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On arithmetic progressions of equal lengths and equal products of terms

by

AJAI CHOUDHRY (Beirut)

The problem of finding arithmetic progressions of equal lengths of positive integers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, such that the products of their terms are equal has been considered by Gabovich [1], Mirkowska and Makowski [2], Szymiczek [5] and by Saradha, Shorey and Tijdeman [3, 4]. When $n = 3$ and $n = 4$, infinitely many examples are already known [1]. When $n > 4$, the only known example of two arithmetic progressions with equal products of terms is given by

$$(n+1)(n+2)(n+3)\ldots(2n) = 2 \cdot 6 \cdot 10 \cdot \ldots \cdot (4n-2).$$

In fact, it is proved in [4] that this example provides the only solution in positive integers of the Diophantine equation

$$x(x+d_1)(x+2d_1)\ldots\{x+(n-1)d_1\} = y(y+d_2)(y+2d_2)\ldots\{y+(n-1)d_2\}$$

with d_1, d_2 being fixed positive integers, $d_1 < d_2$ and n arbitrarily large.

In this paper, we shall obtain for arbitrary n a new solution of two arithmetic progressions in positive integers with equal products of terms. In addition, further examples are given of infinitely many arithmetic progressions with equal products of terms when $n = 4$ and $n = 5$.

Let $t_0, t_1, t_2, \ldots, t_n$ be $n+1$ positive integers in arithmetic progression with common difference d so that $t_n = t_0 + nd$. Let r and s be two positive integers with $r > s$. We define

$$a_i = rt_{i-1} \quad \text{for } i = 1, \ldots, n,$$
$$b_i = st_i \quad \text{for } i = 1, \ldots, n.$$

Clearly $a_1, \ldots, a_n$ are in arithmetic progression with common difference rd while $b_1, \ldots, b_n$ are in arithmetic progression with common difference sd. The products of the terms of these arithmetic progressions will be equal if

$$(rt_0)(rt_1)\ldots(rt_{n-1}) = (st_1)(st_2)\ldots(st_n), \quad \text{or}$$

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$$r^n t_0 = s^n t_n, \quad \text{or}$$
$$r^n t_0 = s^n (t_0 + nd), \quad \text{or}$$
$$(r^n - s^n) t_0 = n s^n d.$$

Thus, if we take $t_0 = ns^n$ and $d = r^n - s^n$, the products of the terms of the two arithmetic progressions $\{a_i\}$ and $\{b_i\}$ will be equal. As $r > s$, all the terms of the two arithmetic progressions are positive integers. Moreover, by choice of r and s, we can readily ensure that the two arithmetic progressions do not have a common term. For the two arithmetic progressions to have a common term, we must have $a_i = b_j$ for some i and j where $1 \le i \le n$ and $1 \le j \le n$. Thus, we must have for some i and j,

$$r t_{i-1} = s t_j, \quad \text{or}$$
$$r\{t_0 + (i-1)d\} = s(t_0 + jd), \quad \text{or}$$
$$r\{ns^n + (i-1)(r^n - s^n)\} = s\{ns^n + j(r^n - s^n)\}.$$

Dividing by s^{n+1}, and writing r/s as θ, we get the equation

$$n\theta + (i-1)\theta(\theta^n - 1) = n + j(\theta^n - 1).$$

For given i, j this has at most $n+1$ rational roots and since $1 \le i \le n$ and also $1 \le j \le n$, there are only a finite number of such equations for any given n. Thus there are only a finite number of rational values of r/s for which we could have $a_i = b_j$ for some i and j. Hence, by choosing r and s such that $r > s$ and r/s is different from any of the finite number of rational values which make $a_i = b_j$ for some i, j, we can ensure that the two arithmetic progressions do not have a common term. The two arithmetic progressions may be explicitly stated as follows:

(i) an arithmetic progression with first term nrs^n and common difference $r(r^n - s^n)$,

(ii) an arithmetic progression with first term $s\{r^n + (n-1)s^n\}$ and common difference $s(r^n - s^n)$.

As an example, when $n = 7$, $r = 2$, $s = 1$, we get the following two arithmetic progressions with equal products of terms:

$$14,\ 268,\ 522,\ 776,\ 1030,\ 1284,\ 1538$$

and

$$134,\ 261,\ 388,\ 515,\ 642,\ 769,\ 896.$$

When $n = 4$, we give below, in parametric form, an example, different from the above solution of two arithmetic progressions with equal products of terms:

(i) $63pq^4,\ 2p(16p^4 + 9q^4),\ p(64p^4 - 27q^4),\ 24p(4p^4 - 3q^4),$

(ii) $3q(16p^4 + 9q^4),\ 56p^4q,\ q(64p^4 - 27q^4),\ 18q(4p^4 - 3q^4).$

It is readily verified that these are indeed arithmetic progressions such that the products of their terms are equal. The solution is in positive integers when p and q are positive integers such that

$$p > (3/4)^{1/4}q, \quad \text{or} \quad p > (0.930\ldots)q.$$

As a numerical example, with $p = 2$, $q = 1$, we get the following two arithmetic progressions with equal products of terms:

$$126,\ 1060,\ 1994,\ 2928 \quad \text{and} \quad 795,\ 896,\ 997,\ 1098.$$

Similarly, when $n = 5$, the following two arithmetic progressions have equal products of terms:

(i) $6p(p^5 - 3q^5),\ p(4p^5 + 27q^5),\ 2p(p^5 + 36q^5),\ 117pq^5,\ 2p(81q^5 - p^5)$,

(ii) $2q(81q^5 - p^5),\ 3q(p^5 + 36q^5),\ 2q(4p^5 + 27q^5),\ 13p^5q,\ 18q(p^5 - 3q^5)$.

The solution is in positive integers when p and q are positive integers such that

$$3^{1/5}q < p < 81^{1/5}q, \quad \text{or} \quad (1.245\ldots)q < p < (2.408\ldots)q.$$

As a numerical example, when $p = 5$ and $q = 3$, we get the following two arithmetic progressions with equal products of terms:

$$71880,\ 95305,\ 118730,\ 142155,\ 165580$$

and

$$99348,\ 106857,\ 114366,\ 121875,\ 129384.$$

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Embassy of India
Sahmarani Building
Kantari Street
P.O. Box 113-5240
Beirut, Lebanon

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