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On the diophantine equation $(x^m+1)(x^n+1)=y^2$

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. In [7], Ribenboim proved that the equation

$$(1)\qquad (x^m+1)(x^n+1)=y^2,\qquad x,y,m,n\in\mathbb{N},\ x>1,\ n>m\ge 1,$$

has no solution (x,y,m,n) with $2\mid x$ and (1) has only finitely many solutions (x,y,m,n) with $2\nmid x$. Moreover, all solutions of (1) with $2\nmid x$ satisfy $\max(x,m,n)<C$, where C is an effectively computable constant. In this paper we completely determine all solutions of (1) as follows.

Theorem. *Equation* (1) *has only the solution* $(x,y,m,n)=(7,20,1,2)$.

2. Preliminaries

Lemma 1 ([4]). *The equation*

$$X^2-2Y^4=1,\qquad X,Y\in\mathbb{N},$$

has no solution (X,Y). *The equation*

$$X^2-2Y^4=-1,\qquad X,Y\in\mathbb{N},$$

has only the solutions $(X,Y)=(1,1)$ *and* $(239,13)$.

Lemma 2 ([5]). *Let* a, D *be positive integers with* $2\nmid a$. *The equation*

$$a^2X^4-DY^2=-1,\qquad X,Y\in\mathbb{N},$$

has at most one solution (X,Y).

Lemma 3 ([1]). *The equation*

$$X^n+1=Y^2,\qquad X,Y,n\in\mathbb{N},\ n>1,$$

has only the solution $(X,Y,n)=(2,3,3)$.

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Lemma 4 ([6]). *The equation*

$$\frac{X^n+1}{X+1}=Y^2,\qquad X,Y,n\in\mathbb{N},\ X>1,\ n>1,\ 2\nmid n,$$

has no solution (X,Y,n).

Let a, b be nonzero integers such that $a>0$, $\gcd(a,b)=1$ and $a-4b>0$. Let α, β be distinct zeros of the polynomial $z^2-\sqrt{a}z+b$. For any odd integer n, let $F(n)=(\alpha^n-\beta^n)/(\alpha-\beta)$. Then $F(n)$ are nonzero integers if $n>0$.

Lemma 5 ([9]). *If* $4\,|\,a$, $b\equiv 1 \pmod 4$ *and* $(a/b)=1$, *where* (a/b) *is Jacobi's symbol*, *then the equation*

$$F(n)=Y^2,\qquad n,Y\in\mathbb{N},\ n>1,\ 2\nmid n,$$

has no solution (n,Y). *If* $4\,|\,a$, $b\equiv 3 \pmod 4$ *and* $(a/b)=1$, *then the equation*

$$F(n)=nY^2,\qquad n,Y\in\mathbb{N},\ n>1,\ 2\nmid n,$$

has no solution (n,Y).

Lemma 6. *The equation*

$$\frac{X^n+1}{X+1}=nY^2,\qquad X,Y,n\in\mathbb{N},\ X>1,\ n>1,\ 2\nmid n,\tag{2}$$

has no solution (X,Y,n) *with* $X\equiv 1 \pmod 4$.

Proof. Let $\alpha=X$ and $\beta=-1$. Then α and β are distinct zeros of $z^2-(X-1)z-X$. Since $X-1\equiv 0 \pmod 4$ and $-X\equiv 3 \pmod 4$ if $X\equiv 1 \pmod 4$, by Lemma 5, (2) is impossible. The lemma is proved.

Lemma 7. *If* (X,Y,n) *is a solution of* (2) *with* $X+1\equiv 0 \pmod n$, *then* n *is squarefree.*

