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The space of period polynomials

by

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1. Introduction. The purpose of this note is to enhance our understanding of the space of period polynomials. The period polynomials has been studied in connection with modular integrals (e.g., Knopp [4]), cusp forms via the Eichler–Shimura isomorphism (e.g., Kohnen–Zagier [5]), and with various other topics of mathematics (Zagier [8]). Let K be a field, and X be an indeterminate. Let $V_w = V_w(K)$ denote the space of polynomials in X of degree $\le w$ (even positive) with coefficients in K. Then $V_w(K)$ is an $(w+1)$-dimensional vector space, which may be identified with the space $\bigoplus_{n=0}^{w} K(X^n)$. Let $G = \mathrm{GL}_2(\mathbb{Z})/\pm 1$. Then G acts on V_w via

$$(P|\gamma)(X) = P\left(\frac{aX+b}{cX+d}\right)(cX+d)^w \tag{1.1}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $P(X) \in V_w$. Now we define the space, W_w, of *period polynomials* of weight w to be the subspace of V_w characterized by the following properties: $W_w = \ker(1+S) \cap \ker(1+U+U^2)$ (see [1, 4, 5, 7]), namely,

$$W_w = \{P \in V_w : P + P|S = P + P|U + P|U^2 = 0\}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Though the space W_w is interesting in its own right, it might be more natural to consider period Laurent polynomials, rather than period polynomials alone. Let $\widehat{V}_w$ be the space $K(X^{-1}) \oplus V_w \oplus K(X^{w+1})$. The space, $\widehat{W}_w$, of *period Laurent polynomials* can be defined in a similar way, and it turns out to be a subspace of $\widehat{V}_w$. In Lemma 2.2, it will be shown that W_w

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is a codimension one subspace of $\widehat{W}_w$. From the definition, clearly $1 - X^w$ belongs to W_w and it represents a "trivial" element. Therefore, it might be more natural to consider the quotient spaces $W_w/\langle 1-X^w\rangle$ and $\widehat{W}_w/\langle 1-X^w\rangle$.

The main purpose of this note is to construct homomorphisms $q\alpha : V_w \to \widehat{V}_w/\langle 1 - X^w\rangle$ and $q\beta : V_w \to \widehat{V}_w/\langle 1 - X^w\rangle$, and describe their images explicitly. (See Theorem 3.3, Lemmas 4.1 and 5.1.) The two mappings $q\alpha$ and $q\beta$ are expressed using Bernoulli polynomials, and their images are indeed identified with $\widehat{W}_w/\langle 1-X^w\rangle$. Consequently, this will yield a spanning set $\{q\alpha(X^n)\}_{n=0}^w$ or $\{q\beta(X^n)\}_{n=0}^w$ of $\widehat{W}_w/\langle 1 - X^w\rangle$. We obtain a relation between the polynomials $\beta(X^n)$ and $r^\pm(R_n)$ of Kohnen–Zagier [5], and thus, we show that

$$\{r^\pm(R_n)\}_{n=0}^w \bmod 1 - X^w \quad \text{spans} \quad W_w/\langle 1 - X^w\rangle. \tag{1.2}$$

In Kohnen–Zagier [5, p. 203], the fact (1.2) was obtained using the isomorphism theorem of Eichler and Shimura for period mappings. In this note, we take a reverse route from that of [5], namely, we first construct a spanning set for the space $W_w/\langle 1 - X^w\rangle$ in terms of the homomorphisms $q\beta$, and as a corollary of this result, we rediscover the isomorphism theorem of Eichler and Shimura for period mappings.

2. Preliminaries. Throughout the paper we assume that w is an even positive integer. For each w, let

$$V_w(K) = \{\text{polynomials of degree} \le w \text{ in } X \text{ with coefficients in } K\}.$$

We often write V_w for $V_w(K)$ when the field K is plain. The action of G on V_w defined in (1.1) can be extended to an action of the group ring $\mathbb{Z}G$ by

$$\Big(P\Big|\sum n_i\gamma_i\Big) = \sum n_i(P|\gamma_i)$$

for $n_i \in \mathbb{Z}$ and $\gamma_i \in G$. Using the action, a subspace W_w of V_w can be described as

$$W_w = \ker(1+S) \cap \ker(1+U+U^2) \tag{2.1}$$

$$= \{P \in V : P + P|S = P + P|U + P|U^2 = 0\} \tag{2.2}$$

for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Following Kohnen–Zagier [5, p. 199], we consider the action of a specific element, i.e., $\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It is shown in [5] that $W|\varepsilon = W$ and there is a direct sum decomposition

$$W_w = W_w^+ \oplus W_w^-$$

of W_w such that $P|\varepsilon = \pm P$ for $P \in W_w^\pm$. More precisely,

$$W_w^+ = \{P \in W_w : P \text{ is an even polynomial}\},$$
$$W_w^- = \{P \in W_w : P \text{ is an odd polynomial}\}.$$

We may also consider the space

$$\widehat{V} = \Big\{ \sum_{i=m}^{n} c_i X^i : m, n \in \mathbb{Z} \text{ such that } m \le n,\ c_i \in K \Big\}$$

of Laurent polynomials in one variable and its subspace

$$\widehat{V}_w = \Big\{ \sum_{i=-1}^{w+1} c_i X^i : c_i \in K \Big\}.$$

For $\gamma \in G$ and $P \in \widehat{V}$, $P|\gamma$ is defined by (1.1). It is no longer an element of $\widehat{V}$, but a rational function. However, the equation $P|\gamma = 0$ will make sense. The space $\widehat{W}_w$ is defined similarly to (2.2), i.e.,

$$\widehat{W}_w = \{P \in \widehat{V} :\ P + P|S = P + P|U + P|U^2 = 0\}.$$

Clearly $W_w \subset \widehat{W}_w$. Moreover, $\widehat{W}_w^{\pm}$ are defined similarly to $W_w^{\pm}$:

$$\widehat{W}_w^{+} = \{P \in \widehat{W}_w : P \text{ is an even Laurent polynomial}\},$$
$$\widehat{W}_w^{-} = \{P \in \widehat{W}_w : P \text{ is an odd Laurent polynomial}\}.$$

It is obvious that $\widehat{W}_w = \widehat{W}_w^{+} \oplus \widehat{W}_w^{-}$. We call an element of W_w (respectively, $\widehat{W}_w$) a *period polynomial* (respectively, *period Laurent polynomial*) of weight w. We also call W_w (respectively, $\widehat{W}_w$) the *space of period polynomials* (respectively, *period Laurent polynomials*) of weight w.

