Errata to the paper

“On a functional equation satisfied by certain Dirichlet series”


by

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We have to point out that formula (5) in [1] is wrong, as well as the formula for \( \Phi_L(s) \) given in the statement of the Theorem in [1]. The following lemma will take the place of formula (5) in [1].

**Lemma 0.1.** The following formula for derivatives of higher order of \( z^{\nu}I_\nu(z) \) holds:

\[
\frac{d^p}{dz^p}(z^{\nu}I_\nu(z)) = \sum_{l=0}^{\lfloor p/2 \rfloor} (2l-1)!! \left(\frac{p}{2l}\right) z^{\nu-l}I_{\nu-(p-l)}(z)
\]

if we put \((-1)!! = 1\).

**Proof.** From the well known formula (see [2])

\[
\frac{d}{dz}(z^{\nu}I_\nu(z)) = z^{\nu}I_{\nu-1}(z)
\]

we derive, by induction, that if \( p \geq 1 \) then

\[
\frac{d^p}{dz^p}(z^{\nu}I_\nu(z)) = \sum_{l=0}^{\lfloor p/2 \rfloor} \beta_{p,l} z^{\nu-l}I_{\nu-(p-l)}(z).
\]

By a direct computation we get \( \beta_{p,0} = 1 \) for all \( p \geq 1 \). Comparing

\[
\frac{d^{p+1}}{dz^{p+1}}(z^{\nu}I_\nu(z)) = \sum_{l=0}^{\lfloor (p+1)/2 \rfloor} \beta_{p+1,l} z^{\nu-l}I_{\nu-(p+1-l)}(z)
\]

with

\[
\frac{d}{dz}\left(\frac{d^p}{dz^p}(z^{\nu}I_\nu(z))\right)
\]
developed by (0.2) from (0.3), we obtain the following recurrence formula:

\[ \beta_{p+1,t} = (p - 2t + 2)\beta_{p,t-1} + \beta_{p,t}, \]

where \( p \geq 1 \), \( 0 \leq t \leq \lfloor (p + 1)/2 \rfloor \) and \( \beta_{p,i} = 0 \) if \( i > \lfloor p/2 \rfloor \) or \( i < 0 \). From (0.4) for \( t \geq 2 \) due to the well known formula

\[ \sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1} \]

we obtain, for all \( p \geq 1 \),

\[ \beta_{p+1,t} = (2t - 1)!! \binom{p+1}{2t}. \]

We note that \( \beta_{1,0} = 1 \), so (0.5) holds if \( p = 0 \). If \( t = 0 \), taking \((-1)!! = 1\) the above formula holds by a direct computation. For \( t = 1 \), (0.5) follows directly from (0.4). ✷

By using formula (0.1) we obtain the corrected form for the function \( \Phi_L(s) \) given in the statement of the Theorem in [1].

In the proof of the Theorem of [1] we have to replace page 270, from the fifth line starting with “By Cauchy’s theorem . . .” up to the end of the page, with the following:

By Cauchy’s theorem we have

\[ I_N(s) = -\sum_{-N \leq 2n \leq N, n \neq 0} \text{Res} \left( H(z)I_{s-1/2} \left( \frac{\delta}{2} z \right) z^{s-1/2} ; 2\pi ni \right). \]

If we put

\[ A(z) = I_{s-1/2} \left( \frac{\delta}{2} z \right) z^{s-1/2}, \]

its Taylor series at \( s = 2\pi ni, n \neq 0 \), is

\[ A(z) = \sum_{m=0}^{\infty} \frac{1}{m!} A^{(m)}(2\pi ni)(z - 2\pi ni)^m. \]

Then we have

\[ \text{Res}(H(z)A(z); 2\pi ni) \]

\[ = \sum_{p=1}^{d} \sum_{l=0}^{(d+1)} \frac{1}{l!} \alpha_p A^{(l)}(2\pi ni) = \sum_{p=0}^{d} \frac{1}{p!} \alpha_p A^{(p)}(2\pi ni). \]
By (0.1),

\[ A(p)(z) = \sum_{l=0}^{[p/2]} (2l - 1)!! \left( \frac{p}{2l} \right) \left( \frac{\delta}{2} \right)^{p-l} p^{s-1/2-l} I_{s-1/2-(p-l)} \left( \frac{\delta}{2} \right)^{p-l} z^{s-1/2-l}. \]

Therefore

\[ I_N(s) = -\sum_{n \in \mathbb{Z}} \sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \sum_{n \neq 0}^{\frac{p}{2l}} \frac{(2l - 1)!!}{p!} \left( \frac{p}{2l} \right) \left( \frac{\delta}{2} \right)^{p-l} \times \alpha_{n-p-1}^{n} (2n\pi i)^{s-1/2-l} I_{s-1/2-(p-l)} (\delta n\pi i). \]

By (2) and (3) of [1] the series

\[ \sum_{n \neq 0}^{\frac{p}{2l}} \alpha_{n-p-1}^{n} (2n\pi i)^{s-1/2-l} I_{s-1/2-(p-l)} (\delta n\pi i) \]

converges absolutely and uniformly on compact subsets of \( \sigma < 0 \). Thus, for \( \sigma < 0 \), we have

\[ I(s) = -\sum_{n \in \mathbb{Z}} \sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \frac{(2l - 1)!!}{p!} \left( \frac{p}{2l} \right) \left( \frac{\delta}{2} \right)^{p-l} \times \alpha_{n-p-1}^{n} (2n\pi i)^{s-1/2-l} I_{s-1/2-(p-l)} (\delta n\pi i). \]

Then we derive the final formula for \( \Phi_L(s) \) in \( \sigma > 1 \):

\[ \Phi_L(s) = I(1 - s) = -\sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \sum_{n \neq 0}^{\frac{p}{2l}} \frac{(2l - 1)!!}{p!} \left( \frac{p}{2l} \right) \left( \frac{\delta}{2} \right)^{p-l} \times \alpha_{n-p-1}^{n} (2\pi n\pi i)^{1/2-s-l} I_{1/2-s-(p-l)} (\delta n\pi i). \]

References


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