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Hyperelliptic modular curves $X_0^*(N)$

by

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1. Introduction. Let N be a positive integer, and let

$$\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})\;\middle|\; c\equiv 0 \bmod N\right\}.$$

For each positive divisor N' of N with $(N', N/N')=1$ (we write $N'\parallel N$), $W_{N'}=W_{N'}^{(N)}$ denotes the corresponding Atkin–Lehner involution defined for $\Gamma_0(N)$. (W_1 is the identity operator.) Then we define the modular group $\Gamma_0^*(N)$ to be the group generated by $\Gamma_0(N)$ and $\{W_{N'}\}_{N'\parallel N}$:

$$\Gamma_0^*(N)=\langle \Gamma_0(N)\cup\{W_{N'}\}_{N'\parallel N}\rangle.$$

It is well known (see [1]) that $\Gamma_0^*(N)$ normalizes $\Gamma_0(N)$ and the factor group $W(N):=\Gamma_0^*(N)/\Gamma_0(N)$ is abelian of type $(2,\ldots,2)$ with order $2^{\omega(N)}$, where $\omega(N)$ denotes the number of the distinct prime divisors of N.

Let $X_0^*(N)$ be the modular curve which corresponds to $\Gamma_0^*(N)$, namely,

$$X_0^*(N):=X_0(N)/W(N)=X_0(N)/\langle\{W_{N'}\}_{N'\parallel N}\rangle.$$

In [7], we proved

Theorem A. *Assume that N is square-free. Then $X_0^*(N)$ is hyperelliptic if and only if $X_0^*(N)$ is of genus two.*

In [7], we were reduced to 56 cases (21 square-free cases and 35 non-square-free cases). The above theorem, conjectured by Kluit [9], is the result for square-free cases.

The purpose of this article is to determine all hyperelliptic curves of type $X_0^*(N)$ with genus ≥ 3, i.e., to check the hyperellipticity of $X_0^*(N)$ for the 35 values of N listed in Table 1. Our result is formulated in Theorem B below.

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Table 1

Genus	N									
3	136	144	152	162	164	171	175	196	207	234
	240	252	270	294	312	315	348	420	476	
4	160	176	264	280	300	306	342			
5	216	279	396	630						
6	336									
7	360	450								
10	840									
19	1680									

Theorem B. *There are sixty-four values of N for which $X_0^*(N)$ is hyperelliptic. Of these, there are only seven values of N for which $X_0^*(N)$ is hyperelliptic with genus $g \geq 3$, namely,* $N = 136$, 171, 207, 252, 315 *for* $g = 3$, $N = 176$ *for* $g = 4$, *and* $N = 279$ *for* $g = 5$.

Remark 1. The 57 values of N for which $X_0^*(N)$ is of genus two are as follows:

67, 73, 85, 88, 93, 103, 104, 106, 107, 112,
115, 116, 117, 121, 122, 125, 129, 133, 134, 135,
146, 147, 153, 154, 158, 161, 165, 166, 167, 168,
170, 177, 180, 184, 186, 191, 198, 204, 205, 206,
209, 213, 215, 221, 230, 255, 266, 276, 284, 285,
286, 287, 299, 330, 357, 380, 390.

Their defining equations are given in [5] (see also [11]).

Notation. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote respectively the ring of rational integers, the field of rational numbers and the field of complex numbers. $\mathbb{F}_{p^\nu}$ denotes the finite field with p^ν elements. $\mathbb{P}^n$ is the n-dimensional projective space. We denote by τ an element of the complex upper half plane, and we put $q = \exp(2\pi i\tau)$.

2. Modular involutions on $X_0^*(N)$ (I). In this section, we treat the case with $8 \mid N$ or $9 \parallel N$. As we shall show, $X_0^*(N)$ has an involution which comes from a matrix when $8 \mid N$ or $9 \parallel N$. We can use this involution to determine the hyperellipticity of $X_0^*(N)$ for some cases. One may refer to [10], [1] for the structure of the normalizer of $\Gamma_0(N)$ in $\mathrm{GL}_2^+(\mathbb{Q})$. But he should be careful to use Theorem 8 of [1], since some errors are included there.

Put $S_\mu = \left(\begin{smallmatrix} \mu & 1 \\ 0 & \mu \end{smallmatrix}\right)$. Then S_2 is in the normalizer of $\Gamma_0(N)$ when N is divisible by 4, and S_3 is in the normalizer of $\Gamma_0(N)$ when N is divisible by 9.

