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A note on Sinnott's index formula

by

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0. Introduction. Let k be an imaginary abelian number field with exactly two ramified primes. The letters E and C denote the group of units and the group of circular units in k respectively. Sinnott's index formula for this case is the following (see Proposition 4.1, Theorem 4.1 and Theorem 5.1 in [6]).

Theorem A (Sinnott). *Let k be an imaginary abelian number field with conductor $m = p_1^{e_1} p_2^{e_2}$, where p_1 and p_2 are distinct prime numbers and both e_1 and e_2 are positive integers. Denote by k_i $(i = 1, 2)$ the maximal subfield of k which is unramified outside $p_i\infty$. Let G be the Galois group of k over $\mathbb{Q}$. Further, T_{p_i} and D_{p_i} denote the inertia group and the decomposition group of p_i in G $(i = 1, 2)$. Then the group C has finite index in E, and*

$$(1) \qquad [E : C] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{-g'} \cdot 2^{\varepsilon_1[G:D_{p_1}]+\varepsilon_2[G:D_{p_2}]+\delta_1+\delta_2-1} \cdot Qh^+,$$

where Q is the unit index of k, h^+ the class number of the maximal real subfield of k and g' some rational integer. Moreover, ε_i and δ_i are defined by

$$\varepsilon_i = \begin{cases} 0 & \text{if } k_{3-i} \text{ is imaginary,} \\ 1 & \text{otherwise,} \end{cases}$$

$$\delta_i = \begin{cases} 0 & \text{if } k_{3-i} \text{ is real and } [D_{p_i} : T_{p_i}] \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$. Finally, the rational integer g' satisfies $\mu \le g' \le \nu$, where

$$\mu = \sharp\{1 \le i \le 2 : k_i \text{ is imaginary}\},$$
$$\nu = \sharp\{1 \le i \le 2 : [k_i : \mathbb{Q}] \text{ is even}\}.$$

In general, the formula (1) contains the unknown factor $2^{-g'}$. But if both k_1 and k_2 are imaginary, then we have $\mu = \nu = 2$ and $g' = 2$. Hence,

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in this case, (1) reads

$$[E:C]=\frac{[k_1:\mathbb{Q}][k_2:\mathbb{Q}]}{[k:\mathbb{Q}]}\cdot 2^{-1}\cdot Qh^{+}. \tag{2}$$

In a previous paper [1], we gave another proof of (2) by constructing a system of fundamental circular units (i.e., a basis of the free part of C) of k. It is the main purpose of this note to prove the following completely explicit version of Theorem A.

Theorem B. *Let the notation be as in Theorem* A. *Then*

$$[E:C]=\frac{[k_1:\mathbb{Q}][k_2:\mathbb{Q}]}{[k:\mathbb{Q}]}\cdot 2^{\varepsilon_1[G:D_{p_1}]+\varepsilon_2[G:D_{p_2}]-1}\cdot Qh^{+}. \tag{3}$$

The proof includes the explicit construction of a system of fundamental circular units of k. By the way, comparing (1) with (3), we obtain

$$g'=\delta_1+\delta_2. \tag{4}$$

Kučera kindly wrote me a direct proof of the inequality $g'\ge\delta_1+\delta_2$. But I have never found any direct proof of (4) .

In the last section, we also mention our result on a real abelian number field with exactly two ramified primes.

1. Notation. Let k be an imaginary abelian number field of conductor $m=p_1^{e_1}p_2^{e_2}$. We note that k is a subfield of the mth cyclotomic field $K=\mathbb{Q}(\zeta_m)$, where $\zeta_m=e^{2\pi\sqrt{-1}/m}$. Let N be the subgroup of $T=(\mathbb{Z}/m\mathbb{Z})^\times$ which corresponds to $\mathrm{Gal}(K/k)$ under the natural isomorphism

$$(\mathbb{Z}/m\mathbb{Z})^\times\ni t \bmod m\mapsto (t,K)\in\mathrm{Gal}(K/\mathbb{Q}),$$

where the automorphism (t,K) maps ζ_m to ζ_m^t. Throughout this paper, we use the following symbols:

