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On the average number of direct factors of finite abelian groups

by

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1. Introduction. Let $t(G)$ denote the number of direct factors of a finite abelian group G. We shall be concerned with obtaining estimates for the sum

$$T(x) = \sum t(G), \tag{1.1}$$

where the summation is taken over all abelian groups of order not exceeding x. The asymptotic behaviour of $T(x)$ was first studied by E. Cohen [2], who derived

$$T(x) = d_1 x(\log x + 2\gamma - 1) + d_2 x + \Delta_0(x), \tag{1.2}$$

where γ is the Euler constant and $\Delta_0(x)$ is estimated by

$$\Delta_0(x) \ll \sqrt{x}\log x.$$

E. Krätzel [5] improved this result to

$$\Delta_0(x) = d_3\sqrt{x}\big(\tfrac{1}{2}\log x + 2\gamma - 1\big) + d_4\sqrt{x} + \Delta_1(x) \tag{1.3}$$

with the new remainder term $\Delta_1(x)$ satisfying

$$\Delta_1(x) \ll x^{5/12}\log^4 x.$$

We remark that in the formulas (1.2) and (1.3) d_1, d_2, d_3, d_4 are effective constants which will be defined by (1.5)–(1.8) below.

The exponent $5/12$ was improved to $83/201, 45/109, 3/8$ respectively by Menzer [9], Menzer and Seibold [10] and Yu Gang [13]. The latest result is due to Liu [7], who proved that

$$\Delta_1(x) \ll x^{7/19+\varepsilon}. \tag{1.4}$$

The aim of this paper is further to improve this result. We have

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THEOREM 1. *Let d_1, d_2, d_3, d_4 be defined by*

$$d_1 = \zeta^2(2) \sum_{n=1}^{\infty} \tau_3(n) n^{-1}, \tag{1.5}$$

$$d_2 = -\sum_{n=1}^{\infty} \tau_3(n) n^{-1} (\zeta^2(2) \log n - 4\zeta(2)\zeta'(2)), \tag{1.6}$$

$$d_3 = \zeta^2\left(\tfrac{1}{2}\right) \sum_{n=1}^{\infty} \tau_3(n) n^{-1/2}, \tag{1.7}$$

$$d_4 = -\sum_{n=1}^{\infty} \tau_3(n) n^{-1/2} \left(\tfrac{1}{2}\zeta^2\left(\tfrac{1}{2}\right) \log n - \zeta\left(\tfrac{1}{2}\right)\zeta'\left(\tfrac{1}{2}\right)\right), \tag{1.8}$$

where $\tau_3(n)$ is defined by

$$\sum_{n=1}^{\infty} \tau_3(n) n^{-s} = \prod_{u=3}^{\infty} \zeta^2(us) \qquad \left(\Re s > \tfrac{1}{3}\right). \tag{1.9}$$

Then we have

$$T(x) = d_1 x(\log x + 2\gamma - 1) + d_2 x + d_3\sqrt{x}\left(\tfrac{1}{2}\log x + 2\gamma - 1\right) + d_4\sqrt{x} + O(x^{4/11+\varepsilon}). \tag{1.10}$$

Following Krätzel [5], we only need to study the asymptotic behaviour of the divisor function $d(1,1,2,2;n)$ which is defined by

$$d(1,1,2,2;n) = \sum_{n=n_1 n_2 n_3^2 n_4^2} 1.$$

Let $\Delta(1,1,2,2;x)$ denote the error term of the summation function

$$D(1,1,2,2;x) = \sum_{n \le x} d(1,1,2,2;n). \tag{1.11}$$

We then have

THEOREM 2. *We have*

$$\Delta(1,1,2,2;x) = O(x^{4/11+\varepsilon}). \tag{1.12}$$

Theorem 1 immediately follows from Theorem 2.