Proposition 1. (i) *Let $2^\nu \parallel N$ with $\nu \geq 3$. Then $V_2 = S_2 W_{2^\nu} S_2$ normalizes $\Gamma_0^*(N)$ and $\Gamma_0(N)$. Further, $V_2^2 \in \Gamma_0(N)$.*

(ii) *Let $9 \parallel N$. Then $V_3 = S_3 W_9 S_3^2$ normalizes $\Gamma_0^*(N)$ and $\Gamma_0(N)$. Further, $V_3^2 \in \Gamma_0(N)$.*

Proof. This follows from a direct calculation. ■

Corollary. *Suppose* $8 \mid N$ (*resp.* $9 \parallel N$). *Then* V_2 (*resp.* V_3) *defines an involution on* $X_0(N)$ *and on* $X_0^*(N)$.

Let $S_2(N)$, $S_2^0(N)$ and $S_2^*(N)$ be respectively the space of cuspforms of weight 2 on $\Gamma_0(N)$, the space spanned by newforms of weight 2 on $\Gamma_0(N)$, and the space of cuspforms of weight 2 on $\Gamma_0^*(N)$. Put $V = V_2$ or V_3 (or $V_2V_3 = V_3V_2$ if $N = 144 = 2^4 3^2$). To calculate the genus $\bar{g} = \bar{g}(N;V)$ of $X_0^*(N)/\langle V\rangle$, it suffices to determine the dimension of the subspace

$$S_2^*(N)^V = \{f \in S_2^*(N) \mid f|V = f\}$$

of $S_2^*(N)$.

Lemma 1. *Let* M *be a positive integer. Let* M' *be a positive divisor of* M *and let* d *be a positive divisor of* M/M'. *For a prime divisor* p *of* M, *define integers* α, β *and* γ *by*

$$p^\alpha \parallel M, \quad p^{\alpha-\beta} \parallel M', \quad p^\gamma \parallel d.$$

(i) *Let* $f(\tau) \in S_2(M')$. *Then*

$$f(d\tau)|W_{p^\alpha}^{(M)} = p^{\beta-2\gamma}(f|W_{p^{\alpha-\beta}}^{(M')})(d'\tau),$$

where $d' = p^{\beta-2\gamma}d$.

(ii) *If* $f \in S_2^0(M')$ *is a newform on* $\Gamma_0(M')$, *then* f *is also an eigenform for all* $W_m^{(M')}$ *with* $m \parallel M'$. *In particular, if* $f|W_{p^{\alpha-\beta}}^{(M')} = \lambda'_p f$ $(= \pm f)$ *and if* $\beta \neq 2\gamma$ (*resp.* $\beta = 2\gamma$), *then*

$$f(d\tau) \pm p^{\beta-2\gamma}\lambda'_p f(d'\tau) \quad (\textit{resp. } f(d\tau))$$

becomes an eigenform for $W_{p^\alpha}^{(M)}$ *with eigenvalue equal to* ± 1 (*resp.* λ'_p).

Proof. See [1]. ■

For simplicity, we will sometimes write $f^{(d)}(\tau) = f(d\tau)$ in the following.

Proposition 2. *Let* N *be a positive integer such that* $8 \mid N$. *Let* N' *be a positive divisor of* N *and let* d *be a positive divisor of* N/N'. *Define integers* α, β *and* γ *by*

$$2^\alpha \parallel N, \quad 2^{\alpha-\beta} \parallel N', \quad 2^\gamma \parallel d,$$

so that $N = 2^\alpha M$ *and* $N' = 2^{\alpha-\beta}M'$ *for some positive odd integers* M, M' *with* $M' \mid M$. *Let* $f = \sum a_n q^n$ *be a newform on* $\Gamma_0(N')$ *such that* $f|W_{2^{\alpha-\beta}}^{(N')} = \lambda f$, *and put*

$$g^{(d)} = f^{(d)} + f^{(d)}|W_{2^{\alpha-\beta}}^{(N')} = f^{(d)} + 2^{\beta-2\gamma}\lambda f^{(d')}$$

with $d' = 2^{\beta-2\gamma}d$.