Now we consider a special element of $\widehat{V}_w$. Let f_w be an element of $\widehat{V}_w$ defined by

$$f_w(X) = \sum_{\substack{n=0 \\ n \text{ even}}}^{w+2} \frac{B_n B_{w+2-n}}{n!(w+2-n)!} X^{n-1}.$$

It was shown in Zagier [7, p. 453] that $f_w \in \widehat{W}_w^{-}$ (the homogeneous version of this fact was also proved in [3]). Let $\langle f_w \rangle$ denote the subspace of $\widehat{W}_w^{-}$ which is spanned by f_w. We are interested in how different $\widehat{W}_w^{\pm}$ and $W_w^{\pm}$ are. This will be answered in Lemmas 2.2 and 2.3. We will also show that $\widehat{W}_w \subset \widehat{V}_w$.

LEMMA 2.1. *For $m \ge 2$, let $P(X) = \sum_{i=-m}^{w+m} c_i X^i$ be a Laurent polynomial such that $P|(1+S) = P|(1+U+U^2) = 0$. Then $c_{-m} = c_{w+m} = 0$.*

Proof. Since $P|(1+S) = 0$, we have

$$\sum_{i=-m}^{w+m} c_i X^i + \sum_{i=-m}^{w+m} c_i \Big(\frac{-1}{X}\Big)^i X^w = 0,$$

namely,

$$c_i + (-1)^{w-i} c_{w-i} = 0 \qquad \text{for } i = -m, \dots, 0, \dots, w+m. \tag{2.3}$$

Since $P|(1+U+U^2) = 0$, we have

$$\sum_{i=-m}^{w+m} c_i X^i + \sum_{i=-m}^{w+m} c_i \left(\frac{X-1}{X}\right)^i X^w + \sum_{i=-m}^{w+m} c_i \left(\frac{-1}{X-1}\right)^i (X-1)^w = 0. \tag{2.4}$$

Multiply (2.4) by $X^m (X-1)^m$ to obtain

$$\sum_{i=-m}^{w+m} c_i X^{m+i} (X-1)^m + \sum_{i=-m}^{w+m} c_i (X-1)^{m+i} X^{w-i+m} + \sum_{i=-m}^{w+m} c_i (-1)^i (X-1)^{w-i+m} X^m = 0. \tag{2.5}$$

We calculate the coefficient of X^1 on the left hand side of (2.5). Since $m \geq 2$ by the assumption, we obtain

$$(-1)^{m-1} m c_{-m} + (-1)(w+2m) c_{w+m} + (-1)^m c_{-m+1} + (-1) c_{w+m-1} = 0. \tag{2.6}$$

The equations (2.6) and (2.3) imply that $(w+m)c_{w+m} = 0$. Hence $c_{w+m} = 0$, and then $c_{-m} = 0$ by (2.3), completing the proof. ∎

Lemma 2.2.

$$\widehat{W}_w = W_w \oplus \langle f_w \rangle.$$

Proof. Let $P(X) = \sum_{i=-m}^{w+m} c_i X^i$ belong to $\widehat{W}_w$. Then we can assume $m = 1$ by Lemma 2.1 above. Set

$$Q(X) = P(X) - c_{-1} \frac{(w+2)!}{B_{w+2}} f_w(X).$$

Then the fact that the coefficients of X^{-1} and X^{w+1} in $Q(X)$ vanish implies that $Q(X) \in W_w$. Hence $\widehat{W}_w \subset W_w \oplus \langle f_w \rangle$. Since it is clear that $W_w \oplus \langle f_w \rangle \subset \widehat{W}_w$, we complete the proof. ∎

Observing $f_w \in \widehat{W}_w^-$, we have the following:

Lemma 2.3. (1) $\widehat{W}_w^+ = W_w^+$.
(2) $\widehat{W}_w^- = W_w^- \oplus \langle f_w \rangle$.

Note that $\widehat{W}_w \subset \widehat{V}_w$ from Lemma 2.2.

3. The mapping α. It is easy to see that $1 - X^w$ belongs to $W_w^+ \subset W_w \subset \widehat{W}_w \subset \widehat{V}_w$. So we may consider the various quotient spaces, e.g., $W_w^+/\langle 1 - X^w \rangle$, $\widehat{W}_w/\langle 1 - X^w \rangle$, $\widehat{V}_w/\langle 1 - X^w \rangle$.

The purpose of this section is to define a map $q\alpha : V_w \to \widehat{V}_w/\langle 1 - X^w\rangle$ whose image is exactly the space $\widehat{W}_w/\langle 1 - X^w\rangle$. Let V'_w denote the space of polynomials of degree $\le w+1$, namely, $V'_w = \{\sum_{n=0}^{w+1} c_n X^n : c_n \in K\}$.

DEFINITION 3.1. (1) Let $u : V_w \to V_w$ be defined by

$$uP(X) = P|(1-U)(X) \quad \text{for } P = P(X) \in V_w.$$

(2) Let $b : V_w \to V'_w$ be defined by

$$bP(X) = \sum_{n=0}^{w} \frac{c_n}{n+1} B_{n+1}(X+1)$$

where we write $P(X) = \sum_{n=0}^{w} c_n X^n \in V_w$, with $B_n(X)$ denoting nth Bernoulli polynomial.