- $q_1=p_1^{e_1}$, $q_2=p_2^{e_2}$;
- $\zeta=\zeta_m$;
- $K_1=\mathbb{Q}(\zeta_{q_1}),\ K_2=\mathbb{Q}(\zeta_{q_2})$;
- $k_1=k\cap K_1,\ k_2=k\cap K_2$;
- W is the group of roots of unity in k;
- $D(q_1)=\langle -1,\mathrm{N}_{K_1/k_1}(1-\zeta^{aq_2}):1\le a<q_1\wedge(a,p_1)=1\rangle$;
- $D(q_2)=\langle -1,\mathrm{N}_{K_2/k_2}(1-\zeta^{q_1b}):1\le b<q_2\wedge(b,p_2)=1\rangle$;
- $D(m)=\langle -1,\mathrm{N}_{K/k}(1-\zeta^{x}):1\le x<m\wedge(x,p_1)=(x,p_2)=1\rangle$;
- $D=D(q_1)D(q_2)D(m)$;
- $C=D\cap E$ is the group of circular units in k;
- $T_1=\{a\,(=a \bmod m)\in T: a\equiv 1 \bmod q_2\}$;
- $T_2=\{b\,(=b \bmod m)\in T: b\equiv 1 \bmod q_1\}$;
- $T_1'=\{a\in T_1:\text{there exists } b\in T_2 \text{ such that } ab\in N\}$;
- $T_2'=\{b\in T_2:\text{there exists } a\in T_1 \text{ such that } ab\in N\}$;

- $J = -1 \bmod m \in T$;
- $J_1 \equiv -1 \bmod q_1 \wedge J_1 \equiv 1 \bmod q_2$;
- $J_2 \equiv 1 \bmod q_1 \wedge J_2 \equiv -1 \bmod q_2$.

LEMMA 1. *With the above notation we have*:

(1) $\mathrm{Gal}(K_1/k_1) = T_1'$,
(2) $\mathrm{Gal}(K_2/k_2) = T_2'$,
(3) $\mathrm{Gal}(k/k_1) = T_1'T_2/N$,
(4) $\mathrm{Gal}(k/k_2) = T_1T_2'/N$.

Proof. Easy. ■

Remark. By the statements (1) and (2) of Lemma 1, we can see that k_i is imaginary if and only if $J_i \notin T_i'$ $(i = 1, 2)$.

In order to prove the theorem, we have to consider the following four cases separately:

I. Both k_1 and k_2 are imaginary ($\Leftrightarrow J_1 \notin T_1' \wedge J_2 \notin T_2'$);
II. k_1 is imaginary and k_2 is real ($\Leftrightarrow J_1 \notin T_1' \wedge J_2 \in T_2'$);
II′. k_1 is real and k_2 is imaginary ($\Leftrightarrow J_1 \in T_1' \wedge J_2 \notin T_2'$);
III. Both k_1 and k_2 are real ($\Leftrightarrow J_1 \in T_1' \wedge J_2 \in T_2'$).

But case I was treated in [1], and cases II and II′ are similar. So it is sufficient to consider cases II and III. In case II, we use the following symbols:

- $a_1 (= 1), a_2, \ldots, a_{r_1}$ a system of representatives for $T_1/\langle J_1\rangle T_1'$;
- $b_1 (= 1), b_2, \ldots, b_{r_2}$ a system of representatives for T_2/T_2';
- $d_1 (= 1), d_2, \ldots, d_s$ a system of representatives for $T_1'T_2'/N$;
- $Y = \{a_{i_1} b_{i_2} d_j : 1 \le i_1 \le r_1 \wedge 1 \le i_2 \le r_2 \wedge 1 \le j \le s\}$;
- $Y' = Y - \{1, a_2, \ldots, a_{r_1}, b_2, \ldots, b_{r_2}\}$;
- $M = \{a_{i_1} q_2 : 2 \le i_1 \le r_1\} \cup \{q_1 b_{i_2} : 2 \le i_2 \le r_2\} \cup Y'$;
- $M' = M \cup \{0\}$.

