We may take for example $x = \sqrt[3]{2}$ and $n = 1, 3, 4, 5$. However, the probability of this to happen decreases quickly to 0 as $n \to \infty$ according to the following theorem.

THEOREM 2. *There exist positive constants C, μ with $0 < \mu < 1$ such that*

$$P\biggl(x - x_n \le \biggl|x - \frac{p_n}{q_n}\biggr|\biggr) \le C\mu^n \qquad (n \ge 1).$$

The proof of the above theorem will show more precisely that

$$\limsup_{n\to\infty} \frac{1}{n}\log P\biggl(x - x_n \le \biggl|x - \frac{p_n}{q_n}\biggr|\biggr) \le \theta$$

with

$$\theta = \inf_{\alpha>0} \frac{1}{\alpha+1}(\alpha\log 10 + \log\lambda(2+2\alpha)) < 0.$$

The following sections are devoted to the proof of Theorems 1 and 2.

1. Conditional probabilities. If $\alpha_1, \ldots, \alpha_i$ are given integers ≥ 1, the set of numbers in $[0,1]$ which have a continued fraction expansion which begins with $\alpha_1, \ldots, \alpha_i$ is an interval (a *fundamental interval*) denoted here as $I(\alpha_1, \ldots, \alpha_i)$. More precisely,

$$I(\alpha_1, \ldots, \alpha_i) = \begin{cases} \left[\dfrac{p_i}{q_i}, \dfrac{p_i + p_{i-1}}{q_i + q_{i-1}}\right] & \text{if } i \text{ is even,} \\ \left[\dfrac{p_i + p_{i-1}}{q_i + q_{i-1}}, \dfrac{p_i}{q_i}\right] & \text{if } i \text{ is odd,} \end{cases}$$

where as usual

$$\frac{p_i}{q_i} = [0; \alpha_1, \ldots, \alpha_i].$$

In the following we will write $I(\alpha_1,\ldots,\alpha_i)=[b_i,c_i]$ for short. Let $r_{ni}=[10^n b_i]$ and $r'_{ni}=[10^n c_i]$, thus

$$\frac{r_{ni}}{10^n}\le b_i<\frac{r_{ni}+1}{10^n}\quad\text{and}\quad\frac{r'_{ni}}{10^n}\le c_i<\frac{r'_{ni}+1}{10^n}.$$

Let $x\in[0,1]$ be an irrational number. If $x\in[b_i,c_i]$, then $k_n(x)\ge i$ only when x_n,y_n both belong to $[b_i,c_i]$. If $(r_{ni}+1)/10^n>c_i$ then $y_n=(r_{ni}+1)/10^n$, thus $y_n\notin[b_i,c_i]$. But if $(r_{ni}+1)/10^n\le c_i$, we will have $x_n,y_n\in[b_i,c_i]$ only when $x\in[(r_{ni}+1)/10^n,r'_{ni}/10^n]$ in the case $r_{ni}/10^n<b_i$ and when $x\in[b_i,r'_{ni}/10^n]$ in the case $r_{ni}/10^n=b_i$. Since

$$c_i-b_i=\frac{1}{q_i(q_i+q_{i-1})},$$

we see that the conditional probability

$$P(k_n<i\mid a_1=\alpha_1,\ldots,a_i=\alpha_i)$$

is given by

$$\begin{cases}1 & \text{if } \dfrac{r_{ni}+1}{10^n}>c_i,\\ \left(\dfrac{r_{ni}+1}{10^n}-b_i+c_i-\dfrac{r'_{ni}}{10^n}\right)q_i(q_i+q_{i-1}) & \text{if } \dfrac{r_{ni}+1}{10^n}\le c_i \text{ and } \dfrac{r_{ni}}{10^n}<b_i,\\ \left(c_i-\dfrac{r'_{ni}}{10^n}\right)q_i(q_i+q_{i-1}) & \text{if } \dfrac{r_{ni}+1}{10^n}\le c_i \text{ and } \dfrac{r_{ni}}{10^n}=b_i.\end{cases}$$

