

Determination of all imaginary abelian sextic number fields with class number ≤ 11

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1. Introduction. Louboutin [L1] and Yamamura [Y] have determined all imaginary abelian sextic number fields with class number one. There are exactly 17 such fields and their conductors are ≤ 129 . The determination of all CM-fields with a given class number and given degree stems from lower bounds for the relative class number. In particular, the lower bounds for the relative class number established by Louboutin [L1] enable us to give reasonable upper bounds for the conductors of abelian sextic CM-fields with small class number. Moreover, these lower bounds for the relative class number can be improved, using Theorem 2 in [L3]: in other words, we need less computer calculations. We thus make a finite list of all possible conductors for a given class number. We shorten this list using the divisibility properties of the relative class number. Thus in this paper we prove the following:

THEOREM 1. *There are precisely 17 imaginary abelian sextic number fields of class number 1; 5 fields of class number 2; 23 fields of class number 3; 15 fields of class number 4; 2 fields of class number 5; 6 fields of class number 6; 14 fields of class number 7; 6 fields of class number 8; 33 fields of class number 9; 2 fields of class number 10; 1 field of class number 11; these fields are listed in Tables 3 and 4.*

THEOREM 2. *There are precisely 26 imaginary abelian sextic number fields of relative class number 1; 7 fields of relative class number 2; 27 fields of relative class number 3; 20 fields of relative class number 4; these fields are listed in Table 3.*

In Section 2, we review some well-known facts about the cyclic cubic number fields which will be used in the next sections. In Section 3, we obtain lower bounds for the relative class number of an imaginary abelian

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sextic number field K in terms of the conductor f , hence an upper bound for the conductors f of K when the relative class number is given. In Section 4, we study the arithmetic properties of the relative class number which allow us to find a finite list of all possible conductors f with a given relative class number. In Section 5, we illustrate how to determine all fields of class number 3; in that case we have several cases to consider. Ultimately, we list all fields with relative class number $h^*(K) \leq 4$ and all fields with class number $h(K) \leq 11$ in Tables 3 and 4, respectively.

2. Cyclic cubic extension of \mathbb{Q} . In this section we recall a few standard facts concerning cyclic cubic extensions of \mathbb{Q} . Let k be a cyclic cubic extension of \mathbb{Q} , and f_k the conductor of k . Then

$$f_k = \begin{cases} p_1 \cdots p_r & \text{if } 3 \nmid f_k, \\ 3^2 p_2 \cdots p_r & \text{if } 3 \mid f_k, \end{cases}$$

where p_i 's are distinct prime numbers with $p_i \equiv 1 \pmod{6}$ and r is the number of prime divisors of f_k . Moreover, there exist 2^{r-1} extensions of the conductor f_k and we have

$$\begin{cases} h(k) \equiv 1 \pmod{3} & \text{if } r = 1, \\ 3^{r-1} \text{ divides } h(k) & \text{if } r \geq 2. \end{cases}$$

Therefore, if 3 does not divide $h(k)$ then $f_k = 3^2$ or $f_k \equiv 1 \pmod{6}$ is prime, and k is well determined by f_k . In addition, we can easily obtain a cubic polynomial defining k as follows. Let χ be a primitive Dirichlet character modulo f_k of order 3 such that the cyclic group $\{\chi^i : 0 \leq i \leq 2\}$ is the group of characters associated with k . For a positive integer l we let $\zeta_l = \exp(2i\pi/l)$. Then the element

$$\theta = \sum_{\substack{g=1 \\ \chi(g)=1}}^{f_k-1} \zeta_{f_k}^g$$

is a primitive element of the extension k of \mathbb{Q} , and

$$\theta^{(i)} = \sum_{\substack{g=1 \\ \chi(g)=\zeta_3^i}}^{f_k-1} \zeta_{f_k}^g, \quad i = 0, 1, 2,$$

are the conjugates of θ . In this way, from the character χ we can calculate explicitly an irreducible polynomial defining the number field k . Note that M.-N. Gras ([M.N.G]) has explained the cyclic cubic fields in detail and has determined all cyclic cubic fields of conductor ≤ 4000 (see also G. Gras [G.G]).

3. Lower bounds for relative class number. Let X be a group of Dirichlet characters and N the associated abelian number field. We assume N is a CM-field. We denote by N_+ its maximal real subfield. The relative class number $h^*(N)$ can be written as

$$\begin{aligned} h^*(N) &= \frac{Qw(N)}{(2\pi)^n} \sqrt{\frac{d(N)}{d(N_+)}} \cdot \frac{\text{Res}_{s=1}(\zeta_N)}{\text{Res}_{s=1}(\zeta_{N_+})} \\ &= \frac{Qw(N)}{(2\pi)^n} \left(\prod_{\chi \text{ odd}} f_\chi \right)^{1/2} \prod_{\chi \text{ odd}} L(1, \chi) \end{aligned}$$

where $2n = [N : \mathbb{Q}]$, $w(N)$ is the number of roots of unity in N , Q is the Hasse unit index of N , f_χ is the conductor of the character χ , and ζ_N and ζ_{N_+} are the Dedekind zeta functions of N and N_+ , respectively (see Chapter 4 of [Ws]). For lower bounds for the relative class number we need upper bounds for $\text{Res}_{s=1}(\zeta_{N_+})$ and lower bounds for $\text{Res}_{s=1}(\zeta_N)$.