Proof. Let (X,Y,n) be a solution of (2) with $X+1\equiv 0 \pmod n$. If n is not squarefree, then there exists an odd prime p satisfying $p^2\,|\,n$. Since $X^{n/p}+1\equiv 0 \pmod{X+1}$ and $X+1\equiv 0 \pmod n$, we derive from (2) that

$$\frac{X^{n/p}+1}{X+1}=ndY_1^2,\qquad \frac{(X^{n/p})^p+1}{X^{n/p}+1}=dY_2^2,\tag{3}$$

where d, Y_1, Y_2 are positive integers satisfying $dY_1Y_2=Y$. Since $d\,|\,p$, we get either $d=1$ or $d=p$. By Lemma 4, (3) is impossible for $d=1$. So we have $d=p$ and

$$\frac{X^{n/p}+1}{X+1}=npY_1^2,\qquad \frac{X^n+1}{X^{n/p}+1}=pY_2^2.\tag{4}$$

By the same argument, we infer from the first equality of (4) that

$$\frac{X^{n/p^2}+1}{X+1}=np^2Y_{11}^2,\qquad \frac{X^{n/p}+1}{X^{n/p^2}+1}=pY_{12}^2,\qquad pY_{11}Y_{12}=Y_1,\ Y_{11},Y_{12}\in\mathbb{N}. \tag{5}$$

Combination of (4) and (5) yields

$$\frac{(X^{n/p^2})^{p^2}+1}{X^{n/p^2}+1}=(pY_{12}Y_2)^2.$$

However, by Lemma 4, this is impossible. The lemma is proved.

LEMMA 8. *Let* $\varrho=1+\sqrt{2}$ *and* $\overline{\varrho}=1-\sqrt{2}$. *For any nonnegative integer* k, *let*

$$U_k=\frac{\varrho^k+\overline{\varrho}^k}{2},\qquad V_k=\frac{\varrho^k-\overline{\varrho}^k}{2\sqrt{2}}. \tag{6}$$

Then U_k *and* V_k *are nonnegative integers satisfying*:

(i) $\gcd(U_k,V_k)=1$.

(ii) $\gcd(U_k,U_{k+1})=\gcd(V_k,V_{k+1})=1$.

(iii) *If* $U_k\equiv 7 \pmod 8$, *then* $k\equiv 3 \pmod 4$.

(iv) *The prime factors* p *of* U_k *satisfy*

$$p\equiv\begin{cases}\pm1 \pmod 8 & \text{if } 2\nmid k,\\ 1,3 \pmod 8 & \text{if } 2\mid k.\end{cases}$$

(v) *If* $2\nmid k$, *then the prime factors* p *of* V_k *satisfy* $p\equiv 1 \pmod 4$.

(vi) U_k *is a square if and only if* $k=1$.

(vii) V_k *is a square if and only if* $k=1,7$.

Proof. Since U_k and V_k are integers satisfying

$$U_k^2-2V_k^2=(-1)^k, \tag{7}$$

we get (i), (iv) and (v) immediately. Moreover, by Lemmas 1 and 2, we obtain (vi) and (vii), respectively.

On the other hand, since U_k and V_k satisfy the recurrences

$$U_0=1,\quad U_1=1,\quad U_{k+2}=2U_{k+1}+U_k,\quad k\ge 0,$$
$$V_0=0,\quad V_1=1,\quad V_{k+2}=2V_{k+1}+V_k,\quad k\ge 0,$$

respectively, we get (ii) and (iii) immediately. The lemma is proved.

LEMMA 9. *The equation*

$$U_{rs}=U_rY^2,\qquad r,s,Y\in\mathbb{N},\ r>1,\ s>1,\ 2\nmid r,\ 2\nmid s, \tag{8}$$

has no solution (r,s,Y).

Proof. Let $\alpha = \varrho^r$ and $\beta = \overline{\varrho}^r$. Then α and β are distinct zeros of $z^2 - 2\sqrt{2}V_r z + 1$. If (r, s, Y) is a solution of (8), then we have

$$\frac{\alpha^s - \beta^s}{\alpha - \beta} = Y^2.$$

However, by Lemma 5, this is impossible.

LEMMA 10. *Let n be a positive integer with $n < 23$. Then the equation*

$$U_r = nY^2, \quad r, Y \in \mathbb{N},\ r > 1,\ 2 \nmid r,\ Y > 1 \tag{9}$$

has no solution (r, Y).

Proof. By (iv) of Lemma 8, we see from (9) that every prime factor p of n satisfies $p \equiv \pm 1 \pmod 8$. Since $n < 23$, we have $n = 7$ or 17. Since $U_3 = 7$, by Lemma 9, (9) is impossible for $n = 7$. Notice that $U_4 = 17$ and $17 \nmid U_j$ for $j = 1, 2, 3$. We see that $17 \nmid U_r$ if $2 \nmid r$. This implies that (9) is impossible for $n = 17$. The lemma is proved.