(i) *If* $\alpha-\beta\geq 2$, *then*

$$\begin{cases} g^{(d)}|V_2=-g^{(d)} & \text{if } \beta>\gamma=0,\\ g^{(d)}|V_2=+g^{(d)} & \text{if } \beta-\gamma>\gamma>0,\\ f^{(d)}|V_2=\lambda f^{(d)} & \text{if } \beta=2\gamma. \end{cases}$$

(ii) *If* $\alpha-\beta=1$, *then*

$$\begin{cases} (g^{(d)}+\lambda g^{(2d)})|V_2=-(g^{(d)}+\lambda g^{(2d)}) & \text{if } \gamma=0,\\ g^{(d)}|V_2=+g^{(d)} & \text{if } \beta-\gamma>\gamma>0,\\ f^{(d)}|V_2=\lambda f^{(d)} & \text{if } \beta=2\gamma. \end{cases}$$

Proof. Write $S=S_2$ and $W=W_{2^\alpha}^{(N)}$. Since $S\tau=\tau+1/2$, we have $f^{(d)}|S=+f^{(d)}$ if $\gamma>0$, and $f^{(d)}|S=-f^{(d)}$ if $\alpha-\beta\geq 2$ and $\gamma=0$ (note that if $\alpha-\beta\geq 2$, then $a_{2m}=0$ for $m=1,2,\ldots$). The assertions, except for the case $\alpha-\beta=1$ and $\gamma=0$, follow from these and Lemma 1. Finally, let $\alpha-\beta=1$ and $\gamma=0$. Then

$$f^{(d)}+f^{(d)}|S=2\sum_{n=1}^{\infty}a_{2n}q^{2dn}=2a_2\sum_{n=1}^{\infty}a_nq^{2dn}=2a_2f^{(2d)}=-2\lambda f^{(2d)},$$

so we have

$$f^{(d)}|V_2=-2\lambda f^{(2d)}|WS-f^{(d)}|WS=-2^{\beta-1}f^{(2^{\beta-1}d)}-2^\beta\lambda f^{(2^\beta d)}.$$

From this, we see that

$$(f^{(d)}-f^{(d)}|V_2)+\lambda(f^{(2d)}-f^{(2d)}|V_2) \\ =(f^{(d)}+2^\beta\lambda f^{(2^\beta d)})+\lambda(f^{(2d)}+2^{\beta-2}\lambda f^{(2^{\beta-1}d)}),$$

hence the assertion follows. ∎

PROPOSITION 3. *Let* $N=9M$ *with* M *a positive integer such that* $3\nmid M$. *Let* M' *be a positive divisor of* M, *and let* d *be a positive divisor of* M/M'.

(i) *Let* $f=\sum a_nq^n$ *be a newform on* $\Gamma_0(9M')$ *such that* $f|W_9^{(9M')}=+f$. *Then* $f^{(d)}$ *is an eigenform of* V_3 *with eigenvalue* $+1$.

(ii) *Let* $f=\sum a_nq^n$ *be a newform on* $\Gamma_0(3M')$ *such that* $f|W_3^{(3M')}=\lambda f$. *Then* $f^{(d)}+3\lambda f^{(3d)}$ *is an eigenform of* V_3 *with eigenvalue* -1.

Proof. Write $S=S_3$ and $W=W_9^{(9M)}$, and put $\zeta=\exp(2\pi i/3)$. Since $S\tau=\tau+1/3$, it follows that

$$f^{(d)}+f^{(d)}|S+f^{(d)}|S^2=\sum(1+\zeta^{dn}+\zeta^{2dn})a_nq^{dn} \\ =3\sum a_{3n}q^{3dn}=3a_3\sum a_nq^{3dn}.$$

(i) In this case, we have

$$(f^{(d)}|S+f^{(d)}|S^2)|W=f^{(d)}|S+f^{(d)}|S^2,$$

since $a_3 = 0$ and $f|W_9^{(9M')} = f$. On the other hand, by Theorem 6 of [1],

$$g := \frac{1}{\sqrt{-3}}(f|S - f|S^2) = \sum_{n=1}^{\infty} \left(\frac{-3}{n}\right) a_n q^n$$

is also a newform on $\Gamma_0(9M')$ with $g|W_9^{(9M')} = g$. Hence $f^{(d)}|SW = f^{(d)}|S$, or equivalently, $f^{(d)}|SWS^2 = f^{(d)}$.