And, in case III, we use the following ones:

- $a_1 (= 1), a_2, \ldots, a_{r_1}$ a system of representatives for T_1/T_1';
- $b_1 (= 1), b_2, \ldots, b_{r_2}$ a system of representatives for T_2/T_2';
- $d_1 (= 1), d_2, \ldots, d_s$ a system of representatives for $T_1'T_2'/\langle J\rangle N$;
- $Y = \{a_{i_1} b_{i_2} d_j : 1 \le i_1 \le r_1 \wedge 1 \le i_2 \le r_2 \wedge 1 \le j \le s\}$;
- $Y' = Y - \{1, a_2, \ldots, a_{r_1}, b_2, \ldots, b_{r_2}\}$;
- $M = \{a_{i_1} q_2 : 2 \le i_1 \le r_1\} \cup \{q_1 b_{i_2} : 2 \le i_2 \le r_2\} \cup Y'$;
- $M' = M \cup \{0\}$.

Remark. We note that Y is a system of representatives for $T/\langle J\rangle N$. Further, the cardinality of M is equal to $\frac{1}{2}[k:\mathbb{Q}]-1$, which is the unit rank of k.

Now let $l : k^\times \to \mathbb{R}[G]$ be the G-module homomorphism defined by

$$l : k^\times \ni \alpha \mapsto \sum_{\sigma\in G} \log|\alpha^\sigma| \cdot \sigma^{-1} \in \mathbb{R}[G].$$

It is easy to see that $\mathrm{Ker}\, l \cap E = \mathrm{Ker}\, l \cap C = W$. Hence

$$[E : C] = [l(E) : l(C)].$$

Furthermore, any finite subset $\{v_1, \ldots, v_r\}$ of C is a system of fundamental circular units of k if and only if $\{l(v_1), \ldots, l(v_r)\}$ is a $\mathbb{Z}$-basis of $l(C)$.

For any $a, b, y \in T$, we define circular units $v_{aq_2}, v_{q_1 b}, v_y$ by

$$v_{aq_2} = \mathrm{N}_{K_1/k_1}\left(\frac{1-\zeta^{aq_2}}{1-\zeta^{q_2}}\right), \quad v_{q_1 b} = \mathrm{N}_{K_2/k_2}\left(\frac{1-\zeta^{q_1 b}}{1-\zeta^{q_1}}\right), \quad v_y = \mathrm{N}_{K/k}(1-\zeta^y).$$

Then we notice the following facts:

(1) If $a \equiv a' \bmod T_2 N$, then $v_{aq_2} = v_{a'q_2}$.
(2) If $b \equiv b' \bmod T_1 N$, then $v_{q_1 b} = v_{q_1 b'}$.
(3) If $y \equiv y' \bmod N$, then $v_y = v_{y'}$.

Let C' be the subgroup of C generated by $\{v_x : x \in M\}$. Later, we shall see that $l(C')$ has finite index in $l(C)$ and $l(E)$. Hence we have

$$[E : C] = [l(E) : l(C)] = \frac{[l(E) : l(C')]}{[l(C) : l(C')]}.$$

2. Computation of $[l(C) : l(C')]$. First, from the definition of C', we can easily see the following lemma.

LEMMA 2. *$l(C')$ is generated by $\{l(v_x) : x \in M\}$.*

We choose two integers l_1 and l_2 such that

$$l_1 \equiv 1 \bmod q_1, \quad p_1 l_1 \equiv 1 \bmod q_2,$$
$$p_2 l_2 \equiv 1 \bmod q_1, \quad l_2 \equiv 1 \bmod q_2.$$

Then $l_1 \in T_2$ and $l_2 \in T_1$. We define f_1, g_1, f_2 and g_2 by

$$f_1 = [\langle l_1\rangle T_2' : T_2'], \quad g_1 = [T_2 : \langle l_1\rangle T_2'],$$
$$f_2 = [\langle l_2\rangle T_1' : T_1'], \quad g_2 = [T_1 : \langle l_2\rangle T_1'].$$

Let $\{1, s_2, \ldots, s_{g_1}\}$ and $\{1, t_2, \ldots, t_{g_2}\}$ be systems of representatives for $T_2/\langle l_1\rangle T_2'$ and $T_1/\langle l_2\rangle T_1'$ respectively.