(3) Let $s : V'_w \to \widehat{V}_w$ be defined by

$$sP(X) = P|(1-S)(X),$$

more explicitly,

$$sP(X) = \sum_{n=0}^{w+1} c_n X^n - \sum_{n=0}^{w+1} (-1)^n c_n X^{w-n}$$

for $P(X) = \sum_{n=0}^{w+1} c_n X^n \in V'_w$.

(4) Let $\alpha : V_w \to \widehat{V}_w$ be defined by $\alpha = sbu$.

(5) Let $q : \widehat{V}_w \to \widehat{V}_w/\langle 1 - X^w\rangle$ be the projection map.

We need two lemmas to prove the main theorem.

LEMMA 3.1.

$$bP(X) - bP(X-1) = P(X).$$

Proof. Set $P(X) = \sum_{n=0}^{w} c_n X^n$. Then

$$\begin{aligned} bP(X) - bP(X-1) &= \sum_{n=0}^{w} \frac{c_n}{n+1}(B_{n+1}(X+1) - B_{n+1}(X)) \\ &= \sum_{n=0}^{w} \frac{c_n}{n+1}(n+1)X^n = P(X). \end{aligned}$$

This follows from the property of Bernoulli polynomials:

$$B_{n+1}(X+1) - B_{n+1}(X) = (n+1)X^n. \quad \blacksquare \tag{3.1}$$

LEMMA 3.2.

$$\alpha(1) = -\frac{1}{w+1}(X^{w+1} - (-1)^{w+1}X^{-1}) + (\textit{terms of degrees from } 0 \textit{ to } w).$$

In particular, the coefficient of X^{-1} in $\alpha(1)$ does not vanish.

Proof.

$$\begin{aligned}\alpha(1) = sbu(1) = sb(1-X^w) &= s\left(\frac{B_1(X+1)}{0+1} - \frac{B_{w+1}(X+1)}{w+1}\right)\\ &= s\left(B_1(X) + X^0 - \frac{B_{w+1}(X)}{w+1} - X^w\right)\\ &= s\left(-\frac{X^{w+1}}{w+1} + (\text{terms of degrees from } 0 \text{ to } w)\right)\\ &= -\frac{1}{w+1}(X^{w+1} - (-1)^{w+1}X^{-1})\\ &\quad + (\text{terms of degrees from } 0 \text{ to } w). \ \blacksquare\end{aligned}$$

Now we are ready to describe the image of the homomorphism

$$q\alpha : V_w \to \widehat{V}_w/\langle 1 - X^w\rangle.$$

THEOREM 3.3.

$$\operatorname{Im} q\alpha = \widehat{W}_w/\langle 1 - X^w\rangle.$$

Proof. Firstly we show the inclusion $\operatorname{Im} q\alpha \subset \widehat{W}_w/\langle 1-X^w\rangle$, by proving that $\operatorname{Im}\alpha \subset \widehat{W}_w$. For $P(X) \in V_w$, setting $P_1(X) = uP(X)$, we have

$$P_1(X) = P|(1-U)(X) = P(X) - P\left(\frac{X-1}{X}\right)X^w.$$

Next let $P_2(X) = bP_1(X)$. Then we have

$$P_2(X) - P_2(X-1) = P_1(X)$$

by Lemma 3.1. Furthermore, let $P_3(X) = sP_2(X)$. Then we have

$$P_3(X) = P_2|(1-S)(X) = P_2(X) - P_2\left(\frac{-1}{X}\right)X^w.$$

By the definition, $P_3(X) = \alpha P(X)$.

Now we claim that $P_3|(1+U+U^2) = 0$. In fact,

$$\begin{aligned}&P_3|(1+U+U^2)(X)\\ &= \left(P_2(X) - P_2\left(\frac{-1}{X}\right)X^w\right)|(1+U+U^2)\\ &= P_2(X) - P_2\left(\frac{-1}{X}\right)X^w + P_2\left(\frac{X-1}{X}\right)X^w - P_2\left(\frac{-1}{\frac{X-1}{X}}\right)\left(\frac{X-1}{X}\right)^w X^w\\ &\quad + P_2\left(\frac{-1}{X-1}\right)(X-1)^w - P_2\left(\frac{-1}{\frac{-1}{X-1}}\right)\left(\frac{-1}{X-1}\right)^w (X-1)^w\end{aligned}$$

$$= P_2(X) - P_2(X-1) + \left\{P_2\left(\frac{X-1}{X}\right) - P_2\left(\frac{X-1}{X}-1\right)\right\}X^w$$
$$+ \left\{P_2\left(\frac{-1}{X-1}\right) - P_2\left(\frac{-1}{X-1}-1\right)\right\}(X-1)^w$$
$$= P_1(X) + P_1\left(\frac{X-1}{X}\right)X^w + P_1\left(\frac{-1}{X-1}\right)(X-1)^w$$
$$= P_1|(1+U+U^2)(X) = P|(1-U)(1+U+U^2)(X)$$
$$= P|(1-U^3)(X) = 0.$$

This shows that

$$\alpha P(X) \in \operatorname{Im}\alpha \subset \{Q \in \widehat{V}_w : Q|(1+U+U^2) = 0\}. \tag{3.2}$$

Now noting that $1 - S^2 = 0$, we have

$$\operatorname{Im} s = \{Q|(1-S) : Q \in \widehat{V}_w\} = \{Q \in \widehat{V}_w : Q|(1+S) = 0\}. \tag{3.3}$$