(ii) In this case, we have

$$f^{(d)} + f^{(d)}|S + f^{(d)}|S^2 = 3a_3 f^{(3d)} = -f^{(d)}|W,$$

so

$$f^{(d)}|WS^2 + f^{(d)}|SWS^2 + f^{(d)}|S^2WS^2 = -f^{(d)}|S^2.$$

Now we have

$$3\lambda f^{(3d)}|SWS^2 = 3\lambda f^{(3d)}|WS^2 = f^{(d)}|S^2$$

and

$$f^{(d)}|WS^2 = 3\lambda f^{(3d)}|S^2 = 3\lambda f^{(3d)}.$$

Also we compute

$$f^{(d)}|S^2WS^2 = f^{(d)}|WSW = 3\lambda f^{(3d)}|SW = 3\lambda f^{(3d)}|W = f^{(d)},$$

since $(WS)^3 \in \Gamma_0(9M)$. Hence we obtain the equality

$$(f + 3\lambda f^{(3)})|SWS^2 = -(f + 3\lambda f^{(3)}),$$

as desired. ∎

As a consequence of these results, we can determine the hyperellipticity of $X_0^*(N)$ for some cases.

Remark 2. To calculate $\bar{g}$, it seems more natural to find the formula for the number of the fixed points for V, but our method has the advantage of giving the defining equation of $X_0^*(N)$ (see also the last section).

EXAMPLE 1. Let $N = 136 = 8 \cdot 17$. Then $S_2^*(136)$ is spanned by

$$f_1 - 4f_1^{(4)}, \quad f_2 - 2f_2^{(2)}, \quad f_3 - 2f_3^{(2)},$$

where f_1, f_2, f_3 are newforms such that $f_1 \in S_2^0(2 \cdot 17)^{(-,+)}$ and $f_2, f_3 \in S_2^0(4 \cdot 17)^{(-,+)}$. Here and in what follows, signatures indicate the eigenvalues of Atkin–Lehner involutions (see [2]). Thus $X_0^*(136)/\langle V_2 \rangle \cong \mathbb{P}^1$, from which we see that $X_0^*(136)$ is hyperelliptic, with hyperelliptic involution V_2.

Example 2. Let $N = 360 = 8 \cdot 45$. A basis $\langle g_1, \ldots, g_7\rangle$ of $S_2^*(360)$ is given by

$$\begin{aligned} g_1 &= f_1 - 2f_1^{(2)} + 9f_1^{(9)} - 18f_1^{(18)},\\ g_2 &= f_1^{(3)} - 2f_1^{(6)},\\ g_3 &= f_2 + 2f_2^{(2)} + 4f_2^{(4)} - 3f_2^{(3)} - 6f_2^{(6)} - 12f_2^{(12)},\\ g_4 &= f_2^{(2)} - 3f_2^{(6)},\\ g_5 &= f_3 - 2f_3^{(2)} + 5f_3^{(5)} - 10f_3^{(10)},\\ g_6 &= f_4 - 4f_4^{(4)},\\ g_7 &= f_5 - 3f_5^{(3)}, \end{aligned}$$

where $f_1 \in S_2^0(4\cdot 5)^{(-,+)}$, $f_2 \in S_2^0(2\cdot 3\cdot 5)^{(+,-,+)}$, $f_3 \in S_2^0(4\cdot 9)^{(-,+)}$, $f_4 \in S_2^0(2\cdot 9\cdot 5)^{(-,+,+)}$ and $f_5 \in S_2^0(8\cdot 3\cdot 5)^{(+,-,+)}$. Then, by Proposition 2, we see that $S_2^*(360)^{V_2} = \langle g_4,\, g_7\rangle_{\mathbb{C}}$, hence the genus $\overline{g} = \overline{g}(360; V_2)$ of $X_0^*(360)/\langle V_2\rangle$ is 2. Therefore $X_0^*(360)$ is not hyperelliptic, by the following proposition.

Proposition 4. *Let $X/\mathbb{C}$ be a hyperelliptic curve of genus g. Let w be an involution on X, and $\overline{g}$ the genus of $X/\langle w\rangle$. Suppose $\overline{g} \neq 0$. If g is even, then $\overline{g} = g/2$, and if g is odd, then $\overline{g} = (g+1)/2$ or $(g-1)/2$.*

Proof. This is a corollary to Proposition 1 of [12], whose statement will be given in the next section. ∎

Example 3. Put $\alpha_n := \begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix}$. An easy calculation shows that

$$\alpha_n \Gamma_0^*(n^2N)\alpha_n^{-1} \subseteq \Gamma_0^*(N).$$

Let $n = 2$. Then $\alpha_2\Gamma_0^*(8N)\alpha_2^{-1}$ is of index 4 in $\Gamma_0^*(2N)$. Now consider the curve $X_0^*(840)$, which is of genus 10. Then there is a covering $X_0^*(840) \to X_0^*(210)$ of degree 4. We can regard this covering as a composition

$$X_0^*(840) \to X' \to X_0^*(210)$$

of coverings of degree 2. Therefore, by the above proposition, we conclude that $X_0^*(840)$ is not hyperelliptic, since $X_0^*(210)$ is of genus 1 (see [2]).