PROPOSITION 1. *Let N be a CM-field of degree $2n$. Then $\beta \in [1 - 2/\log d_N, 1[$ and $\zeta_N(\beta) \leq 0$ imply*

$$\text{Res}_{s=1}(\zeta_N) \geq \varepsilon_N \frac{2}{e \log d_N}$$

where

$$\varepsilon_N = 1 - \frac{2\pi n e^{1/n}}{d_N^{1/(2n)}} \quad \text{or} \quad \frac{2}{5} \exp\left(-\frac{2\pi n}{d_N^{1/(2n)}}\right).$$

Proof. This is the content of Proposition A of [L3]. ■

PROPOSITION 2. *Let χ be a nontrivial even primitive Dirichlet character mod f_χ . Then*

$$|L(1, \chi)| \leq \frac{1}{2}(\log f_\chi + 0.05).$$

Proof. This is the content of Theorem of [L2]. ■

We now turn to the relative class number of an imaginary abelian sextic number field. Let K be an imaginary abelian sextic number field, K_+ the real cubic subfield of K and k_{im} the imaginary quadratic subfield of K . We let f, f_+ and m denote the conductors of K, K_+ and k_{im} , respectively. For a number field F , we let $h(F)$ and $d(F)$ be the class number of F and the discriminant of F , respectively. We have $d(K) = -f^2 \cdot f_+^2 \cdot m$ by the conductor-discriminant formula. Let χ be a primitive odd Dirichlet character modulo f of order 6 such that $\{\chi^i : 0 \leq i \leq 5\}$ is the group of characters associated with K . Let χ_{im} be the odd primitive character modulo m which induces χ^3 and let χ_+ be the even primitive character modulo f_+ which

induces χ^2 . The relative class number $h^*(K)$ is rewritten as

$$h^*(K) = \frac{Qw(K)}{w(k_{\text{im}})} h(k_{\text{im}}) \frac{f|L(1, \chi)|^2}{4\pi^2},$$

where $w(K)$ and $w(k_{\text{im}})$ are the number of roots of unity in K and k_{im} , respectively. Louboutin [L1] has estimated $L(1, \chi)$ from below in terms of the conductor f and obtained the following theorem:

THEOREM 3. *Let K/\mathbb{Q} be an imaginary abelian sextic extension of conductor f . We have the following lower bounds for $h^*(K)$:*

$$\begin{aligned} h^*(K) &\geq \frac{1}{7300} \cdot \frac{f}{\log^2(f/\pi)} && \text{if } f \geq 5 \cdot 10^5; \\ h^*(K) &\geq \frac{1}{4200} \cdot \frac{f}{\log^2(f/\pi)} && \text{if } f \geq 5 \cdot 10^5 \text{ and } 3 \text{ divides } f; \\ h^*(K) &\geq \frac{1}{3300} \cdot \frac{f}{\log^2(f/\pi)} && \text{if } f \geq 5 \cdot 10^5 \text{ and } 2 \text{ divides } f. \end{aligned}$$

Therefore

$$\begin{aligned} h^*(K) &> 11 && \text{if } f \geq 2.0 \cdot 10^7; \\ &&& \text{if } f \geq 1.1 \cdot 10^7 \text{ and } 3 \text{ divides } f; \\ &&& \text{if } f \geq 7.9 \cdot 10^6 \text{ and } 2 \text{ divides } f. \end{aligned}$$

Proof. See Theorem 2 of [L1]. ■

This estimate shows that the computations can be done on a PC or a Workstation. However, we shall show below (Theorem 6) that these upper bounds can be improved using Propositions 1 and 2.

Furthermore, using generalized Bernoulli numbers we can evaluate explicitly the relative class number for K :

$$h^*(K) = \frac{Qw(K)}{w(k_{\text{im}})} h(k_{\text{im}}) |\tau_\chi|^2 \quad \text{with } \tau_\chi = -\frac{1}{2f} \sum_{a=1}^{f-1} a\chi(a).$$

From this formula we obtain

PROPOSITION 3. *Let K/\mathbb{Q} be an imaginary abelian sextic extension of conductor f .*

(a) $h^*(K) = h(k_{\text{im}}) |\tau_\chi|^2$ for $K \neq \mathbb{Q}(\zeta_7), \mathbb{Q}(\zeta_9)$, where $\zeta_n = \exp(2i\pi/n)$ for $n > 2$.

(b) $h(k_{\text{im}})$ divides $h^*(K)$.

Proof. See Lemma A and Corollary D of [L1]. ■

The fields $\mathbb{Q}(\zeta_7)$ and $\mathbb{Q}(\zeta_9)$ have class number one. Thus they can be omitted from all future considerations, and we have $h^*(K) = h(k_{\text{im}}) |\tau_\chi|^2$

from this point on. In order to determine all imaginary abelian sextic number fields having class number less than or equal to 11, we need to determine all imaginary quadratic number fields k_{im} with $h(k_{\text{im}}) \leq 11$ and $|d(k_{\text{im}})| \leq 2 \cdot 10^7$. For convenience we list the computational results here.