Thus

$$\operatorname{Im}\alpha = \operatorname{Im} sbu \subset \{Q \in \widehat{V}_w : Q|(1+S) = 0\}. \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$\operatorname{Im}\alpha \subset \{Q \in \widehat{V}_w : Q|(1+S) = Q|(1+U+U^2) = 0\} = \widehat{W}_w. \tag{3.5}$$

This gives the inclusion that we are after, namely,

$$\operatorname{Im} q\alpha \subset q(\widehat{W}_w) = \widehat{W}_w/\langle 1 - X^w\rangle. \tag{3.6}$$

Secondly we claim that $q(\widehat{W}_w) \subset \operatorname{Im} q\alpha$. Since $\alpha(1) \in \operatorname{Im}\alpha \subset \widehat{W}_w$, and $\alpha(1) \notin W_w$ by Lemma 3.2, we know

$$\widehat{W}_w = W_w \oplus \langle \alpha(1)\rangle$$

noting that W_w is a codimension one subspace of $\widehat{W}_w$. Hence it suffices to show the inclusion $q(W_w) \subset \operatorname{Im} q\alpha$.

Let Q be any element of W_w. We now show $q(Q) \in \operatorname{Im} q\alpha$. Since $Q \in \ker(1+S) = \operatorname{Im}(1-S)$, there is $Q_1 \in V_w$ such that $Q_1|(1-S) = Q$, namely,

$$Q(X) = Q_1(X) - Q_1\left(\frac{-1}{X}\right)X^w.$$

Let $Q_2(X)$ be defined by

$$Q_2(X) = Q_1(X) - Q_1(X-1). \tag{3.7}$$

Note that $Q_2 \in V_w$.

Next we show that $Q_2|(1+U+U^2)=0$. In fact,

(3.8) $$\begin{aligned} &Q_2|(1+U+U^2)(X)\\ &= Q_1(X)+Q_1\left(\frac{X-1}{X}\right)X^w+Q_1\left(\frac{-1}{X-1}\right)(X-1)^w\\ &\quad - Q_1(X-1)-Q_1\left(\frac{X-1}{X}-1\right)X^w-Q_1\left(\frac{-1}{X-1}-1\right)(X-1)^w\\ &= \left\{Q_1(X)-Q_1\left(\frac{-1}{X}\right)X^w\right\}+\left\{Q_1\left(\frac{X-1}{X}\right)X^w-Q_1(X-1)\right\}\\ &\quad+\left\{Q_1\left(\frac{-1}{X-1}\right)(X-1)^w-Q_1\left(\frac{-X}{X-1}\right)(X-1)^w\right\}. \end{aligned}$$

We also have

(3.9) $$\begin{aligned} &Q|(1+U+U^2)(X)\\ &= Q(X)+Q\left(\frac{X-1}{X}\right)X^w+Q\left(\frac{-1}{X-1}\right)(X-1)^w\\ &= Q_1(X)-Q_1\left(\frac{-1}{X}\right)X^w+\left\{Q_1\left(\frac{X-1}{X}\right)-Q_1\left(\frac{-X}{X-1}\right)\left(\frac{X-1}{X}\right)^w\right\}X^w\\ &\quad+\left\{Q_1\left(\frac{-1}{X-1}\right)-Q_1(X-1)\left(\frac{-1}{X-1}\right)^w\right\}(X-1)^w. \end{aligned}$$

Notice that the expressions on the right hand sides of (3.8) and (3.9) do coincide, so that we have

(3.10) $$Q_2|(1+U+U^2)=Q|(1+U+U^2).$$

In particular, since the right hand side of (3.10) is zero by the assumption that $Q\in W_w$, this means $Q_2|(1+U+U^2)=0$ as we required.

Since $\ker(1+U+U^2)=\mathrm{Im}(1-U)$, it follows that $Q_2\in\mathrm{Im}(1-U)$. Hence there is $Q_3(X)\in V_w$ such that $Q_3|(1-U)=Q_2$.

Finally, we show that $q\alpha(Q_3)=q(Q)$. By the definitions of α and Q_2,

(3.11) $$q\alpha(Q_3)=qsbu(Q_3)=qsb(Q_2).$$

Since $Q_1(X)-Q_1(X-1)=Q_2(X)$ by (3.7), and $bQ_2(X)-bQ_2(X-1)=Q_2(X)$ by Lemma 3.1, bQ_2-Q_1 is a constant, say c. Calculate the right hand side of (3.11) to obtain

$$qsb(Q_2)=qs(Q_1+c)=q(sQ_1+c(1-X^w))=q(Q+c(1-X^w))=q(Q)$$

as $q((1-X^w))=0$. This means $q\alpha(Q_3)=q(Q)$.

Thus we have proved that, for any $Q(X)\in W_w$, there is $Q_3(X)\in V_w$ such that $q\alpha(Q_3)=q(Q)$. This implies $q(W_w)\subset\mathrm{Im}\,q\alpha$, completing the proof. ∎

4. Calculation. We calculate $\alpha((X-1)^n)$ for $n=0,\dots,w$. Let $\widetilde{n}$ denote $w-n$ for $n=0,\dots,w$. First note that

$$u((X-1)^n) = (X-1)^n|(1-U) = (X-1)^n - \left(\frac{-1}{X}\right)^n X^w = (X-1)^n - (-1)^n X^{\widetilde{n}};$$

moreover,

$$b(X^n) = \frac{1}{n+1} B_{n+1}(X+1)$$

by the definition of b, and

$$b((X-1)^n) = \frac{1}{n+1} B_{n+1}(X)$$

by Lemma 3.1 and (3.1). Thus we have

$$bu((X-1)^n) = \frac{B_{n+1}(X)}{n+1} - (-1)^n \frac{B_{\widetilde{n}+1}(X+1)}{\widetilde{n}+1}.$$