PROPOSITION 3. *In case* II, *$l(C)$ is generated by*

$$l(v_{a_{i_1} q_2}) \quad (2 \le i_1 \le r_1), \qquad l(v_y) \quad (y \in Y')$$

and

$$\tfrac{1}{2} l(v_{q_1 l_1}), \ldots, \tfrac{1}{2} l(v_{q_1 l_1^{f_1 - 1}}),$$

$$l(v_{q_1 s}), \tfrac{1}{2}(l(v_{q_1 s}) - l(v_{q_1 s l_1})), \ldots, \tfrac{1}{2}(l(v_{q_1 s l_1^{f_1-2}}) - l(v_{q_1 s l_1^{f_1-1}})) \quad (s = s_2, \ldots, s_{g_1}).$$

Moreover, $[l(C) : l(C')] = 2^{r_2 - g_1}$.

Proposition 4. *In case* III, $l(C)$ *is generated by*

$$l(v_y) \quad (y \in Y')$$

and

$$\tfrac{1}{2}l(v_{q_1 l_1}), \ldots, \tfrac{1}{2}l(v_{q_1 l_1^{f_1-1}}),$$
$$l(v_{q_1 s}), \tfrac{1}{2}(l(v_{q_1 s}) - l(v_{q_1 s l_1})), \ldots, \tfrac{1}{2}(l(v_{q_1 s l_1^{f_1-2}}) - l(v_{q_1 s l_1^{f_1-1}})) \quad (s = s_2, \ldots, s_{g_1})$$

and

$$\tfrac{1}{2}l(v_{l_2 q_2}), \ldots, \tfrac{1}{2}l(v_{l_2^{f_2-1} q_2}),$$
$$l(v_{t q_2}), \tfrac{1}{2}(l(v_{t q_2}) - l(v_{t l_2 q_2})), \ldots, \tfrac{1}{2}(l(v_{t l_2^{f_2-2} q_2}) - l(v_{t l_2^{f_2-1} q_2})) \quad (t = t_2, \ldots, t_{g_2}).$$

Moreover, $[l(C) : l(C')] = 2^{r_1 + r_2 - g_1 - g_2}$.

Remark. In the next section, we shall see that $l(C')$ has finite index in $l(E)$. Hence

$$\operatorname{rank}_{\mathbb{Z}} l(C') = \operatorname{rank}_{\mathbb{Z}} l(C) = \operatorname{rank}_{\mathbb{Z}} l(E).$$

On the other hand, we notice that the cardinality of every system of generators for $l(C')$ or $l(C)$ stated in the above propositions is equal to the unit rank of k. This implies that these systems of generators are bases.

We can prove Propositions 3 and 4 in a similar fashion. So we only prove Proposition 3. Let L be the subgroup of $\mathbb{R}[G]$ generated by the elements stated in the proposition.

First we prove $L \subset l(C)$. Fix a b_{i_2} $(1 \le i_2 \le r_2)$. Then

$$\begin{aligned}
&\sum_{\substack{1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(v_{a_{i_1} b_{i_2} d_j}) \\
&\quad = \sum_{\substack{1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(\mathrm{N}_{K/k}(1 - \zeta^{a_{i_1} b_{i_2} d_j})) \\
&\quad = \frac{1}{2} \sum_{\substack{1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(\mathrm{N}_{K/k}(1 - \zeta^{a_{i_1} b_{i_2} d_j})) + \frac{1}{2} \sum_{\substack{1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(\mathrm{N}_{K/k}(1 - \zeta^{J_1 a_{i_1} b_{i_2} d_j}))
\end{aligned}$$

$$= \frac{1}{2} \sum_{\substack{0 \le h_1 \le 1 \\ 1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(\mathrm{N}_{K/k}(1 - \zeta^{J_1^{h_1} a_{i_1} b_{i_2} d_j}))$$

$$= \frac{1}{2} l\Big(\prod_{\substack{0 \le h_1 \le 1 \\ 1 \le i_1 \le r_1 \\ 1 \le j \le s}} \mathrm{N}_{K/k}(1 - \zeta^{J_1^{h_1} a_{i_1} b_{i_2} d_j})\Big).$$