Notations. $e(t) = \exp(2\pi i t)$. $[t]$ is the integer part of t, and $\{t\} = t - [t]$, $\|t\| = \min(\{t\}, 1 - \{t\})$, $n \sim N$ means $N < n \le 2N$, $n \cong N$ means $C_1 N \le n \le C_2 N$ for some constants C_1 and C_2. ε is a sufficiently small number which may be different at each occurrence. $\Delta(t)$ always denotes the error term of the Dirichlet divisor problem.

2. A non-symmetric expression of $\Delta(1,1,2,2;x)$. In this paper we do not use the symmetric expression of $\Delta(1,1,2,2;x)$ due to Vogts [12] (also [10], Lemma 2). We shall use a non-symmetric expression of $\Delta(1,1,2,2;x)$ which is easier and simpler. We have the following basic lemma.

BASIC LEMMA. *We have*

$$\Delta(1,1,2,2;x) = \sum_{m\le x^{1/3}} d(m)\Delta\Big(\frac{x}{m^2}\Big) + \sum_{m\le x^{1/3}} d(m)\Delta\Big(\sqrt{\frac{x}{m}}\Big) + O(x^{1/3}\log x). \tag{2.1}$$

Proof. We only sketch the proof since it is elementary and direct.

We begin with

$$\begin{aligned} D(1,1,2,2;x) &= \sum_{n\le x} d(1,1,2,2;n) = \sum_{n_1 n_2 n_3^2 n_4^2\le x} 1 = \sum_{nm^2\le x} d(n)d(m) \\ &= \sum_{n\le x^{1/3}} d(n)D\Big(\sqrt{\frac{x}{n}}\Big) + \sum_{m\le x^{1/3}} d(m)D\Big(\frac{x}{m^2}\Big) - D^2(x^{1/3}) \\ &= \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3, \end{aligned} \tag{2.2}$$

where $D(u) = \sum_{n\le u} d(n)$.

Now we use the well-known abelian partial summation formula

$$\sum_{n\le u} d(n)f(n) = D(u)f(u) - \int_1^u D(t)f'(t)\,dt \tag{2.3}$$

to $\sum_1$ and $\sum_2$, and utilize the well-known formula

$$D(x) = x\log x + (2\gamma-1)x + \Delta(x) \tag{2.4}$$

with $\Delta(x) \ll x^{1/3}$. We get

$$D(1,1,2,2;x) = \text{main terms } + \sum_{n\le x^{1/3}} d(n)\Delta\Big(\sqrt{\frac{x}{n}}\Big) + \sum_{m\le x^{1/3}} d(m)\Delta\Big(\frac{x}{m^2}\Big) + O(x^{1/3}\log x), \tag{2.5}$$

whence our lemma follows.

3. Some preliminary lemmas. In this section we quote some lemmas to be used later.

LEMMA 1. *Suppose* $0 < c_1\lambda_1 \le |f'(n)| \le c_2\lambda_1$ *and* $|f''(n)| \cong \lambda_1 N^{-1}$ *for* $N \le n \le CN$. *Then*

$$\sum_{n\cong N} e(f(n)) \ll \lambda_1^{1/2}N^{1/2} + \lambda_1^{-1}.$$

If $c_2\lambda_1 \le 1/2$, *then*

$$\sum_{n\cong N} e(f(n)) \ll \lambda_1^{-1}.$$

LEMMA 2. *Let* α, β *be real numbers*, $\alpha\beta(\alpha+\beta-1)(\alpha+\beta-2) \ne 0$. *Let* $f(x,y) = Ax^\alpha y^\beta$, $D \subset \{(x,y) \mid x \sim X,\ y \sim Y\}$, $X \ge Y$, $F \equiv AX^\alpha Y^\beta$, $N \equiv XY$. *Then*

$$\begin{aligned} S \equiv (NF)^{-\varepsilon} &\sum_{(x,y)\in D} e(f(x,y)) \\ &\ll \sqrt[6]{F^2N^3} + N^{5/6} + \sqrt[8]{F^{-1}N^8X^{-1}} + NF^{-1/4} + NY^{-1/2}. \end{aligned}$$