Here we adopt Kohnen–Zagier's notation $B_n^0(X)$ for the nth Bernoulli polynomial without its B_1-term ([5, p. 208]):

$$B_n^0(X) = \sum_{\substack{i=0\\ i\neq 1}}^{n} \binom{n}{i} B_i X^{n-i} = \sum_{\substack{0\le i\le n\\ i \text{ even}}} \binom{n}{i} B_i X^{n-i}.$$

Then we have

$$\begin{aligned}
\alpha((X-1)^n) &= sbu((X-1)^n)\\
&= \frac{B_{n+1}(X)}{n+1} - \frac{B_{n+1}(-1/X)}{n+1} X^w\\
&\quad - (-1)^n \left\{ \frac{B_{\widetilde{n}+1}(X+1)}{\widetilde{n}+1} - \frac{B_{\widetilde{n}+1}(-1/X+1)}{\widetilde{n}+1} X^w \right\}\\
&= \frac{1}{n+1} \left\{ B_{n+1}(X) - B_{n+1}\left(\frac{-1}{X}\right) X^w \right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1} \left\{ B_{\widetilde{n}+1}(X) + (\widetilde{n}+1) X^{\widetilde{n}} \right.\\
&\quad \left. - B_{\widetilde{n}+1}\left(\frac{-1}{X}\right) X^w - (\widetilde{n}+1)\left(\frac{-1}{X}\right)^{\widetilde{n}} X^w \right\}\\
&= \frac{1}{n+1} \left\{ B^0_{n+1}(X) - B^0_{n+1}\left(\frac{-1}{X}\right) X^w \right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1} \left\{ B^0_{\widetilde{n}+1}(X) - B^0_{\widetilde{n}+1}\left(\frac{-1}{X}\right) X^w \right\} - \frac{1}{n+1} \cdot \frac{n+1}{2} X^n
\end{aligned}$$

$$+\frac{1}{n+1}\cdot\frac{n+1}{2}\left(\frac{-1}{X}\right)^n X^w+\frac{(-1)^n}{\widetilde{n}+1}\cdot\frac{\widetilde{n}+1}{2}X^{\widetilde{n}}$$
$$-\frac{(-1)^n}{\widetilde{n}+1}\cdot\frac{\widetilde{n}+1}{2}\left(\frac{-1}{X}\right)^{\widetilde{n}} X^w-(-1)^n X^{\widetilde{n}}+(-1)^n\left(\frac{-1}{X}\right)^{\widetilde{n}} X^w$$
$$=\frac{1}{n+1}\left\{B^0_{n+1}(X)-B^0_{n+1}\left(\frac{-1}{X}\right)X^w\right\}$$
$$-\frac{(-1)^n}{\widetilde{n}+1}\left\{B^0_{\widetilde{n}+1}(X)-B^0_{\widetilde{n}+1}\left(\frac{-1}{X}\right)X^w\right\}.$$

Summarizing the above calculation, we obtain

Lemma 4.1.

$$\alpha((X-1)^n)=\frac{1}{n+1}\left\{B^0_{n+1}(X)-B^0_{n+1}\left(\frac{-1}{X}\right)X^w\right\}$$
$$-\frac{(-1)^n}{\widetilde{n}+1}\left\{B^0_{\widetilde{n}+1}(X)-B^0_{\widetilde{n}+1}\left(\frac{-1}{X}\right)X^w\right\}.$$

5. The mapping β. In Section 3, we defined the mapping α. In this section, we will define and study a similar mapping $\beta : V_w \to \widehat{V}_w$. First let us define auxiliary mappings $s' : V_w \to V_w$, $b' : V_w \to V'_w$ and $u' : V'_w \to \widehat{V}_w$ as follows:

$$s'P(X)=P|(1-S)(X),$$
$$b'P(X)=\sum_{n=0}^{w}\frac{c_n}{n+1}B_{n+1}(X)\qquad\text{for } P(X)=\sum_{n=0}^{w}c_nX^n,$$
$$u'P(X)=P|(1-U)(X)=P(X)-P\left(\frac{X-1}{X}\right)X^w.$$

Note that, if $P(X)=\sum_{n=0}^{w+1}c_nX^n$ is an element of V'_w, then

$$P\left(\frac{X-1}{X}\right)X^w=\sum_{n=0}^{w+1}c_n(X-1)^nX^{w-n},$$

and it has terms of degree ranging from -1 to $w+1$. This implies $u'P\in\widehat{V}_w$. Now let us define the map $\beta : V_w\to\widehat{V}_w$ by letting $\beta=u'b's'$.

Lemma 5.1.

(5.1) $$\beta(X^n)=\alpha((X-1)^n)\qquad \textit{for } n=0,\ldots,w.$$

P r o o f. We calculate $\beta(X^n)$:

$$\beta(X^n)=u'b's'(X^n)=u'b'\left(X^n-\left(\frac{-1}{X}\right)^nX^w\right)=u'b'(X^n-(-1)^nX^{\widetilde{n}})$$