LEMMA 11 ([8]). *Let p be an odd prime with $p < 1000$. If $V_r = pY^2$ for some positive integers r, Y with $2 \nmid r$, then $Y = 1$.*

Let α be an algebraic number with the minimal polynomial over $\mathbb{Z}$

$$a_0 z^d + a_1 z^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i \alpha), \qquad a_0 > 0,$$

where $\sigma_1\alpha, \sigma_2\alpha, \ldots, \sigma_d\alpha$ are all conjugates of α. Then

$$h(\alpha) = \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log \max(1, |\sigma_i \alpha|)\Big)$$

is called the *absolute logarithmic height* of α.

LEMMA 12 ([2, Corollary 2]). *Let α_1, α_2 be real algebraic numbers which exceed one and are multiplicatively independent. Further, let $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2$ for some positive integers b_1, b_2. Then*

$$\log |\Lambda| \geq -24.34 D^4 (\log A_1)(\log A_2)(\max(1/2, 21/D, 0.14 + \log B))^2,$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$, $\log A_j = \max(h(\alpha_j), |\log \alpha_j|/D, 1/D)$ $(j = 1, 2)$, $B = b_1/(D \log A_2) + b_2/(D \log A_1)$.

LEMMA 13. *If (X, Y, n) is a solution of the equation*

$$X^n + 1 = 2Y^2, \quad X, Y, n \in \mathbb{N},\ X > 1,\ Y > 1,\ n > 1,\ 2 \nmid n, \tag{10}$$

then there exist suitable positive integers X_1, Y_1, s such that

$$-X = X_1^2 - 2Y_1^2, \quad \gcd(X_1, Y_1) = 1, \quad 1 < \left|\frac{X_1 + Y_1\sqrt{2}}{X_1 - Y_1\sqrt{2}}\right| < (3 + 2\sqrt{2})^2, \tag{11}$$

$$\left| n\log\frac{X_1+Y_1\sqrt{2}}{-X_1+Y_1\sqrt{2}} - s\log(3+2\sqrt{2})^2\right| < \frac{2\sqrt{2}}{Y} < \frac{4}{X^{n/2}}, \quad s<n. \tag{12}$$

P r o o f. Since $2\nmid n$, we see from (10) that

$$(-X)^n = 1-2Y^2. \tag{13}$$

Notice that the class number of $\mathbb{Q}(\sqrt{2})$ is equal to one. By much the same argument as in the proof of [3, Theorem 2], we can obtain (11) and (12) from (13). The lemma is proved.

Lemma 14. *All solutions* (X,Y,n) *of* (10) *satisfy* $n<330000$.

P r o o f. Let (X,Y,n) be a solution of (10). By Lemma 13, there exist positive integers X_1, Y_1, s satisfying (11) and (12). Let $\alpha_1=(X_1+Y_1\sqrt{2})/(-X_1+Y_1\sqrt{2})$, $\alpha_2=(3+2\sqrt{2})^2$ and $\Lambda=n\log\alpha_1-s\log\alpha_2$. Then α_1 and α_2 are multiplicatively independent and satisfy $X\alpha_1^2-2(X_1^2+2Y_1^2)\alpha_1+X=0$ and $\alpha_2^2-34\alpha_2+1=0$ respectively. So we have

$$h(\alpha_1)=\frac{1}{2}\left(\log X+\log\left|\frac{X_1+Y_1\sqrt{2}}{X_1-Y_1\sqrt{2}}\right|\right), \qquad h(\alpha_2)=\log(3+2\sqrt{2}). \tag{14}$$