THEOREM 4. *There are 497 imaginary quadratic fields of conductor $m \leq 2 \cdot 10^7$ with class number less than or equal to 11, and their conductors are less than or equal to 15667.*

REMARK. Some of the class number problems for imaginary quadratic number fields have been solved: Stark [S1, S2] for class number 1 and 2; Montgomery and Weinberger [MW] for class number 3; Arno [A1] for class number 4; Wagner [Wg] for class numbers 5, 6 and 7 and Arno [A2] for all odd class numbers from 5 to 23.

We now turn our attention to the evaluation of the L -functions at $s = 1$.

THEOREM 5. *Let χ be an odd quadratic Dirichlet character of conductor f . If $f \leq 593000$, then $L(s, \chi) > 0$ for $s > 0$.*

PROOF. See [Low]. ■

COROLLARY 1. *Let K/\mathbb{Q} be an imaginary abelian sextic extension.*

- (i) *If $h(K) \leq 11$ then $|d(k_{\text{im}})| \leq 15667$.*
- (ii) *If $h(K) \leq 11$ then $L(s, \chi^3) > 0$ for $s > 0$.*

PROOF. (i) follows from Theorem 4.

(ii) follows from (i) and Theorem 5. ■

Consider the Dedekind zeta function

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(s, \chi^3)L(s, \chi^2)L(s, \chi^4)L(s, \chi)L(s, \chi^5).$$

It is known that $\zeta_{\mathbb{Q}}(s) \leq 0$ on $]0, 1[$. For real s , we have

$$L(s, \chi^4) = \overline{L(s, \chi^2)} \quad \text{and} \quad L(s, \chi^5) = \overline{L(s, \chi)}.$$

Therefore, $\zeta_K(s)$ has the same sign as $\zeta_{\mathbb{Q}}(s)L(s, \chi^3) = \zeta_{k_{\text{im}}}(s)$. By Corollary 1, if $h(K) \leq 11$, then $\zeta_K(s) \leq 0$ for $0 < s < 1$. At this point, we apply Proposition 1 to improve the lower bound for $\text{Res}_{s=1}(\zeta_K)$, so we obtain the following:

THEOREM 6 (Louboutin). *Let K/\mathbb{Q} be an imaginary abelian sextic extension. If $h(k_{\text{im}}) \leq 11$, then*

$$h^*(K) > \frac{w(K)\varepsilon_K}{5e\pi^3} \cdot \frac{f\sqrt{m}}{(\log f + 0.05)^3}.$$

Here, $\varepsilon_K = 1 - 6\pi\sqrt[3]{e}/d(K)^{1/6}$. Thus $h^*(K) > 11$ if $f \geq 3 \cdot 10^6$.

PROOF. This follows from Propositions 1 and 2. ■

4. Arithmetic properties of the relative class number. In this section, we shall study some divisibility properties of the relative class number, which allow us to reduce the amount of the computations.

LEMMA 1. *Let N be a CM-field and N_+ its maximal totally real subfield. Let t be the number of prime ideals of N that are ramified in the quadratic extension N/N_+ . Then 2^{t-1} divides $h^*(N)$.*

PROOF. See Proposition 2 of [LO]. ■

PROPOSITION 4. *If a prime number q is ramified in k_{im}/\mathbb{Q} and if q splits in K_+/\mathbb{Q} , then $4|h^*(K)$. Consequently, if $h^*(K)$ is not divisible by 4 and if q is ramified in k_{im}/\mathbb{Q} , then $\chi_+(q) \neq 1$.*

PROOF. This is clear by Lemma 1. ■

For the fields having class number 3, 6 and 9, we make use of the following result.

THEOREM 7. *Let K be an imaginary abelian sextic number field. Let T denote the number of primes dividing f_+ which split in k_{im} and set $\varepsilon = 1$ or 0 according as $k_{\text{im}} = \mathbb{Q}(\sqrt{-3})$ or not. Then $3^{T-\varepsilon}h(k_{\text{im}})$ divides $h^*(K)$.*

PROOF. See Proposition 8 of [LOO]. ■

According to Proposition 3, in order to determine all imaginary abelian sextic number fields K with $h(K) \leq 11$ we need to consider the following 28 cases:

Table 1

$h(K)$	1	2	3	3	3	4	4	4	5	6	6	6	6
$h(K_+)$	1	1	1	1	3	1	1	4	1	1	1	3	6
$h^*(K)$	1	2	3	3	1	4	4	1	5	6	6	2	1
$ \tau_\chi ^2$	1	1	1	3	1	1	4	1	1	1	3	1	1
$h(k_{\text{im}})$	1	2	3	1	1	4	1	1	5	6	2	1	1

Table 2

$h(K)$	7	7	7	8	8	8	9	9	9	9	9	10	10	11
$h(K_+)$	1	1	7	1	1	4	1	1	1	3	3	9	1	10
$h^*(K)$	7	7	1	8	8	2	9	9	9	3	3	1	10	1
$ \tau_\chi ^2$	1	7	1	1	4	1	1	3	9	1	3	1	1	1
$h(k_{\text{im}})$	7	1	1	8	2	2	9	3	1	3	1	1	10	1