$$\begin{aligned}
&= u'\left(\frac{B_{n+1}(X)}{n+1} - (-1)^n \frac{B_{\widetilde{n}+1}(X)}{\widetilde{n}+1}\right)\\
&= \frac{1}{n+1}\left\{B_{n+1}(X) - B_{n+1}\left(\frac{X-1}{X}\right)X^w\right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1}\left\{B_{\widetilde{n}+1}(X) - B_{\widetilde{n}+1}\left(\frac{X-1}{X}\right)X^w\right\}\\
&= \frac{1}{n+1}\left\{B_{n+1}(X) - B_{n+1}\left(\frac{-1}{X}\right)X^w - (n+1)\left(\frac{-1}{X}\right)^n X^w\right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1}\left\{B_{\widetilde{n}+1}(X) - B_{\widetilde{n}+1}\left(\frac{-1}{X}\right)X^w - (\widetilde{n}+1)\left(\frac{-1}{X}\right)^{\widetilde{n}} X^w\right\}\\
&= \frac{1}{n+1}\left\{B^0_{n+1}(X) - B^0_{n+1}\left(\frac{-1}{X}\right)X^w\right.\\
&\quad \left. - \frac{n+1}{2}X^n + \frac{n+1}{2}\left(\frac{-1}{X}\right)^n X^w - (n+1)(-1)^n X^{\widetilde{n}}\right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1}\left\{B^0_{\widetilde{n}+1}(X) - B^0_{\widetilde{n}+1}\left(\frac{-1}{X}\right)X^w\right.\\
&\quad \left. - \frac{\widetilde{n}+1}{2}X^{\widetilde{n}} + \frac{\widetilde{n}+1}{2}\left(\frac{-1}{X}\right)^{\widetilde{n}} X^w - (\widetilde{n}+1)(-1)^{\widetilde{n}} X^n\right\}\\
&= \frac{1}{n+1}\left\{B^0_{n+1}(X) - B^0_{n+1}\left(\frac{-1}{X}\right)X^w\right\}\\
&\quad - \frac{(-1)^n}{\widetilde{n}+1}\left\{B^0_{\widetilde{n}+1}(X) - B^0_{\widetilde{n}+1}\left(\frac{-1}{X}\right)X^w\right\}\\
&= \alpha((X-1)^n). \ \blacksquare
\end{aligned}$$

As a corollary of the above relation between the mappings α and β, we can determine the image of $q\beta$ where $q : \widehat{V}_w \to \widehat{V}_w/\langle 1-X^w\rangle$ is the projection map as before.

COROLLARY 5.2.

$$\operatorname{Im} q\beta = \operatorname{Im} q\alpha = \widehat{W}_w/\langle 1 - X^w\rangle. \tag{5.2}$$

Proof. Let $t : V_w \to V_w$ be an isomorphism determined by $t(X^n) = (X-1)^n$ for $n = 0, \ldots, w$. Then, by Lemma 5.1, we have $\beta = \alpha t$. It follows that $\operatorname{Im} q\beta = \operatorname{Im} q\alpha$ as t is an isomorphism. $\blacksquare$

Note that

$$B^0_{n+1}\left(\frac{-1}{X}\right) = \sum_{\substack{0\le i\le n+1\\ i \text{ even}}} \binom{n+1}{i} B_i \left(\frac{-1}{X}\right)^{n+1-i} = (-1)^{n+1} B^0_{n+1}\left(\frac{1}{X}\right)$$

$$= \begin{cases} B^0_{n+1}(1/X), & n \text{ odd}, \\ -B^0_{n+1}(1/X), & n \text{ even}. \end{cases}$$

Then we obtain the following description for $\beta(X^n)$ from Lemmas 4.1 and 5.1:

LEMMA 5.3. (1) *For n even and $0 \le n \le w$,*

$$\beta(X^n) = \frac{1}{n+1}B^0_{n+1}(X) + \frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right) - \frac{1}{\tilde{n}+1}B^0_{\tilde{n}+1}(X) - \frac{X^w}{\tilde{n}+1}B^0_{\tilde{n}+1}\left(\frac{1}{X}\right).$$

(2) *For n odd and $1 \le n \le w-1$,*

$$\beta(X^n) = \frac{1}{n+1}B^0_{n+1}(X) - \frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right) + \frac{1}{\tilde{n}+1}B^0_{\tilde{n}+1}(X) - \frac{X^w}{\tilde{n}+1}B^0_{\tilde{n}+1}\left(\frac{1}{X}\right).$$

6. Spanning sets of $W^{\pm}_w$ and $\widehat{W}^{\pm}_w$. From Corollary 5.2, we know that both $\{q\alpha(X^n)\}^w_{n=0}$ and $\{q\beta(X^n)\}^w_{n=0}$ span $\widehat{W}_w/\langle 1 - X^w\rangle$.

Since $\beta(X^n)$ is an even (respectively, odd) Laurent polynomial depending on n being odd (respectively, even), we can derive the following fact rather plainly.

LEMMA 6.1. (1) *$\beta(X^n) \in \widehat{W}^+_w$ for n odd.*
(2) *$\beta(X^n) \in \widehat{W}^-_w$ for n even.*

In what follows, $\langle\beta(X^n)\rangle_{0\le n\le w,\, n\,\text{odd (resp. even)}}$, denotes the space spanned by $\beta(X^n)$ for n odd (resp. even) and $0 \le n \le w$. (The notation $\langle r^{\pm}(R_n)\rangle_{0\le n\le w,\, n\,\text{odd(even)}}$ will be used in the next section denoting similar spaces.) For subspaces V and W, $V + W$ denotes the subspace spanned by V and W.

LEMMA 6.2. (1) *$q(\langle\beta(X^n)\rangle_{0\le n\le w,\, n\, odd}) = q(\widehat{W}^+_w)$.*
(2) *$\langle\beta(X^n)\rangle_{0\le n\le w,\, n\, even} = \widehat{W}^-_w$.*

Proof. We know by Corollary 5.2 that $\operatorname{Im} q\beta = q(\widehat{W}_w) = q(\widehat{W}^+_w) \oplus q(\widehat{W}^-_w)$. Hence, by Lemma 6.1, we have

$$q(\langle\beta(X^n)\rangle_{0\le n\le w,\, n\,\text{odd}}) = q(\widehat{W}^+_w) \tag{6.1}$$

and

$$q(\langle\beta(X^n)\rangle_{0\le n\le w,\, n\,\text{even}}) = q(\widehat{W}^-_w). \tag{6.2}$$

Note that $q|\widehat{W}_w^- : \widehat{W}_w^- \to q(\widehat{W}_w^-)$ is an isomorphism. This is because $\widehat{W}_w = \widehat{W}_w^+ \oplus \widehat{W}_w^-$ and $\langle 1 - X^w \rangle \subset \widehat{W}_w^+$. Thus we also have

$$\langle \beta(X^n) \rangle_{0\le n\le w,\, n\ \mathrm{even}} = \widehat{W}_w^- \tag{6.3}$$

from (6.2). ∎

By Lemma 6.2, we obtain spanning sets for W_w^-, $\widehat{W}_w^-$, and $\widehat{W}_w^+/\langle 1-X^w\rangle$.