Now, by Lemma 1(4), we have

$$\prod_{\substack{0 \le h_1 \le 1 \\ 1 \le i_1 \le r_1 \\ 1 \le j \le s}} \mathrm{N}_{K/k}(1 - \zeta^{J_1^{h_1} a_{i_1} b_{i_2} d_j}) = \mathrm{N}_{k/k_2}(\mathrm{N}_{K/k}(1 - \zeta^{b_{i_2}})).$$

So we get

$$\sum_{\substack{1 \le i_1 \le r_1 \\ 1 \le j \le s}} l(v_{a_{i_1} b_{i_2} d_j})$$

$$\begin{aligned}
&= \frac{1}{2} l(\mathrm{N}_{k/k_2}(\mathrm{N}_{K/k}(1 - \zeta^{b_{i_2}}))) \\
&= \frac{1}{2} l(\mathrm{N}_{K_2/k_2}(\mathrm{N}_{K/K_2}(1 - \zeta^{b_{i_2}}))) \\
&= \frac{1}{2} l\left(\mathrm{N}_{K_2/k_2}\left(\frac{1 - \zeta^{q_1 b_{i_2}}}{1 - \zeta^{q_1 l_1 b_{i_2}}}\right)\right) \\
&= \frac{1}{2} l\left(\mathrm{N}_{K_2/k_2}\left(\frac{1 - \zeta^{q_1 b_{i_2}}}{1 - \zeta^{q_1}}\right)\right) - \frac{1}{2} l\left(\mathrm{N}_{K_2/k_2}\left(\frac{1 - \zeta^{q_1 l_1 b_{i_2}}}{1 - \zeta^{q_1}}\right)\right) \\
&= \frac{1}{2}(l(v_{q_1 b_{i_2}}) - l(v_{q_1 l_1 b_{i_2}})).
\end{aligned}$$

Hence

$$\frac{1}{2}(l(v_{q_1 b_{i_2}}) - l(v_{q_1 l_1 b_{i_2}})) \in l(C).$$

From this, we can easily deduce that $L \subset l(C)$.

Next we prove $l(C) \subset L$. For this purpose, it is sufficient to prove that $l(v_y) \in L$ for $y \in Y - Y'$, because it is obvious that $l(v_{q_1 b_{i_2}}) \in L$ for $2 \le i_2 \le r_2$. Fix an a_{i_1} $(2 \le i_1 \le r_1)$. Then

$$\begin{aligned}
\sum_{\substack{1 \le i_2 \le r_2 \\ 1 \le j \le s}} l(v_{a_{i_1} b_{i_2} d_j}) &= l(\mathrm{N}_{k/k_1}(\mathrm{N}_{K/k}(1 - \zeta^{a_{i_1}}))) \\
&= l(\mathrm{N}_{K_1/k_1}(\mathrm{N}_{K/K_1}(1 - \zeta^{a_{i_1}}))) \\
&= l\left(\mathrm{N}_{K_1/k_1}\left(\frac{1 - \zeta^{a_{i_1} q_2}}{1 - \zeta^{a_{i_1} q_2 l_2}}\right)\right)
\end{aligned}$$

$$= l\left(\mathrm{N}_{K_1/k_1}\left(\frac{1-\zeta^{a_{i_1}q_2}}{1-\zeta^{q_2}}\right)\right) - l\left(\mathrm{N}_{K_1/k_1}\left(\frac{1-\zeta^{a_{i_1}q_2 l_2}}{1-\zeta^{q_2}}\right)\right)$$

$$= l(v_{a_{i_1}q_2}) - l(v_{a_{i_1}q_2 l_2}) \in L.$$

If $a_{i_1}b_{i_2}d_j \neq a_{i_1}$, then $a_{i_1}b_{i_2}d_j \in Y'$ by the definition of Y'. Hence $l(v_{a_{i_1}}) \in L$. Now we fix a b_{i_2} $(1 \le i_2 \le r_2)$. Then, as we have seen above,

$$\sum_{\substack{1\le i_1\le r_1\\1\le j\le s}} l(v_{a_{i_1}b_{i_2}d_j}) = \frac{1}{2}(l(v_{q_1 b_{i_2}}) - l(v_{q_1 l_1 b_{i_2}})).$$

If $a_{i_1}b_{i_2}d_j \neq b_{i_2}$, then $l(v_{a_{i_1}b_{i_2}d_j}) \in L$. Hence $l(v_{b_{i_2}}) \in L$. We have thus proved that $l(C) \subset L$.