For all $n\ge1$, let t_n and v_n be the functions defined by

$$t_n(y)=10^n y-[10^n y]\quad\text{and}\quad v_n(y)=1-t_n(y).$$

Since

$$\frac{r_{ni}+1}{10^n}-b_i=\frac{v_n(b_i)}{10^n},$$

we can write $P(k_n<i\mid a_1=\alpha_1,\ldots,a_i=\alpha_i)$ as

$$(1)\qquad \begin{cases}1 & \text{if } v_n(b_i)\dfrac{q_i(q_i+q_{i-1})}{10^n}>1,\\ (v_n(b_i)+t_n(c_i))\dfrac{q_i(q_i+q_{i-1})}{10^n} & \\ & \text{if } v_n(b_i)\dfrac{q_i(q_i+q_{i-1})}{10^n}\le1 \text{ and } \dfrac{r_{ni}}{10^n}<b_i,\\ t_n(c_i)\dfrac{q_i(q_i+q_{i-1})}{10^n} & \text{if } v_n(b_i)\dfrac{q_i(q_i+q_{i-1})}{10^n}\le1 \text{ and } \dfrac{r_{ni}}{10^n}=b_i.\end{cases}$$

Note that $P(k_n < i \mid a_1 = \alpha_1, \dots, a_i = \alpha_i)$ is also equal to

$$(2)\qquad \begin{cases} 1 & \text{if } \dfrac{q_i(q_i+q_{i-1})}{10^n} > 1, \\ (v_n(b_i)+t_n(c_i))\dfrac{q_i(q_i+q_{i-1})}{10^n} & \text{if } \dfrac{q_i(q_i+q_{i-1})}{10^n} \le 1 \text{ and } \dfrac{r_{ni}}{10^n} < b_i, \\ t_n(c_i)\dfrac{q_i(q_i+q_{i-1})}{10^n} & \text{if } \dfrac{q_i(q_i+q_{i-1})}{10^n} \le 1 \text{ and } \dfrac{r_{ni}}{10^n} = b_i. \end{cases}$$

In fact, if $v_n(b_i)q_i(q_i+q_{i-1})/10^n \le 1$ and $q_i(q_i+q_{i-1})/10^n > 1$, or equivalently if

$$\frac{r_{ni}+1}{10^n} \le c_i \quad \text{and} \quad c_i - b_i < \frac{1}{10^n},$$

then we will necessarily have $r_{ni}/10^n < b_i$ and $(r_{ni}+1)/10^n = r'_{ni}/10^n$, thus

$$\frac{v_n(b_i)+t_n(c_i)}{10^n} = c_i - b_i = \frac{1}{q_i(q_i+q_{i-1})}.$$

Let T_{ni} be the random variable

$$T_{ni} = P(k_n < i \mid a_1, \dots, a_i),$$

so, for the expectation of T_{ni} we have

$$E(T_{ni}) = P(k_n < i).$$

2. Transfer operators. Let $E = A_\infty(D)$ be the Banach space of bounded holomorphic functions on the disk $D = \{z : |z-1| < 3/2\}$. The space E is naturally endowed with the supremum norm $\|f\|_\infty = \sup_{z\in D}|f(z)|$. For each complex number s with $\mathrm{Re}(s) > 1$, we consider the following operator on E:

$$L_s(f)(z) = \sum_{n=1}^{\infty} \frac{1}{(n+z)^s} f\left(\frac{1}{n+z}\right) \qquad (z \in D).$$

Note that for $s = 2$, L_s is the "analogue in E" of the Perron–Frobenius operator of the Gauss transformation of continued fractions.

We recall in the following theorem some known properties of these operators L_s (see for example [4] and [1]).

THEOREM 3. (a) *L_s is a nuclear operator of order* 0 (*hence it is compact in particular*).

(b) *For all real $s > 1$, L_s has a dominant eigenvalue $\lambda(s) > 0$ of multiplicity* 1.

(c) *The map $s \to \lambda(s)$ is analytic.*

(d) $\lambda(2) = 1$ *and* $\lambda'(2) = -\pi^2/(12\log 2)$.

A computation shows that the iterates of L_s are given by the formula

$$L_s^n(f)(z) = \sum_{k_1,\ldots,k_n} \frac{1}{(zq_{n-1}+q_n)^s} f\left(\frac{zp_{n-1}+p_n}{zq_{n-1}+q_n}\right),$$

where $k_1,\ldots,k_n$ run over the integers ≥ 1 and

$$\frac{p_n}{q_n} = [0; k_1,\ldots,k_n].$$

In particular, we have

$$L_s^n(f)(0) = \sum_{k_1,\ldots,k_n} \frac{1}{q_n^s} f\left(\frac{p_n}{q_n}\right).$$

Using the well-known formula

$$\frac{q_{n-1}}{q_n} = [0; k_n,\ldots,k_1],$$

we see by inverting the order of summation that we also have

$$L_s^n(f)(0) = \sum_{k_1,\ldots,k_n} \frac{1}{q_n^s} f\left(\frac{q_{n-1}}{q_n}\right). \tag{3}$$

We use the operators L_s to prove some probabilistic estimates about the denominators of the convergents q_n which will be useful later. The letter E denotes as usual the expectation operator.