LEMMA 2. *Assume that τ_χ is an algebraic integer. For any positive prime q , let $v_q(n)$ denote the exponent of q in the prime factorization of $n \geq 1$. If $|\tau_\chi|^2 = n$, q divides n and if $\left(\frac{-3}{q}\right) = -1$, then $v_q(n) \equiv 0 \pmod{2}$. Note that $\left(\frac{-3}{q}\right) = -1$ if and only if $q = 2$ or $q \equiv -1 \pmod{6}$. Therefore,*

$$\begin{aligned}
|\tau_\chi|^2 = 1 & \text{ implies } |\tau_\chi + \bar{\tau}_\chi| = 1 \text{ or } 2; \\
|\tau_\chi|^2 = 3 & \text{ implies } |\tau_\chi + \bar{\tau}_\chi| = 0 \text{ or } 3; \\
|\tau_\chi|^2 = 4 & \text{ implies } |\tau_\chi + \bar{\tau}_\chi| = 2 \text{ or } 4; \\
|\tau_\chi|^2 = 7 & \text{ implies } |\tau_\chi + \bar{\tau}_\chi| = 1, 4 \text{ or } 5; \\
|\tau_\chi|^2 = 9 & \text{ implies } |\tau_\chi + \bar{\tau}_\chi| = 3 \text{ or } 6.
\end{aligned}$$

Proof. Set $\tau_\chi = (a + b\sqrt{-3})/2$, $a, b \in \mathbb{Z}$. Then $4|\tau_\chi|^2 = |\tau_\chi + \bar{\tau}_\chi|^2 + 3b^2$. ■

PROPOSITION 5 (see [L1]). Assume $f_+ = p \equiv 1 \pmod{6}$ and $\gcd(m, p) = 1$. Then

$$\tau_\chi + \bar{\tau}_\chi \equiv \begin{cases} h(k_{\text{im}}) \frac{\chi_{\text{im}}(p) - 1}{2} \pmod{3} & \text{if } m \neq 3, 4, \\ \frac{p-1}{6} \pmod{3} & \text{if } m = 3, \\ \frac{\chi_{\text{im}}(p) - p}{4} \pmod{3} & \text{if } m = 4. \end{cases}$$

Therefore,

- (1) if 3 divides $h(k_{\text{im}})$ then 9 divides $h^*(K)$;
- (2) if $\chi_{\text{im}}(p) = 1$ and $m \neq 3$ then 3 divides $h^*(K)$;
- (3) if $p \equiv 1 \pmod{18}$ and $m = 3$ then 3 divides $h^*(K)$.

Moreover, τ_χ is an algebraic integer and we have the following formula suitable for computations:

$$\tau_\chi + \bar{\tau}_\chi = h(k_{\text{im}}) \frac{\chi_{\text{im}}(p) - 1}{w(k_{\text{im}})} - \frac{1}{m} \sum_{x=1}^{(p-1)/2} S_m(\bar{x}^{3^{(p)}}), p)$$

where $\bar{x}^{(p)}$ is the congruent class of x modulo p and

$$S_m(\alpha, \beta) = \sum_{b=0}^{m-1} b \chi_{\text{im}}(\alpha + b\beta)$$

depends on α and β modulo m only.

Note that our formula given in Proposition 5 makes it much easier to compute $\tau_\chi + \bar{\tau}_\chi$ than to compute τ_χ . Therefore, according to Tables 1 and 2, in using Lemma 2 as a necessary condition, and since we will be able to reduce the determination of all the imaginary abelian sextic number fields of class number prime to 3 to those of conductor $f = mf_+$ with $f_+ = p \equiv 1 \pmod{6}$ and $\gcd(m, f_+) = 1$, Proposition 5 and Theorem 6 will enable us to get fast a very short list of possible fields, and we will have to compute $h^*(K)$ only for the few fields of that list.

THEOREM 8 (Louboutin). *Assume $\gcd(m, f_+) = 1$ and for any relative integer $a \in \mathbb{Z}$ let a_+ denote the only relative integer such that $f_+ a_+ \equiv a \pmod{m}$ and $0 \leq a_+ \leq m - 1$. Then*

$$\begin{aligned} \tau_\chi + \bar{\tau}_\chi &= -\frac{h(k_{\text{im}})}{w(k_{\text{im}})} \prod_{p|f_+} (1 - \chi_{\text{im}}(p)) + \frac{\phi(f_+)h(k_{\text{im}})}{w(k_{\text{im}})} \chi_{\text{im}}(f_+) \\ &\quad - 3\chi_{\text{im}}(f_+) \sum_{\substack{1 \leq a \leq f_+/2 \\ \chi_+(a)=1}} \sum_{b=1}^{a_+-1} \chi_{\text{im}}(b). \end{aligned}$$