Finally, by computing the determinant of the transition matrix, we can see that

$$[l(C) : l(C')] = \frac{1}{\left(\frac{1}{2}\right)^{f_1-1}\cdot\left(\left(\frac{1}{2}\right)^{f_1-1}\right)^{g_1-1}} = 2^{f_1 g_1 - g_1} = 2^{r_2 - g_1}.$$

This completes the proof of Proposition 3.

3. Computation of $[l(E) : l(C')]$**.** For each $t \in T$, we let

$$c_0 = 1,$$

$$c_{tq_2} = \begin{cases} 1 & \text{if } t \in T_2 N,\\ 0 & \text{otherwise,}\end{cases}$$

$$c_{tq_1} = \begin{cases} 1 & \text{if } t \in T_1 N,\\ 0 & \text{otherwise,}\end{cases}$$

$$c_t = \begin{cases} 1 & \text{if } t \in N,\\ 0 & \text{otherwise.}\end{cases}$$

Further, we define

$$\begin{aligned} b_{0,t} &= 2,\\ b_{aq_2,t} &= (c_{(aq_2)t} + c_{J(aq_2)t}) - (c_{tq_2} + c_{Jtq_2}) \quad (a \in T),\\ b_{q_1 b,t} &= (c_{(q_1 b)t} + c_{J(q_1 b)t}) - (c_{tq_1} + c_{Jtq_1}) \quad (b \in T),\\ b_{y,t} &= (c_{yt} + c_{Jyt}) - \frac{1}{[k:k_2]}(c_{ytl_1q_1} + c_{Jytl_1q_1})\\ &\quad - \frac{1}{[k:k_1]}(c_{ytl_2q_2} + c_{Jytl_2q_2}) + \frac{2}{[k:\mathbb{Q}]} \quad (y \in T). \end{aligned}$$

We denote by σ_t $(t \in T)$ the automorphism of k over $\mathbb{Q}$ which is the image of (t, K) under the canonical surjection

$$\mathrm{Gal}(K/\mathbb{Q}) \to G = \mathrm{Gal}(k/\mathbb{Q}).$$

We omit the proof of the next lemma because it is just the same as those of Lemmas 3.5 and 3.6 in [1].

LEMMA 5. *Let Q be the unit index of k and h^+ the class number of k^+. Then*

$$[l(E):l(C')] = \frac{1}{[k:\mathbb{Q}]} \cdot |\det(b_{x,t})_{x\in M',t\in Y}| \cdot Qh^+.$$

In what follows, we compute the determinant of the matrix $(b_{x,t})_{x\in M',t\in Y}$. We define

$$A = \{J_1^{h_1} a_{i_1} q_2 : 0 \le h_1 \le 1 \wedge 1 \le i_1 \le r_1\}, \quad B = \{q_1 b_{i_2} : 1 \le i_2 \le r_2\}$$

in case II, and

$$A = \{a_{i_1} q_2 : 1 \le i_1 \le r_1\}, \quad B = \{q_1 b_{i_2} : 1 \le i_2 \le r_2\}$$

in case III. Then it is easy to see that

$$\sum_{y\in A} c_{yt} = \sum_{y\in B} c_{yt} = 1$$

for any $t \in T$.

LEMMA 6. (1) *For any $x \in A$, there exist rational numbers α_{i_1} $(1 \le i_1 \le r_1)$ such that*

$$c_{xt} + c_{Jxt} = \alpha_1(c_0 + c_0) + \sum_{i_1=2}^{r_1} \alpha_{i_1}(c_{(a_{i_1}q_2)t} + c_{J(a_{i_1}q_2)t})$$

for any $t \in T$.