PROPOSITION 1. (i) *For each* $\alpha > 0$, *there exists a constant* $C = C_\alpha$ *such that*

$$E\left(\frac{1}{q_n^{2\alpha}}\right) \leq C\lambda^n(2\alpha+2) \qquad (n \geq 1).$$

(ii) *For each* $t < 1/2$, *there exists a constant* $C = C_t$ *such that*

$$E(q_n^{2t}) \leq C\lambda^n(2-2t) \qquad (n \geq 1).$$

Proof. (i) The expectation of $1/q_n^{2\alpha}$ is given by

$$E\left(\frac{1}{q_n^{2\alpha}}\right) = \sum_{k_1,\ldots,k_n} \frac{1}{q_n^{2\alpha}} \cdot \frac{1}{q_n(q_n+q_{n-1})} = \sum_{k_1,\ldots,k_n} \frac{1}{q_n^{2\alpha+2}} \cdot \frac{1}{1+q_{n-1}/q_n},$$

thus from (3), $E(q_n^{-2\alpha}) = L_{2\alpha+2}^n(f)(0)$, where $f(z) = 1/(1+z)$. From (b) of Theorem 3, we deduce that $|L_{2\alpha+2}^n(f)(0)| \leq C\lambda^n(2\alpha+2)$ for some constant $C > 0$, thus (i) is proved.

(ii) Following the lines of (i), the expectation of q_n^{2t} is given by $E(q_n^{2t}) = L_{2-2t}^n(f)(0)$ for $t<1/2$, with the same function f. This proves the result. ∎

3. Proof of the main theorem. First part. Since $0 \le T_{ni} \le 1$, we have for all $a > 0$,

$$E(T_{ni}) \le a + P(T_{ni} \ge a). \tag{4}$$

From (1), we have

$$T_{ni} \le (v_n(b_i) + t_n(c_i)) \frac{q_i(q_i + q_{i-1})}{10^n},$$

thus

$$T_{ni} \le \frac{4q_i^2}{10^n}.$$

Note that $E(q_i^2) = \infty$. Hence we cannot obtain a majorization of $E(T_{ni})$ directly from the above inequality by taking expectations. However, we deduce

$$P(T_{ni} \ge a) \le P\left(\frac{q_i^2}{10^n} \ge \frac{a}{4}\right).$$

From the Markov inequality, for all $t > 0$,

$$P\left(\frac{q_i^2}{10^n} \ge \frac{a}{4}\right) \le \left(\frac{4}{a}\right)^t 10^{-nt} E(q_i^{2t}).$$

Hence from (4) and Proposition 1, where we restrict $0 < t < 1/2$, we get the inequality

$$P(k_n < i) = E(T_{ni}) \le a + \frac{C4^t 10^{-nt} \lambda^i (2-2t)}{a^t}.$$

Taking $a = A^{1/(t+1)}$ with $A = C4^t 10^{-nt} \lambda^i (2-2t)$, we obtain

$$P(k_n < i) \le 2A^{1/(t+1)}.$$

Let (i_n) be a sequence of integers ≥ 1 such that

$$\lim_{n\to\infty} \frac{i_n}{n} = z_0 - \varepsilon$$

and

$$\frac{i_n}{n} > z_0 - \varepsilon \quad \text{for all } n \ge 1.$$

From the last inequality for $P(k_n < i)$ we obtain for all $0 < t < 1/2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(k_n < i_n) \le \frac{1}{t+1}(-t \log 10 + (z_0 - \varepsilon) \log \lambda(2-2t)).$$