THEOREM 6.3. (1) W_w^- *(respectively*, $\widehat{W}_w^-$*) is spanned by*

$$\begin{aligned}\beta(X^n) = {}& \frac{1}{n+1}B^0_{n+1}(X) + \frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right)\\ & - \frac{1}{\widetilde{n}+1}B^0_{\widetilde{n}+1}(X) - \frac{X^w}{\widetilde{n}+1}B^0_{\widetilde{n}+1}\left(\frac{1}{X}\right)\end{aligned}$$

for n even and $2 \le n \le w-2$ (respectively, $0 \le n \le w$).

(2) $W_w^+/\langle 1 - X^w\rangle = \widehat{W}_w^+/\langle 1 - X^w\rangle$ *is spanned by*

$$\begin{aligned}q\beta(X^n) = q\biggl\{ & \frac{1}{n+1}B^0_{n+1}(X) - \frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right)\\ & + \frac{1}{\widetilde{n}+1}B^0_{\widetilde{n}+1}(X) - \frac{X^w}{\widetilde{n}+1}B^0_{\widetilde{n}+1}\left(\frac{1}{X}\right)\biggr\}\end{aligned}$$

for n odd and $1 \le n \le w-1$.

Proof. By Lemma 6.2, the theorem is obvious except for the case of W_w^-. Since $\beta(X^0) = -\beta(X^w) \notin W_w^-$, and $\beta(X^n) \in W_w^-$ for n even and $2 \le n \le w-2$, we know that $\{\beta(X^n)\}_{2\le n\le w-2,\, n\ \mathrm{even}}$ spans W_w^-. ∎

7. Relations between $\beta(X^n)$ and $r^{\pm}(R_n)$. In this section we will show that $\beta(X^n)$ is related to the polynomial $r^{\pm}(R_n)$ studied by Kohnen–Zagier [5]. This fact leads us to an alternative proof of the theorem of Eichler and Shimura on period mappings.

Let us recall Kohnen–Zagier's polynomials $r^{\pm}(R_n)$. Let S_{w+2} denote the space of cusp forms of weight $w+2$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. First let $r_n : S_{w+2} \to \mathbb{C}$ be the mapping defined by

$$r_n(f) = \int_0^\infty f(it)t^n\,dt,$$

which is called the nth *period mapping.*

Let $r^{\pm}(f)$ and $r(f)$ be polynomials defined by

$$r^{+}(f)(X)=\sum_{\substack{0\le n\le w\\ n\text{ even}}}(-1)^{n/2}\binom{w}{n}r_n(f)X^{w-n},$$

$$r^{-}(f)(X)=\sum_{\substack{0\le n\le w\\ n\text{ odd}}}(-1)^{(n-1)/2}\binom{w}{n}r_n(f)X^{w-n},$$

$$r(f)(X)=\int_0^{i\infty}f(z)(X-z)^w\,dz$$

for $f\in S_{w+2}$. Then clearly $r(f)=ir^{+}(f)+r^{-}(f)$. Let $R_n\in S_{w+2}$ be defined by

$$(f,R_n)=r_n(f)\qquad\text{for any } f\in S_{w+2}$$

where (,) denotes Petersson product.

The following is a result of Kohnen–Zagier [5] which was proved applying Cohen's [2] representation of R_n.

THEOREM 7.1 (Kohnen–Zagier). (1) *For n even*, $0\le n\le w$,

$$\begin{aligned}(-1)^{(w+2)/2+n/2}&2^{-w}r^{-}(R_n)(X)\\ &=-\frac{1}{n+1}B^0_{n+1}(X)-\frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right)\\ &\quad+\frac{1}{\widetilde{n}+1}B^0_{\widetilde{n}+1}(X)+\frac{X^w}{\widetilde{n}+1}B^0_{\widetilde{n}+1}\left(\frac{1}{X}\right)\\ &\quad-(\delta_{\widetilde{n},0}-\delta_{n,0})\frac{(w+2)!}{(w+1)B_{w+2}}\sum_{\substack{m=-1\\ m\ odd}}^{w+1}\frac{B_{m+1}}{(m+1)!}\cdot\frac{B_{\widetilde{m}+1}}{(\widetilde{m}+1)!}X^m.\end{aligned}$$

(2) *For n odd*, $0\le n\le n$,

$$\begin{aligned}(-1)^{(w+2)/2+(n-1)/2}&2^{-w}r^{+}(R_n)(X)\\ &=\frac{1}{n+1}B^0_{n+1}(X)-\frac{X^w}{n+1}B^0_{n+1}\left(\frac{1}{X}\right)\\ &\quad+\frac{1}{\widetilde{n}+1}B^0_{\widetilde{n}+1}(X)-\frac{X^w}{\widetilde{n}+1}B^0_{\widetilde{n}+1}\left(\frac{1}{X}\right)\\ &\quad-\frac{w+2}{B_{w+2}}\cdot\frac{B_{n+1}}{n+1}\cdot\frac{B_{\widetilde{n}+1}}{\widetilde{n}+1}(X^w-1).\end{aligned}$$

Comparing Theorem 7.1 and Lemma 5.3, we obtain relations between $\beta(X^n)$ and $r^{\pm}(R_n)$:

PROPOSITION 7.2. (1) *For n even*, $0 \le n \le w$,

$$\beta(X^n) = -(-1)^{(w+2)/2+n/2} 2^{-w} r^-(R_n)(X) - (\delta_{\tilde{n},0} - \delta_{n,0}) \frac{(w+2)!}{(w+1)B_{w+2}} f_w(X).$$

(2) *For n odd*, $0 \le n \le w$,

$$\beta(X^n) = (-1)^{(w+2)/2+(n-1)/2} 2^{-w} r^+(R_n)(X) + \frac{w+2}{B_{w+2}} \cdot \frac{B_{n+1}}{n+1} \cdot \frac{B_{\tilde{n}+1}}{\tilde{n}+1} (X^w - 1).$$

We study relations between the polynomials $\beta(X^n)$ and $r^{\pm}(R_n)$ further.