In particular, τ_χ is an algebraic integer in $\mathbb{Q}(\sqrt{-3})$. Therefore 3 divides $|\tau_\chi|^2$ if and only if 3 divides $\tau_\chi + \bar{\tau}_\chi$. We then have $\tau_\chi + \bar{\tau}_\chi \equiv 0 \pmod{3}$ if and only if we are in one of the following five cases:

$$\left\{ \begin{array}{ll} 3 \mid h(k_{\text{im}}) & \text{and } m \neq 3, 4, \\ \{p : p \mid f_+ \text{ and } \chi_{\text{im}}(p) = 1\} \neq \emptyset & \text{and } m \neq 3, 4, \\ |\{p : p \mid f_+\}| \geq 2 & \text{and } m = 3, \\ \{p : p \mid f_+ \text{ and } p \equiv 1 \pmod{18}\} \neq \emptyset & \text{and } m = 3, \\ \{p : p \mid f_+ \text{ and } p \equiv 1 \pmod{12}\} \neq \emptyset & \text{and } m = 4. \end{array} \right.$$

Proof. We argue as in [L1]. Since $\gcd(m, f_+) = 1$ implies $\chi(x) = \chi_+(x)\chi_{\text{im}}(x)$, we get

$$\gcd(x, f_+) = 1 \text{ implies } \chi(x) + \bar{\chi}(x) + \chi_{\text{im}}(x) = \begin{cases} 3\chi_{\text{im}}(x) & \text{if } \chi_+(x) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \tau_\chi + \bar{\tau}_\chi &= -\frac{1}{2f} \sum_{\substack{x=1 \\ \gcd(x, f_+)=1}}^{f-1} x(\chi(x) + \bar{\chi}(x)) \\ &= \frac{1}{2f} \sum_{\substack{x=1 \\ \gcd(x, f_+)=1}}^{f-1} x\chi_{\text{im}}(x) - \frac{3}{2f} \sum_{\substack{x=1 \\ \chi_+(x)=1}}^{f-1} x\chi_{\text{im}}(x). \end{aligned}$$

First, we have

$$\begin{aligned} \sum_{\substack{x=1 \\ \gcd(x, f_+)=1}}^{f-1} x\chi_{\text{im}}(x) &= \sum_{d|f_+} \mu(d) \sum_{\substack{x=1 \\ d|x}}^{f-1} x\chi_{\text{im}}(x) \\ &= \sum_{d|f_+} d\chi_{\text{im}}(d)\mu(d) \sum_{x=1}^{mf_+/d-1} x\chi_{\text{im}}(x). \end{aligned}$$

However, for any $k \geq 1$ we have

$$\begin{aligned} \sum_{x=1}^{km-1} x\chi_{\text{im}}(x) &= \sum_{a=1}^{m-1} \sum_{b=0}^{k-1} (a+mb)\chi_{\text{im}}(a+mb) \\ &= k \sum_{a=1}^{m-1} a\chi_{\text{im}}(a) = -\frac{2kmh(k_{\text{im}})}{w(k_{\text{im}})}. \end{aligned}$$

Therefore, using $f = f_+m$, we get

$$\begin{aligned} \frac{1}{2f} \sum_{\substack{x=1 \\ \gcd(x, f_+)=1}}^{f-1} x\chi_{\text{im}}(x) &= -\frac{h(k_{\text{im}})}{w(k_{\text{im}})} \sum_{d|f_+} \chi_{\text{im}}(d)\mu(d) \\ &= -\frac{h(k_{\text{im}})}{w(k_{\text{im}})} \prod_{p|f_+} (1 - \chi_{\text{im}}(p)). \end{aligned}$$

Second, we have

$$\begin{aligned} -\frac{3}{2f} \sum_{\substack{x=1 \\ \chi_+(x)=1}}^{f-1} x\chi_{\text{im}}(x) &= -\frac{3}{2f_+m} \sum_{\substack{a=1 \\ \chi_+(a)=1}}^{f_+-1} \sum_{b=0}^{m-1} (a+bf_+)\chi_{\text{im}}(a+bf_+) \\ &= -\frac{3}{2m} \sum_{\substack{a=1 \\ \chi_+(a)=1}}^{f_+-1} \sum_{b=0}^{m-1} b\chi_{\text{im}}(a+bf_+) \\ &= -\frac{3}{2m} \sum_{\substack{a=1 \\ \chi_+(a)=1}}^{f_+-1} S_m(a, f_+). \end{aligned}$$

Since $S_m(f_+ - a, f_+) = S_m(a, f_+)$ we get

$$-\frac{3}{2f} \sum_{\substack{x=1 \\ \chi_+(x)=1}}^{f-1} x\chi_{\text{im}}(x) = -\frac{3}{m} \sum_{\substack{1 \leq a \leq f_+/2 \\ \chi_+(a)=1}}^{f_+-1} S_m(a, f_+).$$