Further, by (11) and (14), we get

$$h(\alpha_1)<\frac{1}{2}(\log X+2\log(3+2\sqrt{2})). \tag{15}$$

Since $[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]=[\mathbb{Q}(\sqrt{2}):\mathbb{Q}]=2$, by Lemma 12, we deduce from (14) and (15) that

$$\begin{aligned}\log|\Lambda|\geq & -389.44\left(\frac{1}{2}\log X+\log(3+2\sqrt{2})\right)\\ & \times(\log(3+2\sqrt{2}))(\max(10.5, 0.14+\log B))^2,\end{aligned} \tag{16}$$

where

$$B=\frac{n}{2\log(3+2\sqrt{2})}+\frac{s}{\log X+2\log(3+2\sqrt{2})}. \tag{17}$$

Since $s<n$ by (12), if $n\geq 330000$, then from (17) we get

$$\log B>\log(n/2\log(3+2\sqrt{2}))\geq\log(330000/2\log(3+2\sqrt{2}))>10.5.$$

Therefore, by (16) and (17), we obtain

$$\log|\Lambda|>-343.24(\log X+2\log(3+2\sqrt{2}))(0.14+\log 0.5673n)^2. \tag{18}$$

Combination of (12) and (18) yields

$$\frac{2\log 4}{\log X}+686.48\left(1+\frac{2\log(3+2\sqrt{2})}{\log X}\right)(0.14+\log 0.5673n)^2>n. \tag{19}$$

Since every prime factor p of X satisfies $p\equiv\pm1 \pmod 8$ by (11), we get $X\geq 7$ and (19) is impossible for $n\geq 330000$. The lemma is proved.

LEMMA 15. *The equation*

$$X^n + 1 = 2Y^2, \quad X, Y, n \in \mathbb{N},\ X > 1,\ Y > 1,\ n > 2,\ 2 \mid n, \tag{20}$$

has no solution (X, Y, n).

Proof. Let (X, Y, n) be a solution of (20). Notice that $(X', Y') = (1, 1)$ is a solution of the equation

$$X'^4 - 2Y'^2 = 1, \quad X', Y' \in \mathbb{N}.$$

Hence, by Lemma 2, we have $4 \nmid n$. This implies that $n = 2t$, where t is an odd integer with $t > 1$. Since t has an odd prime factor p if $t > 1$, we see from (20) that

$$X^{n/p} + 1 = 2dY_1^2, \tag{21}$$

$$\frac{X^n + 1}{X^{n/p} + 1} = dY_2^2, \tag{22}$$

where d, Y_1, Y_2 are positive integers satisfying $dY_1Y_2 = Y$. By Lemma 4, if $d = 1$, then (22) is impossible. On the other hand, since $X^{n/p} \equiv -1 \pmod{d}$ by (21), we see from (22) that $d \mid p$. So we have $d = p$ and

$$\frac{(X^{n/p})^p + 1}{X^{n/p} + 1} = pY_2^2, \tag{23}$$

by (22). Since $2 \mid n$ and $X^{n/p} \equiv 1 \pmod{4}$, by Lemma 6, (23) is impossible. The lemma is proved.

3. Proof of Theorem. By [7], it suffices to consider the solutions (x, y, m, n) of (1) with $2 \nmid x$.

Let (x, y, m, n) be a solution of (1) with $2 \nmid x$. Then we have

$$x^m + 1 = dy_1^2, \quad x^n + 1 = dy_2^2, \quad 1 \le m < n, \tag{24}$$

where d, y_1, y_2 are positive integers satisfying $dy_1y_2 = y$ and d is squarefree. By Lemma 3, (24) is impossible for $d = 1$. If $d > 1$ and d has an odd factor d_1 with $d_1 > 1$, let r denote the least positive integer with $x^r + 1 \equiv 0 \pmod{d_1}$. Then from (24) we get

$$m = rm_1, \quad n = rn_1, \tag{25}$$

where m_1, n_1 are odd positive integers with $1 \le m_1 < n_1$. Further, let

$$s = \gcd(m, n), \quad m = sm', \quad n = sn'. \tag{26}$$

We see from (25) and (26) that $r \mid s$ and m', n' are odd positive integers satisfying $1 \le m' < n'$ and $\gcd(m', n') = 1$. Let $z = x^s$. Then (24) can be written as

$$z^{m'} + 1 = dy_1^2, \quad z^{n'} + 1 = dy_2^2. \tag{27}$$

Since $r \mid s$, $2 \nmid s/r$ and $z+1 = x^s+1 \equiv 0 \pmod{x^r+1}$, we derive from (27) that $z+1 = d_1 d' y'^2$ and

$$\frac{z^{m'}+1}{z+1} = d' y_1'^2, \qquad \frac{z^{n'}+1}{z+1} = d' y_2'^2, \tag{28}$$