LEMMA 3. *Let* $f(x)$ *and* $g(x)$ *be algebraic functions for* $x \in [a,b]$, *satisfying*

$$|f''(x)| \cong R^{-1}, \qquad f'''(x) \ll (RU)^{-1},$$
$$|g(x)| \le H, \qquad g'(x) \ll HU_1^{-1}, \qquad U, U_1 \ge 1.$$

Then

$$\begin{aligned} \sum_{a<n\le b} g(n)e(f(n)) = &\sum_{\alpha<u\le\beta} b_u \frac{g(n(u))}{\sqrt{f''(n(u))}} e(f(n(u)) - un(u) + 1/8) \\ &+ O(H\log(\beta-\alpha+2) + H(b-a+R)(U^{-1}+U_1^{-1})) \\ &+ O\left(H\min\left(R^{1/2}, \max\left(\frac{1}{\langle\alpha\rangle}, \frac{1}{\langle\beta\rangle}\right)\right)\right), \end{aligned}$$

where $[\alpha,\beta]$ *is the image of* $[a,b]$ *under the mapping* $y = f'(x)$, $n(u)$ *is determined by the equation* $f'(n(u)) = u$, *and*

$$b_u = \begin{cases} 1 & \text{for } \alpha < u < \beta, \\ \frac{1}{2} & \text{for } u = \alpha = \text{integer or } u = \beta = \text{integer}; \end{cases}$$

the function $\langle x\rangle$ *is defined as follows*:

$$\langle x\rangle = \begin{cases} \|x\| & \text{if } x \text{ is not an integer}, \\ \beta-\alpha & \text{otherwise}, \end{cases}$$

$$\sqrt{f''} = \begin{cases} \sqrt{f''} & \text{if } f'' > 0, \\ i\sqrt{|f''|} & \text{if } f'' < 0. \end{cases}$$

Lemma 4. *Suppose $f(n) \ll P$ and $f'(n) \gg \Delta$ for $n \cong N$. Then*

$$\sum_{n \cong N} \min\left(D, \frac{1}{\|f(n)\|}\right) \ll (P+1)(D+\Delta^{-1}) \log(2+\Delta^{-1}).$$

Lemma 5. *Suppose $a(n) = O(1)$, $0 < L \le M < N \le cL$, $L \gg 1$, $T \ge 2$. Then*

$$\sum_{M<n\le N} a(n) = \frac{1}{2\pi i} \int_{-T}^{T} \sum_{L<l\le cL} \frac{a(l)}{l^{it}} \cdot \frac{N^{it}-M^{it}}{t}\, dt + O\left(\min\left(1, \frac{L}{T\|M\|}\right) + \min\left(1, \frac{L}{T\|N\|}\right)\right) + O\left(\frac{L\log(1+L)}{T}\right).$$

Lemma 6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite sets of real numbers, $\mathcal{X} \subset [-X, X]$, $\mathcal{Y} \subset [-Y, Y]$. Then for any complex functions $u(x)$ and $v(y)$, we have*

$$\Big|\sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} u(x)v(y)e(xy)\Big|^2 \le 2\pi^2(1+XY) \mathop{\sum\sum}_{\substack{x\in\mathcal{X},\, x'\in\mathcal{X} \\ 2Y|x-x'|\le 1}} |u(x)u(x')| \mathop{\sum\sum}_{\substack{y\in\mathcal{Y},\, y'\in\mathcal{Y} \\ 2X|y-y'|\le 1}} |v(y)v(y')|.$$

Lemma 7. *Let $\alpha\beta \ne 0$, $m \ge 1$ and $N \ge 1$. Let $\mathcal{A}(M, N; \Delta)$ be the number of quadruples $(m, \widetilde{m}, n, \widetilde{n})$ such that*

$$\left|\left(\frac{\widetilde{m}}{m}\right)^{\alpha} - \left(\frac{\widetilde{n}}{n}\right)^{\beta}\right| < \Delta$$

with $M \le m, \widetilde{m} \le 2M$ and $N \le n, \widetilde{n} \le 2N$. Then

$$\mathcal{A}(M, N; \Delta) \ll MN \log 2MN + \Delta M^2 N^2.$$

Lemma 8. *We have*

$$\psi(t) = \sum_{1\le|j|\le J} a(j)e(jt) + O\Big(\sum_{|j|\le J} b(j)e(jt)\Big)$$

with $a_j \ll |j|^{-1}$ and $b_j \ll J^{-1}$.