Now, we claim that

$$S_m(a, f_+) = \chi_{\text{im}}(f_+) \left(-\frac{2mh(k_{\text{im}})}{w(k_{\text{im}})} + m \sum_{b=1}^{a_+-1} \chi_{\text{im}}(b) \right),$$

which provides us with the desired first result (upon using $|\{a : 1 \leq a \leq f_+/2 \text{ and } \chi_+(a) = 1\}| = \phi(f_+)/6$). Indeed, we have

$$\begin{aligned}
S_m(a, f_+) &= \chi_{\text{im}}(f_+) \sum_{b=0}^{m-1} b \chi_{\text{im}}(a_+ + b) \\
&= \chi_{\text{im}}(f_+) \sum_{b=0}^{m-1} (a_+ + b) \chi_{\text{im}}(a_+ + b) \\
&= \chi_{\text{im}}(f_+) \sum_{b=a_+}^{a_++m-1} b \chi_{\text{im}}(b) \\
&= \chi_{\text{im}}(f_+) \sum_{b=a_+}^{m-1} b \chi_{\text{im}}(b) + \chi_{\text{im}}(f_+) \sum_{b=m+1}^{a_++m-1} b \chi_{\text{im}}(b) \\
&= \chi_{\text{im}}(f_+) \sum_{b=a_+}^{m-1} b \chi_{\text{im}}(b) + \chi_{\text{im}}(f_+) \sum_{b=1}^{a_+-1} (b+m) \chi_{\text{im}}(b+m) \\
&= \chi_{\text{im}}(f_+) \sum_{b=1}^{m-1} b \chi_{\text{im}}(b) + m \sum_{b=1}^{a_+-1} \chi_{\text{im}}(b) \\
&= \chi_{\text{im}}(f_+) \left(-\frac{2mh(k_{\text{im}})}{w(k_{\text{im}})} + m \sum_{b=1}^{a_+-1} \chi_{\text{im}}(b) \right).
\end{aligned}$$

To get the second desired result, we notice that 6 always divides $\phi(f_+)$, which yields

$$\tau_\chi + \bar{\tau}_\chi \equiv -\frac{h(k_{\text{im}})}{w(k_{\text{im}})} \prod_{p|f_+} (1 - \chi_{\text{im}}(p)) + \frac{\phi(f_+)h(k_{\text{im}})}{w(k_{\text{im}})} \chi_{\text{im}}(f_+) \pmod{3},$$

and implies

$$\tau_\chi + \bar{\tau}_\chi \equiv \begin{cases} -\frac{h(k_{\text{im}})}{2} \prod_{p|f_+} (1 - \chi_{\text{im}}(p)) \pmod{3} & \text{if } m \neq 3, 4, \\ \frac{\phi(f_+)}{6} \pmod{3} & \text{if } m = 3, \\ -\frac{1}{4} \prod_{p|f_+} (1 - \chi_{\text{im}}(p)) + \frac{\phi(f_+)}{4} \chi_{\text{im}}(f_+) \pmod{3} & \text{if } m = 4, \end{cases}$$

(for $m = 3$ implies $\chi_{\text{im}}(p) = 1$ for all primes p dividing f_+). ■

5. Numerical computations and proofs of Theorems 1 and 2.

We show how to determine all imaginary abelian sextic number fields K having $h(K) = 3$. In a similar fashion we will obtain all imaginary abelian sextic number fields K having $h(K) \leq 11$, and those with $h^*(K) \leq 4$. From

Table 3. $h^*(K) \leq 4$

$h^*(K) = 1$									
f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
7	7	1	7	1	67	67	1	67	1
9	9	1	3	1	76	19	1	4	1
19	19	1	19	1	77	7	1	11	1
21	7	1	3	1	91	13	1	7	1
28	7	1	4	1	91	91	3	7	1
						$x^3 - x^2 - 30x - 27$			
36	9	1	4	1	93	31	1	3	1
39	13	1	3	1	104	13	1	8	1
43	43	1	43	1	117	117	3	3	1
						$x^3 - 39x + 26$			
56	7	1	8	1	129	43	1	3	1
63	9	1	7	1	133	133	3	7	1
						$x^3 - x^2 - 44x - 69$			
63	63	3	3	1	171	171	3	19	1
		$x^3 - 21x - 28$				$x^3 - 57x + 152$			
63	63	3	3	1	217	217	3	7	1
		$x^3 - 21x + 35$				$x^3 - x^2 - 72x + 225$			
63	63	3	7	1	247	247	3	19	1
		$x^3 - 21x + 35$				$x^3 - x^2 - 82x + 64$			

$h^*(K) = 2$									
f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
35	7	1	35	2	91	91	3	91	2
						$x^3 - x^2 - 30x - 27$			
45	9	1	15	2	91	91	3	91	2
						$x^3 - x^2 - 30x + 64$			
52	13	1	52	2	105	7	1	15	2
72	9	1	24	2					

Theorem 6 we obtain

$$h^*(K) > 3 \quad \text{if } f \geq 5.9 \cdot 10^5.$$

We consider two cases:

- (A) $(m, f_+) = 1$, and
- (B) $(m, f_+) > 1$.

For each case we consider three possible types in Table 1. Our strategy is now as follows. First, for a given m we compute an upper bound for f . Second, we find all possible conductors f_+ (Propositions 4, 5 and Theorems 7, 8). Third, we compute $|\tau_\chi + \bar{\tau}_\chi|$ and $|\tau_\chi|^2$. Finally, we verify the class

Table 3 (cont.)