LEMMA 7.3. (1) $q(\langle r^+(R_n)\rangle_{0\le n\le w,\, n\ odd}) = q(\langle \beta(X^n)\rangle_{0\le n\le w,\, n\ odd})$.
(2) $\langle r^-(R_n)\rangle_{0\le n\le w,\, n\ even} + \langle f_w\rangle = \langle \beta(X^n)\rangle_{0\le n\le w,\, n\ even}$.

Proof. We first show (1). The equation in (2) of Proposition 7.2 gives rise to the following congruence:

$$(-1)^{(w+2)/2+(n-1)/2} 2^{-w} r^+(R_n)(X) \equiv \beta(X^n) \bmod \langle 1 - X^w\rangle$$

for n odd. This implies (1).

Next we show (2). Observing that $f_w \in \widehat{W}_w$ and using Lemma 6.2(2), we have

$$\langle \beta(X^n)\rangle_{0\le n\le w,\, n\,\text{even}} + \langle f_w\rangle = \langle \beta(X^n)\rangle_{0\le n\le w,\, n\,\text{even}}. \tag{7.1}$$

It is clear that

$$\langle \beta(X^n)\rangle_{0\le n\le w,\, n\,\text{even}} + \langle f_w\rangle = \langle r^-(R_n)\rangle_{0\le n\le w,\, n\,\text{even}} + \langle f_w\rangle \tag{7.2}$$

from Proposition 7.2(1). From (7.1) and (7.2) we obtain

$$\langle r^-(R_n)\rangle_{0\le n\le w,\, n\,\text{even}} + \langle f_w\rangle = \langle \beta(X^n)\rangle_{0\le n\le w,\, n\,\text{even}}$$

completing the proof of (2). ∎

Combining Lemmas 6.2 and 7.3 we obtain:

LEMMA 7.4. (1) $q(\widehat{W}_w^+) = q(\langle r^+(R_n)\rangle_{0\le n\le w,\, n\ even})$.
(2) $\widehat{W}_w^- = \langle r^-(R_n)\rangle_{0\le n\le w,\, n\ even} + \langle f_w\rangle$.

We also obtain the following lemma:

LEMMA 7.5. (1) $q(W_w^+) = q(\langle r^+(R_n)\rangle_{0\le n\le w,\, n\ even})$.
(2) $W_w^- = \langle r^-(R_n)\rangle_{0\le n\le w,\, n\ even}$.

Proof. Since $W_w^+ = \widehat{W}_w^+$, we have (1) from Lemma 7.4(1). From Lemma 7.4(2), we know that $\widehat{W}_w^-$ is spanned by $\{r^-(R_n)\}_{0\le n\le w,\, n\,\text{even}}$ and f_w. Observing that $r^-(R_n)$ are polynomials, and that W_w^- is a codimension one subspace of $\widehat{W}_w^-$, we obtain (2). ∎

Remark 7.1. (a) In [5], the fact that $\{qr^{\pm}(R^n)\}_{n=0}^{w}$ is a spanning set of $q(W_w^{\pm})$ (Lemma 7.5) is a consequence of the Eichler–Shimura isomorphism for period mappings.

(b) In our proof presented above, we do not need to invoke the theorem of Eichler and Shimura. As a matter of fact, our Lemma 7.5 yields an alternative proof to the theorem of Eichler and Shimura on period mappings.

8. The theorem of Eichler and Shimura

COROLLARY 8.1. *$r^-: S_{w+2} \to W_w^-$ and $qr^+ : S_{w+2} \to W_w^+/\langle 1 - X^w\rangle$ are isomorphisms.*

Proof. From Lemma 7.5, we know r^- and qr^+ are surjective. It is well known that the dimension of S_{w+2} is as follows:

$$\dim S_{w+2} = \begin{cases} \left[\dfrac{w+2}{12}\right], & w+2 \not\equiv 10 \pmod{12}, \\ \left[\dfrac{w+2}{12}\right] + 1, & w+2 \equiv 10 \pmod{12}. \end{cases}$$

On the other hand, as in Lang [6], a linear algebra argument shows

$$\dim W_w^- = \dim W_w^+/\langle 1 - X^w\rangle = \begin{cases} \left[\dfrac{w+2}{12}\right], & w+2 \not\equiv 10 \pmod{12}, \\ \left[\dfrac{w+2}{12}\right] + 1, & w+2 \equiv 10 \pmod{12}. \end{cases}$$

This implies r^- and qr^+ are isomorphisms. ∎

Remark 8.1. Let M_{w+2} denote the space of modular forms of weight $w+2$. Zagier [7] "extended" the Eichler–Shimura isomorphism to isomorphisms $r^+ : M_{w+2} \to W_w^+$ and $qr^- : M_{w+2} \to \widehat{W}_w^-$. As Lemma 7.5 gives rise to the Eichler–Shimura isomorphism (Corollary 8.1), Lemma 7.4 gives rise to Zagier's isomorphisms.

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