where d', y', y_1', y_2' are positive integers satisfying $d'y'y_1' = y_1$ and $d'y'y_2' = y_2$. Since $\gcd(m', n') = 1$, we have $\gcd((z^{m'}+1)/(z+1), (z^{n'}+1)/(z+1)) = 1$. Hence, by (28), we get $d' = 1$ and

$$\frac{z^{n'}+1}{z+1} = y_2'^2, \qquad n' > 1,\ 2 \nmid n'. \tag{29}$$

However, by Lemma 3, (29) is impossible. So we have $d = 2$. Then (24) can be written as

$$x^m + 1 = 2y_1^2, \qquad x^n + 1 = 2y_2^2. \tag{30}$$

Let s, m', n' be defined as in (26), and let $z = x^s$. If $m \equiv n \equiv 1 \pmod 2$, then from (30) we get

$$z+1 = 2dy_{11}^2, \qquad \frac{z^{m'}+1}{z+1} = dy_{12}^2, \qquad \frac{z^{n'}+1}{z+1} = dy_{22}^2, \tag{31}$$

where d, y_{11}, y_{12}, y_{22} are positive integers satisfying $dy_{11}y_{12} = y_1$ and $dy_{11}y_{22} = y_2$. Since $\gcd(m', n') = 1$ and $\gcd((z^{m'}+1)/(z+1), (z^{n'}+1)/(z+1)) = 1$, we see from (31) that $d = 1$. Since $n' > 1$ and $2 \nmid n'$, by Lemma 3, (31) is impossible for $d = 1$. On the other hand, if $m \equiv n \equiv 0 \pmod 2$, then $n > 2$ and $2 \mid n$. By Lemma 15, this is impossible. Therefore, the parities of m and n are distinct. Furthermore, by Lemma 15, we conclude from (30) that either $m = 1$ and $n = 2$, or $m = 2$ and $2 \nmid n$.

If $m = 1$ and $n = 2$, then from (30) we get

$$x + 1 = 2y_1^2, \qquad x^2 + 1 = 2y_2^2. \tag{32}$$

Let $\varrho = 1 + \sqrt{2}$ and $\overline{\varrho} = 1 - \sqrt{2}$. For any nonnegative integer k, let U_k, V_k be defined as in (6). We see from the second equality of (32) that

$$x = \frac{\varrho^k + \overline{\varrho}^k}{2} = \frac{\varrho^k + \overline{\varrho}^k}{\varrho + \overline{\varrho}}, \tag{33}$$

for some odd positive integers k. From (33), we get

$$\begin{aligned} x + 1 &= \frac{\varrho^k + \overline{\varrho}^k}{\varrho + \overline{\varrho}} + (-\varrho\overline{\varrho})^{(k-1)/2} \\ &= \begin{cases} 2U_{(k+1)/2}U_{(k-1)/2} & \text{if } k \equiv 1 \pmod 4, \\ 4V_{(k+1)/2}V_{(k-1)/2} & \text{if } k \equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{34}$$

Notice that $\gcd(U_{(k+1)/2}, U_{(k-1)/2}) = 1$ by (ii) of Lemma 8. If $k \equiv 1 \pmod 4$, then from (34) and the first equality of (32) we get

$$(35)\quad U_{(k+1)/2}=y_{11}^2,\quad U_{(k-1)/2}=y_{12}^2,\quad y_{11}y_{12}=y_1,\quad y_{11},y_{12}\in\mathbb{N}.$$

However, by (vi) of Lemma 8, (35) is impossible. Similarly, if $k\equiv 3 \pmod 4$, then from (32) and (35) we get

$$(36)\quad V_{(k+1)/2}=2y_{11}^2,\quad V_{(k-1)/2}=y_{12}^2,\quad 2y_{11}y_{12}=y_1,\quad y_{11},y_{12}\in\mathbb{N},$$

since $2\nmid(k-1)/2$ and $2\nmid V_{(k-1)/2}$. Therefore, by (vii) of Lemma 8, we see from (36) that $(k,y_{11},y_{12})=(3,1,1)$. Further, by (32), we obtain $(x,y,m,n)=(7,20,1,2)$.