$h^*(K) = 3$									
f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
52	13	1	4	1	273	91	3	3	1
							$x^3 - x^2 - 30x + 64$		
57	19	1	3	1	292	73	1	4	1
72	9	1	8	1	301	301	3	7	1
							$x^3 - x^2 - 100x - 223$		
99	9	1	11	1	301	301	3	7	1
							$x^3 - x^2 - 100x + 379$		
111	37	1	3	1	327	109	1	3	1
133	133	3	19	1	333	333	3	3	1
							$x^3 - 111x - 370$		
133	133	3	19	1	341	31	1	11	1
							$x^3 - x^2 - 44x + 64$		
133	7	1	19	1	364	91	3	4	1
							$x^3 - x^2 - 30x - 27$		
148	37	1	4	1	381	127	1	3	1
152	19	1	8	1	399	133	3	3	1
							$x^3 - x^2 - 44x + 64$		
171	171	3	3	1	469	67	1	7	1
							$x^3 - 57x - 19$		
171	171	3	3	1	553	553	3	7	1
							$x^3 - 57x + 152$		
244	61	1	4	1	657	657	9	3	1
							$x^3 - 219x - 730$		
259	259	3	7	1					
							$x^3 - x^2 - 86x + 211$		

number $h(K_+)$ by Gras's Table [M.N.G] and compute the cubic polynomials if 3 divides $h(K_+)$.

Case (A): $(m, f_+) = 1$

(i) If $h(K_+) = 1$, $h^*(K) = 3$ and $h(k_{\text{im}}) = 3$, then $h(K) = 3$ is impossible: by Proposition 5(1), $f_+ = 3^2$, so it suffices to compute τ_χ for the 16 conductors $f = 3^2m$, with $h(\mathbb{Q}(\sqrt{-m})) = 3$.

(ii) If $h(K_+) = 1$, $h^*(K) = 3$ and $h(k_{\text{im}}) = 1$, then for each one of the 9 imaginary quadratic fields having class number one we compute upper bounds for the conductor f of K/\mathbb{Q} :

$$|\tau_\chi|^2 > 3 \quad \text{for } f \geq 3.5 \cdot 10^5 \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-1});$$

$$|\tau_\chi|^2 > 3 \quad \text{for } f \geq 5.4 \cdot 10^5 \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-2});$$

Table 3 (cont.)

$h^*(K) = 4$									
f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
39	13	1	39	4	172	43	1	4	1
56	7	1	56	4	183	61	1	3	1
84	7	1	84	4	201	67	1	3	1
117	9	1	39	4	209	19	1	11	1
117	117	3	39	4	248	31	1	8	1
		$x^3 - 39x - 91$							
124	31	1	4	1	252	63	3	4	1
							$x^3 - 21x - 28$		
133	19	1	7	1	259	259	3	259	4
							$x^3 - x^2 - 86x - 48$		
155	31	1	155	4	473	43	1	11	1
163	163	4	163	1	511	73	1	7	1
171	9	1	19	1	711	711	12	3	1
							$x^3 - 237x - 316$		

$$\begin{aligned}
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 2.5 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-3}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 5.9 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-7}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 4.5 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-11}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 3.2 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-19}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 1.9 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-43}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 1.4 \cdot 10^5 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-67}); \\
 |\tau_\chi|^2 > 3 & \quad \text{for } f \geq 8.0 \cdot 10^4 & \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-163}).
 \end{aligned}$$

We summarize our computational results when $k_{\text{im}} = \mathbb{Q}(\sqrt{-1})$. (For the other fields the computation is exactly the same.) By Theorem 8 we have $f = 4p$, $p \equiv 1 \pmod{12}$.

- 1) There are 2098 prime p 's such that $p \equiv 1 \pmod{12}$ and $p \leq 87500$.
- 2) There are 1413 prime p 's such that $2^{(p-1)/3} \not\equiv 1 \pmod{p}$ (Proposition 4).
- 3) There are 57 prime p 's such that $|\tau_\chi + \bar{\tau}_\chi| = 0$ or 3 (Lemma 2).
- 4) For these 57 prime p 's we compute $|\tau_\chi|^2$.

There are exactly 4 fields having class number 3: $f = 52, 148, 244$ and 292.

(iii) If $h(K_+) = 3$, $h^*(K) = 1$ and $h(k_{\text{im}}) = 1$, the same argument applies to each of the nine fields. We proceed as follows: for example let

Table 4. $h(K) \leq 11$ The fields with $h(K) \leq 4$ are listed in Table 3.

f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
$h(K) = 5$									
79	79	1	79	5	103	103	1	103	5
$h(K) = 6$									
91	91	3	91	2	168	7	1	24	2
		$x^3 - x^2 - 30x - 27$							
91	91	3	91	2	180	9	1	20	2
		$x^3 - x^2 - 30x + 64$							
140	7	1	20	2	285	19	1	15	2
$h(K) = 7$									
143	13	1	11	1	471	157	1	3	1
151	151	1	151	7	589	31	1	19	1
237	79	1	3	1	604	151	1	4	1
268	67	1	4	1	687	229	1	3	1
296	37	1	8	1	721	103	1	7	1
412	103	1	4	1	1199	109	1	11	1
427	61	1	7	1	1371	457	1	3	1
$h(K) = 8$									
91	7	1	91	2	153	9	1	51	2
95	19	1	95	8	195	13	1	15	2
111	37	1	111	8	260	13	1	20	2