Lemmas 1, 5, 4 are Lemmas 1, 2, 3 of C.-H. Jia [4] respectively. Lemmas 6 and 7 are Proposition 1 and Lemma 1 of [3]. Lemma 2 is Lemma 9 of H.-Q. Liu [6]. Lemma 3 is Lemma 1 of [8]. For Lemma 8 see Vaaler [11].

4. Proof of Theorem 2. In order to prove Theorem 2, we only need to estimate the two sums in the Basic Lemma of Section 2.

We first estimate the sum $\sum_1 = \sum_{m\le x^{1/3}} d(m)\Delta(x/m^2)$. We have

Proposition 1. $\sum_1 = O(x^{4/11+\varepsilon})$.

Proof. We only need to show that

$$S(M)=\sum_{m\sim M} d(m)\Delta\left(\frac{x}{m^2}\right)\ll x^{4/11+\varepsilon} \tag{4.1}$$

for any $1\ll M\ll x^{1/3}$.

Case 1. $M\ll x^{3/11}$. Let

$$S(M,N)=\sum_{m\sim M} A_m\sum_{n\sim N} b_n e\left(\frac{2\sqrt{nx}}{m}\right) \tag{4.2}$$

with $A_m=d(m)m^{-1/2}$ and $b_n=d(n)n^{-3/4}$.

By the well-known Voronoï formula for $\Delta(u)$ we have

$$\begin{aligned}S(M)&\ll x^{1/4}\left|\sum_{m\sim M} A_m\sum_{n\le x^{3/11}} B_n e\left(\frac{2\sqrt{nx}}{m}\right)\right|+O(x^{4/11+\varepsilon})\\&\ll x^{1/4}\log x|S(M,N)|+x^{4/11+\varepsilon}\end{aligned}$$

for some $N\le x^{3/11}$ by a splitting-up argument.

R. C. Baker [1] have estimated the sum $S(M,N)$ with A_m replaced by $\mu(m)m^{-1/2}$. Applying the same arguments of Baker we can obtain

$$S(M,N)\ll x^{5/44+\varepsilon}$$

if $N\gg \max(1,M^2x^{-5/11})$.

Now we suppose $M\gg x^{5/22}$ and $N\ll M^2x^{-5/11}$. It suffices for us to bound

$$T(M,N)=\sum_{m\sim M} a_m\sum_{n\sim N} b_n e\left(\frac{2\sqrt{nx}}{m}\right)$$

with $a_m=d(m)M^{1/2}m^{-1/2}$ and $b_n=d(n)N^{3/4-\varepsilon}N^{-3/4}$. We have

$$\begin{aligned}T(M,N)&\ll N\left|\sum_{\substack{M<uv\le 2M\\ u\ge v}} e\left(\frac{2\sqrt{nx}}{uv}\right)\right|\\&\ll N\sum_{V=2^j<(2M)^{1/2}}\left|\sum_{\substack{M<uv\le 2M\\ u\ge v\\ V<v\le 2V}} e\left(\frac{2\sqrt{nx}}{uv}\right)\right|,\end{aligned} \tag{4.3}$$

where the sum

$$\left|\sum_{\substack{M<uv\le 2M\\ u\ge v}} e\left(\frac{2\sqrt{nx}}{uv}\right)\right|$$

takes the maximal value at n. Let $\phi(V)$ denote the inner sum of (4.3). By Lemma 2 we get

$$(4.4)\quad x^{-\varepsilon}\phi(V) \ll N^{1/6}x^{1/6}M^{1/6}+M^{5/6}+(Nx)^{-1/16}M^{17/16}+(Nx)^{-1/8}M^{5/4}+MV^{-1/2}.$$