(2) *For any $x \in B$, there exist rational numbers β_{i_2} $(1 \le i_2 \le r_2)$ such that*

$$c_{xt} + c_{Jxt} = \beta_1(c_0 + c_0) + \sum_{i_2=2}^{r_2} \beta_{i_2}(c_{(q_1 b_{i_2})t} + c_{J(q_1 b_{i_2})t})$$

for any $t \in T$.

P r o o f. As cases II and III are similar, we prove the lemma for case II. If $x \in M$, there is nothing to prove. So we suppose that $x \notin M$.

(1) Since

$$c_{(J_1 a_{i_1} q_2)t} + c_{J(J_1 a_{i_1} q_2)t} = c_{(J_1 a_{i_1} q_2)t} + c_{(a_{i_1} q_2)t} = c_{(a_{i_1} q_2)t} + c_{J(a_{i_1} q_2)t},$$

it suffices to consider the case $x = q_2$. Let us define

$$A_0 = \{a_{i_1} q_2 : 1 \le i_1 \le r_1\}, \quad A_1 = \{J_1 a_{i_1} q_2 : 1 \le i_1 \le r_1\}.$$

Then

$$\sum_{y\in A_0} (c_{yt} + c_{Jyt}) = \sum_{y\in A} (c_{yt} + c_{Jyt}) - \sum_{y\in A_1} (c_{yt} + c_{Jyt}).$$

On the other hand, we can see that

$$\sum_{y\in A_1}(c_{yt}+c_{Jyt})=\sum_{y\in A_1}(c_{(Jy)t}+c_{J(Jy)t})=\sum_{y\in A_0}(c_{yt}+c_{Jyt}).$$

Hence

$$\sum_{y\in A_0}(c_{yt}+c_{Jyt})=\frac{1}{2}\sum_{y\in A}(c_{yt}+c_{Jyt})=\frac{1}{2}(1+1)=\frac{1}{2}(c_0+c_0).$$

Consequently, we obtain

$$c_{q_2t}+c_{Jq_2t}=\frac{1}{2}(c_0+c_0)-\sum_{i_1=2}^{r_1}(c_{(a_{i_1}q_2)t}+c_{J(a_{i_1}q_2)t}).$$

(2) From $x\notin M$, we have $x=q_1$. Since

$$\sum_{i_2=1}^{r_2}(c_{(q_1b_{i_2})t}+c_{J(q_1b_{i_2})t})=1+1=c_0+c_0,$$

we get

$$c_{q_1t}+c_{Jq_1t}=(c_0+c_0)-\sum_{i_2=2}^{r_2}(c_{(q_1b_{i_2})t}+c_{J(q_1b_{i_2})t}).\ \blacksquare$$

Using the above lemma, we can deduce that

$$|\det(b_{x,t})_{x\in M',t\in Y}|=r_1r_2\cdot|\det(c_{xt}+c_{Jxt})_{x\in M',t\in Y}|$$

by the same argument as in the proof of Lemma 3.8 in [1]. So, in order to accomplish our purpose, we have to compute the determinant of the matrix $(c_{xt}+c_{Jxt})_{x\in M',t\in Y}$.

For each $t\in Y$, there exists exactly one $u\in Y$ such that $tu\in\langle J\rangle N$, because Y is a system of representatives for $T/\langle J\rangle N$. We denote this u by t'. Then the map $Y\ni t\mapsto t'\in Y$ is bijective.

LEMMA 7. *Suppose* $x\in M$, $t\in Y$ *and* $c_{xt'}+c_{Jxt'}\neq 0$. *Then*:

(1) *if* $x\in Y'$, *then* $t=x$;
(2) *if* $x=a_{i_1}q_2$ $(2\le i_1\le r_1)$, *then* $t\in Y'$ *or* $t=a_{i_1}$;
(3) *if* $x=q_1b_{i_2}$ $(2\le i_2\le r_2)$, *then* $t\in Y'$ *or* $t=b_{i_2}$.

Proof. (1) Straightforward.

(2) If $c_{(a_{i_1}q_2)t'}+c_{J(a_{i_1}q_2)t'}\neq 0$, then $a_{i_1}t'\in\langle J\rangle T_2N$. Hence there is a $y=b_{i_2}J^hd_j$ such that $a_{i_1}yt'=a_{i_1}b_{i_2}J^hd_jt'\in\langle J\rangle N$, and so we have $a_{i_1}b_{i_2}d_jt'\in\langle J\rangle N$. Therefore $t=a_{i_1}b_{i_2}d_j$, and we obtain $t=a_{i_1}$ or $t\in Y'$.