$k_{\text{im}} = \mathbb{Q}(\sqrt{-7})$. We have

$$|\tau_\chi|^2 > 1 \quad \text{for } f \geq 1.5 \cdot 10^5 \quad \text{if } k_{\text{im}} = \mathbb{Q}(\sqrt{-7}).$$

1) There are 650 conductors f_+ such that

$$\begin{cases} \gcd(f_+, 7) = 1, \\ f_+ \leq 2.2 \cdot 10^4, \\ f_+ = 3^2 p \text{ or } f_+ = p_1 p_2, \quad p, p_1, p_2 \equiv 1 \pmod{6}. \end{cases}$$

2) There are 267 f_+ 's such that

$$\begin{cases} \chi_{\text{im}}(p) = \left(\frac{-7}{p}\right) = -1 & \text{if } f_+ = 3^2 p, \\ \chi_{\text{im}}(p_1) = -1 \text{ and } \chi_{\text{im}}(p_2) = -1 & \text{if } f_+ = p_1 p_2 \text{ (Theorem 7)}. \end{cases}$$

For a given $f_+ = 3^2 p$ or $p_1 p_2$, with $p, p_1, p_2 \equiv 1 \pmod{6}$, there are then two

Table 4 (cont.)

f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$	f	f_+	$h(K_+)$	m	$h(k_{\text{im}})$
		polynomial defining K_+					polynomial defining K_+		
$h(K) = 9$									
31	31	1	31	3	307	307	1	307	3
133	133	3	19	1	333	333	3	3	1
		$x^3 - x^2 - 44x - 69$					$x^3 - 111x - 370$		
133	133	3	19	1	364	91	3	4	1
		$x^3 - x^2 - 44x + 64$					$x^3 - x^2 - 30x - 27$		
139	139	1	139	3	388	97	1	4	1
161	7	1	23	3	399	133	3	3	1
							$x^3 - x^2 - 44x + 64$		
171	171	3	3	1	437	19	1	23	3
		$x^3 - 57x - 19$							
171	171	3	3	1	499	499	1	499	3
		$x^3 - 57x + 152$							
207	9	1	23	3	553	79	1	7	1
217	7	1	31	3	553	553	3	7	1
							$x^3 - x^2 - 184x - 41$		
259	37	1	7	1	597	199	1	3	1
259	259	3	7	1	643	643	1	643	3
		$x^3 - x^2 - 86x + 211$							
273	91	3	3	1	772	193	1	4	1
		$x^3 - x^2 - 30x + 64$							
279	9	1	31	3	817	43	1	19	1
283	283	1	283	3	1057	151	1	7	1
299	13	1	23	3	1727	157	1	11	1
301	301	3	7	1	2453	223	1	11	1
		$x^3 - x^2 - 100x - 223$							
301	301	3	7	1					
		$x^3 - x^2 - 100x + 379$							
$h(K) = 10$									
119	7	1	119	10	143	13	1	143	10
$h(K) = 11$									
271	271	1	271	11					

nonconjugate cubic characters, i.e.

$$\begin{cases} \chi_+ = \chi_{3^2} \chi_p \text{ or } \chi_{3^2} \chi_p^2 & \text{if } f_+ = 3^2 p, \\ \chi_+ = \chi_{p_1} \chi_{p_2} \text{ or } \chi_+ = \chi_{p_1} \chi_{p_2}^2 & \text{if } f_+ = p_1 p_2. \end{cases}$$

Here, for $q \in \{3^2, p, p_1, p_2\}$ we let $\chi_q(g) = \exp(2i\pi/3)$, with g a primitive root modulo q .

Consequently, for 534 pairs (f_+, χ_+) we have to test whether $\chi_+(7) \neq 1$ or not.

3) There are 358 pairs (f_+, χ_+) such that $\chi_+(7) \neq 1$ (Proposition 4).

4) Using Theorem 8 we compute $\tau_\chi + \bar{\tau}_\chi$ and choose $\tau_\chi + \bar{\tau}_\chi$ such that $|\tau_\chi + \bar{\tau}_\chi| = 1$ or 2. (Since $h^*(K) = 1$, we have $|\tau_\chi + \bar{\tau}_\chi| = 1$ or 2 by Lemma 2.) There are 11 pairs (f_+, χ_+) such that $|\tau_\chi + \bar{\tau}_\chi| = 1$ or 2.

5) Finally, we compute $|\tau_\chi|^2$ for these 11 pairs (f_+, χ_+) of 4).

We verify that there are no such fields having class number 3.

Case (B): $(m, f_+) > 1$. We have $f \leq 5.9 \cdot 10^5$. First, we make a finite list of possible conductors f_+ which are less than $5.9 \cdot 10^5$. Second, we select those satisfying Proposition 4 and Theorem 7. Finally, we compute $|\tau_\chi|^2$.

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