Now we use Lemma 1 to estimate the sum over u and the sum over v trivially. We can obtain

$$(4.5)\qquad \phi(V) \ll \frac{M^2}{\sqrt{Nx}}+\frac{N^{1/4}x^{1/4}V}{M^{1/2}}.$$

From (4.4) and (4.5) we have

$$\begin{aligned}(4.6)\quad x^{-\varepsilon}\phi(V) &\ll N^{1/6}x^{1/6}M^{1/6}+M^{5/6}+(Nx)^{-1/16}M^{17/16}+(Nx)^{-1/8}M^{5/4}\\ &\quad+\frac{M^2}{\sqrt{Nx}}+\min(MV^{-1/2},x^{1/4}N^{1/4}M^{-1/2}V)\\ &\ll N^{1/6}x^{1/6}M^{1/6}+M^{5/6}+(Nx)^{-1/16}M^{17/16}+(Nx)^{-1/8}M^{5/4}\\ &\quad+\frac{M^2}{\sqrt{Nx}}+x^{1/12}N^{1/12}M^{1/2}.\end{aligned}$$

From (4.3), (4.6) and the definition of $S(M,N)$ we get

$$\begin{aligned}(4.7)\quad x^{-\varepsilon}S(M,N) &\ll x^{1/6}M^{-1/3}N^{5/12}+M^{1/3}N^{1/4}\\ &\quad+x^{-1/16}M^{9/16}N^{3/16}+x^{-1/8}M^{3/4}N^{1/8}+x^{1/12}\\ &\ll x^{5/44},\end{aligned}$$

where the facts $N\ll M^2x^{-5/11}$ and $M\ll x^{3/11}$ are used.

Thus in any case we always have

$$S(M,N)\ll x^{5/44+\varepsilon},$$

whence (4.1) follows for the case $M\ll x^{3/11}$.

C a s e 2. $x^{3/11}\ll M\ll x^{1/3}$. By the formula

$$\Delta(u)=-2\sum_{n\le u^{1/2}}\psi(u/n)+O(1)$$

we have

$$\begin{aligned}(4.8)\qquad S(M) &= -2\sum_{M<m\le 2M}d(m)\sum_{mn\le x^{1/2}}\psi\left(\frac{x}{m^2n}\right)+O(x^{3/11})\\ &= -2\sum_{\substack{M<uv\le 2M\\ v\le u}}\ \sum_{uvn\le x^{1/2}}\psi\left(\frac{x}{u^2v^2n}\right)+O(x^{1/3})\end{aligned}$$

$$= -2\Big(\sum_{v\le u\le n} + \sum_{\substack{v\le u\\ n<u}} \Big) + O(x^{1/3})$$

$$= -2\Big(\sum_{v\le u\le n} + \sum_{v\le n<u} + \sum_{n<v\le u} \Big) + O(x^{1/3})$$

$$= -2\Big(\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 \Big) + O(x^{1/3}\log x),$$

say, where we used the fact that if $u=v$ and $n<u$, then $un<x^{1/3}$.

We shall only estimate $\sum_1$; $\sum_2$ and $\sum_3$ can be estimated in the same way.

$\sum_1$ can be divided into $O(\log^2 x)$ sums of the form

$$\sum\nolimits_1(V,N) = \sum_{(v,u,n)\in D} \psi\Big(\frac{x}{u^2v^2n}\Big), \tag{4.9}$$

where

$$D = \{(u,v,n) \mid M < uv \le 2M,\ uvn \le x^{1/2},\ v\le u\le n,\ V<v\le 2V,\ N<n\le 2N\}.$$

Let $U = M/V$. By Lemma 8 we get

$$\begin{aligned} \sum\nolimits_1(V,N) &\ll \frac{VUN}{J} + \sum_{h\le J} h^{-1} \Big| \sum_{(v,u,n)\in D} e\Big(\frac{hx}{u^2v^2n}\Big)\Big| \\ &\ll \frac{VUN}{J} + \sum_{H=2^j} H^{-1} \sum_{h\sim H} \Big| \sum_{(v,u,n)\in D} e\Big(\frac{hx}{u^2v^2n}\Big)\Big|. \end{aligned} \tag{4.10}$$