If $m=2$ and $2\nmid n$, then from (30) we get

$$(37)\quad x^2+1=2y_1^2,$$

$$(38)\quad x^n+1=2y_2^2,\quad n>2,\ 2\nmid n.$$

By the proof of [3, Theorem 2], we see from (38) that $x+1=2ny'^2$ for some positive integer y'. Further, by Lemma 6, we get $2\mid y'$ and

$$(39)\quad x+1=8ny_{21}^2,\quad \frac{x^n+1}{x+1}=ny_{22}^2,\quad 2ny_{21}y_{22}=y_2,\quad y_{21},y_{22}\in\mathbb{N}.$$

By Lemma 7 we find from (39) that n is squarefree.

On the other hand, since $x\equiv 7 \pmod 8$, by (iii) of Lemma 8, we see from (37) that

$$(40)\quad x=U_{4k+3},\quad k\ge 0,\ k\in\mathbb{Z}.$$

Combination of the first equality of (39) and (40) yields

$$(41)\quad V_{2k+1}V_{2k+2}=2ny_{21}^2.$$

Notice that $\gcd(V_{2k+1},V_{2k+2})=1$ and $2\nmid V_{2k+1}$. From (39) we get

$$(42)\quad V_{2k+2}=2n_1y_3^2,$$

$$(43)\quad V_{2k+1}=n_2y_4^2,$$

where n_1, n_2, y_3, y_4 are positive integers satisfying

$$(44)\quad n_1n_2=n,\quad y_3y_4=y_{21}.$$

Since $n\ge 3$, by (vii) of Lemma 8, we see from (39) and (41)–(44) that if $n_2=1$, then $k=3$, $n=n_1=51$, $y_3=2$, $y_4=13$ and $x=275807$. Then the second equality of (39) is false. Moreover, if $n_1=1$, then from (42) we get

$$(45)\quad 2y_3^2=V_{2k+2}=2U_{k+1}V_{k+1}.$$

Since $\gcd(U_{k+1},V_{k+1})=1$, we find from (45) that U_{k+1} and V_{k+1} are both squares. Hence, by (vi) and (vii) of Lemma 8, we deduce from (39), (40), (42), (43) and (45) that $k=0$, $U_{k+1}=V_{k+1}=y_3=y_4=1$, $x=U_3=7$ and $n=1$, a contradiction. So we have $n_1>1$ and $n_2>1$.

From (42), we get

$$U_{k+1} = n_3 y_5^2, \tag{46}$$

$$V_{k+1} = n_4 y_6^2, \tag{47}$$

where n_3, n_4, y_5, y_6 are positive integers satisfying

$$\gcd(n_3, n_4) = 1, \quad n_3 n_4 = n_1, \quad y_5 y_6 = y_3. \tag{48}$$

By using the same method, we can prove that $n_3 > 1$ and $n_4 > 1$.

We now consider the case where $2 \mid k$. By Lemma 10, we see from (46) that $n_3 \geq 23$. Moreover, we observe that if $y_4 = 1$ and n_2 is a prime with $n_2 < 1000$, then (39) is false. Recall that n is squarefree. Therefore, by Lemma 11, either n_2 has at least two distinct prime factors or n_2 is a prime with $n_2 > 1000$. Similarly, we see from (47) that n_4 has the same property. Since $\gcd(V_{2k+1}, V_{2k+2}) = 1$, we have $\gcd(V_{k+1}, V_{2k+1}) = 1$. Hence, by (43) and (47), we get $\gcd(n_2, n_4) = 1$. Notice that every prime factor p of $V_{k+1}V_{2k+1}$ satisfies $p \equiv 1 \pmod 4$. So we have

$$n = n_1 n_2 = n_2 n_3 n_4 \geq 23 n_2 n_4 \geq 23 \min(5 \cdot 13 \cdot 17 \cdot 29, 10^6) \geq 482885. \tag{49}$$

But, by Lemma 14, from (38) and (39) we get $n < 330000$, a contradiction.

For the case where $2 \nmid k$, we have $V_{k+1} = 2U_{(k+1)/2}V_{(k+1)/2}$. Therefore, by much the same argument as in the proof of the case where $2 \mid k$, we can obtain a lower bound $n \geq 482885$ as in (49). By Lemma 14, this is impossible. The proof is complete.

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