(3) In case III, the proof is similar to (2). So we consider case II. If $c_{(q_1b_{i_2})t'}+c_{J(q_1b_{i_2})t'}\neq 0$, then $b_{i_2}t'\in\langle J\rangle T_1N$. Hence there is a $w=J_1^{h_1}a_{i_1}d_j$ such that $wb_{i_2}t'=J_1^{h_1}a_{i_1}b_{i_2}d_jt'\in\langle J\rangle N$, and so $a_{i_1}b_{i_2}J_2^{h_1}d_jt'\in\langle J\rangle N$. As

$J_2 \in T_2'$, there exists a $d_{j'}$ such that $J_2^{h_1} d_j \equiv d_{j'} \bmod \langle J \rangle N$, and then $a_{i_1} b_{i_2} d_{j'} t' \in \langle J \rangle N$. Therefore $t = b_{i_2}$ or $t \in Y'$. ∎

From this, we can obtain the following conclusion.

Proposition 8. *Let Q be the unit index of k and h^+ the class number of k^+.*

(1) *In case* II, *we have*

$$[l(E) : l(C')] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{r_2 - 1} \cdot Qh^+.$$

(2) *In case* III, *we have*

$$[l(E) : l(C')] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{r_1 + r_2 - 1} \cdot Qh^+.$$

In particular, $l(C')$ *and* $l(C)$ *have finite indices in* $l(E)$.

4. Proof of Theorem B. First we consider case II. By Propositions 3 and 8(1), we have

$$[E : C] = \frac{[l(E) : l(C')]}{[l(C) : l(C')]} = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot \frac{2^{r_2 - 1}}{2^{r_2 - g_1}} \cdot Qh^+$$
$$= \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{g_1 - 1} \cdot Qh^+.$$

Now, since σ_{l_1} is the inverse of the Frobenius automorphism for p_1 in k, we have

$$g_1 = [T : \langle l_1 \rangle T_1 N] = [G : D_{p_1}].$$

Therefore

$$[E : C] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{[G : D_{p_1}] - 1} \cdot Qh^+.$$

Next we consider case III. By Propositions 4 and 8(2), we have

$$[E : C] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{g_1 + g_2 - 1} \cdot Qh^+.$$

As also $g_1 = [G : D_{p_1}]$ and $g_2 = [G : D_{p_2}]$, we obtain

$$[E : C] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{[G : D_{p_1}] + [G : D_{p_2}] - 1} \cdot Qh^+.$$

This completes the proof of the theorem.

5. The real case. Let k be a real abelian number field with conductor $p_1^{e_1} p_2^{e_2}$. Let T_1, T_2, T_1', T_2', N be the same as in Section 1, and

$\{a_{i_1}\}, \{b_{i_2}\}, \{d_j\}$ be systems of representatives for $T_1/T_1', T_2/T_2', T_1'T_2'/N$ respectively. Then

$$\{v_{a_{i_1}q_2} : 2 \le i_1 \le r_1\} \cup \{v_{q_1 b_{i_2}} : 2 \le i_2 \le r_2\} \cup \{v_y : y \in Y'\}$$

is a system of fundamental circular units of k, where

$$\begin{aligned} Y' = \{&a_{i_1} b_{i_2} d_j : 1 \le i_1 \le r_1 \wedge 1 \le i_2 \le r_2 \wedge 1 \le j \le s\} \\ &- \{1, a_2, \ldots, a_{r_1}, b_2, \ldots, b_{r_2}\}. \end{aligned}$$

We can deduce Sinnott's index formula

$$[E : C] = \frac{[k_1 : \mathbb{Q}][k_2 : \mathbb{Q}]}{[k : \mathbb{Q}]} \cdot 2^{[k:\mathbb{Q}]-1} \cdot h$$

by computing the regulator of the system of fundamental units. We omit the proof because it is very similar to that of the imaginary case.

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