Thus it suffices to bound

$$\phi_1(H,V,U,N) = \sum_{h\sim H} \Big| \sum_{(v,u,n)\in D} e\Big(\frac{hx}{u^2v^2n}\Big)\Big|. \tag{4.11}$$

Now put

$$a = \max(N,u), \quad b = \min\Big(2N, \frac{x^{1/2}}{uv}\Big), \quad \beta = \frac{hx}{u^2v^2a^2}, \quad \alpha = \frac{hx}{u^2v^2b^2}.$$

Then Lemma 3 yields

$$\begin{aligned} \sum_{a\le n\le b} e\Big(\frac{-hx}{u^2v^2n}\Big) &= c_0 \sum_{\alpha<r\le\beta} b_r \frac{h^{1/4}x^{1/4}}{u^{1/2}v^{1/2}r^{3/4}} e\Big(\frac{-2\sqrt{rhx}}{uv}\Big) \\ &\quad + O\Big(\log x + \min\Big(\frac{N}{H^{1/2}F^{1/2}}, \frac{1}{\langle\alpha\rangle}\Big) \\ &\quad + \min\Big(\frac{N}{H^{1/2}F^{1/2}}, \frac{1}{\langle\beta\rangle}\Big)\Big), \end{aligned} \tag{4.12}$$

where $F = x/(V^2U^2N)$.

We first consider the contribution of the error term of (4.12) to $\phi_1(H,V,U,N)$. Obviously, the contribution of $\log x$ is

$$HVU\log x \ll Hx^{1/3}\log x. \tag{4.13}$$

If $b = x^{1/2}/(u^2v^2)$, then α is an integer. By Lemma 1, $1/\langle\alpha\rangle \ll 1$, hence the contribution of $\min(N/(H^{1/2}F^{1/2}), 1/\langle\alpha\rangle)$ to $\phi_1(H,V,U,N)$ is $O(Hx^{1/3})$. If $b = 2N$, then $\alpha = hx/(4u^2v^2N^2)$, by Lemma 3 (we sum over u), the contribution of $\min(N/(H^{1/2}F^{1/2}), 1/\langle\alpha\rangle)$ is

$$HV\sum_{u\sim U}\min\left(\frac{N}{H^{1/2}F^{1/2}}, \frac{1}{\|\frac{hx}{4u^2v^2N^2}\|}\right) \ll H^{3/2}VF^{1/2}\log x. \tag{4.14}$$

Similarly, the contribution of $\min(N/(H^{1/2}F^{1/2}), 1/\langle\beta\rangle)$ is

$$Hx^{1/3}\log x + H^{3/2}VF^{1/2}\log x. \tag{4.15}$$

From (4.11)–(4.15) we have

$$\begin{aligned}\phi_1(H,V,U,N) &= \sum_{h\sim H}c(h)\sum_{(v,u,n)\in D}e\left(\frac{hx}{u^2v^2n}\right)\\ &= c_0\sum_{h\sim H}c(h)\sum_{(v,u)}\sum_{\alpha<r\le\beta}\frac{b_rh^{1/4}x^{1/4}}{u^{1/2}v^{1/2}r^{3/4}}e\left(\frac{-2\sqrt{rhx}}{uv}\right)\\ &\quad + O(Hx^{1/3}\log x + H^{3/2}VF^{1/2}\log x),\end{aligned} \tag{4.16}$$

where $|c(h)| \le 1$.

Now we first use Lemma 5 to the variable r and then to the variable u (or we can use the same argument of (13) of Liu [8]). We get

$$\begin{aligned}\phi_1(H,V,U,N) &\ll \frac{N}{H^{1/2}F^{1/2}}\sum_{h\sim H}\sum_{u\sim U}\left|\sum_{(v,r)\in D_2}C(v,r)e\left(\frac{-2\sqrt{rhx}}{uv}\right)\right|\\ &\quad + O(Hx^{1/3}\log x + H^{3/2}VF^{1/2}\log x),\end{aligned} \tag{4.17}$$

where we used the fact that the contribution of the error term when we used Lemma 5 is $O(Hx^{1/3}\log x + H^{3/2}VF^{1/2}\log x)$ and $D_2 = \{(v,r) \mid v \sim V, r \cong HFN^{-1} = R\}$.