Thus

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{k_n}{n} \le z_0 - \varepsilon\right) \le \theta_1(\varepsilon)$$

with

$$\theta_1(\varepsilon) = \inf_{0<t<1/2} \frac{1}{t+1}(-t \log 10 + (z_0 - \varepsilon) \log \lambda(2-2t)),$$

since $P(k_n/n \le z_0 - \varepsilon) \le P(k_n < i_n)$ from the choice of (i_n). Now we show that $\theta_1(\varepsilon) < 0$. In fact, consider for $u < 1/2$ the function h defined by

$$h(u) = -u \log 10 + (z_0 - \varepsilon) \log \lambda(2 - 2u).$$

By (d) of Theorem 3, $h(0) = 0$ and $h'(0) = -\log 10 + (z_0 - \varepsilon)\pi^2/(6 \log 2) < 0$. Thus if t is sufficiently small, then $h(t) < 0$, which implies that $\theta_1(\varepsilon) < 0$ as asserted.

4. Proof of the main theorem. Second part. From (2) we have

$$P\left(\frac{q_i(q_i + q_{i-1})}{10^n} > 1\right) \le P(T_{ni} = 1) \le E(T_{ni}) = P(k_n < i),$$

thus

$$P(k_n \ge i) \le P\left(\frac{10^n}{q_i(q_i + q_{i-1})} \ge 1\right).$$

This last inequality can also be proved by noticing that if $k_n \ge i$ then x_n, y_n are in the same i-fundamental interval as x, thus

$$y_n - x_n = \frac{1}{10^n} \le \frac{1}{q_i(q_i + q_{i-1})},$$

and this gives as above

$$P(k_n \ge i) \le P\left(\frac{10^n}{q_i(q_i + q_{i-1})} \ge 1\right).$$

We can write

$$P(k_n \ge i) \le P\left(\frac{10^n}{q_i^2} \ge 1\right).$$

From the Markov inequality and Proposition 1, we get for all $\alpha > 0$,

$$P(k_n \ge i) \le 10^{n\alpha} E\left(\frac{1}{q_i^{2\alpha}}\right) \le C 10^{n\alpha} \lambda^i(2 + 2\alpha).$$

Now take a sequence (i_n) of integers ≥ 1 such that

$$\lim_{n\to\infty} \frac{i_n}{n} = z_0 + \varepsilon$$

and

$$\frac{i_n}{n} \le z_0 + \varepsilon \quad \text{for all } n \ge 2.$$

We have

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{k_n}{n} \ge z_0 + \varepsilon\right) \le \theta_2(\varepsilon)$$

with

$$\theta_2(\varepsilon) = \inf_{\alpha > 0} (\alpha \log 10 + (z_0 + \varepsilon) \log \lambda(2 + 2\alpha)).$$

Now we prove that $\theta_2(\varepsilon) < 0$. As in the first part of the proof, consider the function

$$h(u) = u \log 10 + (z_0 + \varepsilon) \log \lambda(2 + 2u) \quad (u > -1/2),$$

and note that $h(0) = 0$ and $h'(0) < 0$, thus $h(\alpha) < 0$ for α sufficiently close to 0 and $\theta_2(\varepsilon) < 0$.

5. Proof of Theorem 2. From $x - x_n = t_n(x)/10^n$ and

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2},$$

we deduce

$$P\left(x - x_n \le \left| x - \frac{p_n}{q_n} \right| \right) \le P\left(t_n < \frac{10^n}{q_n^2} \right).$$

For all $\varepsilon > 0$ and $\alpha > 0$, we can write

$$P\left(t_n < \frac{10^n}{q_n^2} \right) \le P(t_n \le \varepsilon) + P\left(\frac{10^n}{q_n^2} > \varepsilon \right) \le \varepsilon + \frac{C 10^{n\alpha} \lambda^n (2 + 2\alpha)}{\varepsilon^\alpha}.$$

The last inequality follows from the Markov inequality, Proposition 1, and the fact that for all $n \ge 1$, t_n is distributed according to the uniform law on $[0, 1]$. Taking

$$\varepsilon = (C 10^{n\alpha} \lambda^n (2 + 2\alpha))^{1/(\alpha+1)},$$

we have

$$P\left(t_n < \frac{10^n}{q_n^2} \right) \le 2(C 10^{n\alpha} \lambda^n (2 + 2\alpha))^{1/(\alpha+1)},$$

thus

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(t_n < \frac{10^n}{q_n^2} \right) \le \theta$$

with

$$\theta = \inf_{\alpha>0} \frac{1}{\alpha+1} (\alpha \log 10 + \log \lambda(2 + 2\alpha)) < 0,$$

which proves the theorem. ■

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