By Lemma 6 we get

$$\sum_{h\sim H}\sum_{u\sim U}\left|\sum_{(v,r)\in D_2}C(v,r)e\left(\frac{-2\sqrt{rhx}}{uv}\right)\right| \ll (HFB_1B_2)^{1/2}, \tag{4.18}$$

where B_1 is the number of lattice points $(h,u,\widetilde{h},\widetilde{u})$ such that

$$h,\widetilde{h}\sim H, \quad u,\widetilde{u}\cong U, \quad \left|\frac{\sqrt{h}}{u} - \frac{\sqrt{\widetilde{h}}}{\widetilde{u}}\right| \ll \frac{V}{\sqrt{Rx}},$$

and where B_2 is the number of lattice points $(v,r,\widetilde{v},\widetilde{r})$ such that

$$v,\widetilde{v}\sim V, \quad r,\widetilde{r}\cong R, \quad \left|\frac{\sqrt{r}}{v}-\frac{\sqrt{\widetilde{r}}}{\widetilde{v}}\right| \ll \frac{U}{\sqrt{Hx}}.$$

By Lemma 7 we have

$$B_1 \ll HU\log x+\frac{1}{HF}U^2H^2 \ll HU\log x, \tag{4.19}$$

$$B_2 \ll RV\log x+\frac{1}{HF}R^2V^2 \ll RV\log x. \tag{4.20}$$

Combining (4.17)–(4.20) we get

$$x^{-\varepsilon}\phi_1(H,V,U,N) \ll H(FVUN)^{1/2}+x^{1/3}+H^{3/2}VF^{1/2}. \tag{4.21}$$

Inserting (4.21) into (4.10) and choosing $J=(F^{-1}U^2N^3)^{1/3}$, we get

$$\begin{aligned} x^{-\varepsilon}\textstyle\sum_1(V,N) &\ll (FVUN)^{1/2}+\sqrt[3]{FV^3UN}+x^{1/3} \\ &\ll (FVUN)^{1/2}+x^{1/3} \ll (x/M)^{1/2}+x^{1/3} \ll x^{4/11}. \end{aligned} \tag{4.22}$$

In the last step the fact that $M\gg x^{3/11}$ was used. From (4.22) we immediately have

$$\textstyle\sum_1 \ll x^{4/11+\varepsilon}.$$

In the same way we can show that

$$x^{-\varepsilon}\Big(\textstyle\sum_2+\sum_3\Big) \ll x^{1/2}M^{-1/2}+x^{1/3} \ll x^{4/11}. \tag{4.23}$$

Now (4.1) follows from (4.22) and (4.23) in the case $x^{3/11}\ll M\ll x^{1/3}$. This completes the proof of Proposition 1.

As for $\sum_2=\sum_{m\le x^{1/3}} d(m)\Delta(\sqrt{x/m})$, we have the following

PROPOSITION 2. *$\sum_2=O(x^{1/3+\varepsilon})$.*

Proof. The proof is the same as the proof of Case 2 in Proposition 1, so we omit the details. Actually, similar to the proof of Case 2, we can get (for $x^{1/5}\ll M\ll x^{1/3}$)

$$T(M)=\sum_{m\sim M} d(m)\Delta(\sqrt{x/m}) \ll x^{1/4+\varepsilon}M^{1/4}+x^{1/3} \ll x^{1/3+\varepsilon}.$$

For $M\ll x^{1/5}$ we use $\Delta(t)\ll t^{1/3}$ and obtain

$$T(M)\ll \sum_{m\sim M} d(m)x^{1/6}M^{-1/6} \ll x^{1/3}\log x.$$

Thus Proposition 2 holds.

Now, Theorem 2 follows immediately from the Basic Lemma and the two propositions.

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