By the same reason, $X_0^*(1680)$ is not hyperelliptic, since $X_0^*(1680)$ is of genus 19 and $X_0^*(420)$ is of genus 3.

To calculate $\overline{g}(N; V)$ in this way, we need information on the W-splitting of $S_2^0(N)$. For $N \le 300$, such data are given in [2]. For $N \ge 301$, we include here a table of the W-splitting of $S_2^0(N)$ to the extent of our needs. (Remark. The third column of Table 2 gives dimensions of direct summands $S_2^0(N)^{(\pm,\ldots,\pm)}$ of $S_2^0(N)$, ordered lexicographically.)

Table 2. The W-splitting of $S_2^0(N)$

N	$p\,\vert\,N$	The W-splitting of $S_2^0(N)$	N	$p\vert N$	The W-splitting of $S_2^0(N)$
306	2,3,17	0,2,1,1,2,0,0,2	360	2,3,5	0,1,1,0,1,0,1,1
312	2,3,13	1,1,1,0,1,0,1,1	396	2,3,11	0,0,0,0,0,0,1,2
315	3,5,7	2,0,2,0,2,1,0,3	450	2,3,5	1,0,2,1,1,0,0,2
336	2,3,7	0,1,1,0,1,1,0,2	630	2,3,5,7	0,1,1,0,1,1,1,1,
342	2,3,19	1,0,2,1,1,0,0,2			1,0,0,1,0,1,1,0

Table 3. Genera of $X_0^*(N)$ and $X_0^*(N)/\langle V\rangle$

N	V	Genus of $X_0^*(N)$	Genus of $X_0^*(N)/\langle V\rangle$	N	V	Genus of $X_0^*(N)$	Genus of $X_0^*(N)/\langle V\rangle$
136	V_2	3	0	176	V_2	4	0
144	V_2	3	1	264	V_2	4	1
	V_3	3	1	280	V_2	4	1
	V_2V_3	3	1	306	V_3	4	1
152	V_2	3	1	342	V_3	4	1
171	V_3	3	0	216	V_2	5	2
207	V_3	3	2	279	V_3	5	0
234	V_3	3	1	396	V_3	5	3
240	V_2	3	1	630	V_3	5	2
252	V_3	3	0	336	V_2	6	2
312	V_2	3	1	360	V_2	7	2
315	V_3	3	2	450	V_3	7	3
160	V_2	4	2				

We know from Table 3 that $X_0^*(N)$ is hyperelliptic with hyperelliptic involution V for $N = 136$, 171, 252, 176 and 279. We also know that $X_0^*(N)$ is *not* hyperelliptic for $N = 264$, 280, 306, 342, 336 and 360, by virtue of Proposition 4. Further, we use the following fact to conclude that $X_0^*(207)$ and $X_0^*(315)$ are hyperelliptic.

PROPOSITION 5. *Let X, Y be curves over $\mathbb{C}$ of genus* 3, 2, *respectively. If there is a covering $\pi:\ X\to Y$, then X is hyperelliptic.*

Proof. See [6]. ■

Determination of the hyperellipticity of $X_0^*(N)$ for $N = 144$, 152, 234, 240, 312, 160, 216, 396, 630 and 450 will be postponed to the following sections.

3. Fixed points of V_2. In this section, we always assume that $8\,|\,N$ except for Remark 3. The important fact is that V_2 is defined over $\mathbb{Q}$, so that the following Ogg's observations [12] are applicable.

Lemma 2 ([12], Prop. 1). *Let $X/\mathbb{C}$ be a hyperelliptic curve, and v the hyperelliptic involution on X. Let w be another involution, and put $u = vw$, which is also an involution. Then the fixed-point sets of u, v, and w are disjoint. If g is even, then w and u have two fixed points each. If g is odd, then w has four fixed points, and u has none, or vice versa.*

Proposition 6. *Let X be a curve defined over $\mathbb{Q}$, and let w be a non-hyperelliptic involution defined over $\mathbb{Q}$ on X. If w has only one rational fixed point, then X is non-hyperelliptic.*

Proof. Suppose X is hyperelliptic. Then the hyperelliptic involution v, which is in the center of $\operatorname{Aut} X$, acts on the set of fixed points of w. But v is defined over $\mathbb{Q}$, so v must fix the (unique) rational fixed point of w. This is a contradiction. ■

Observe that $S_2 = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ commutes with all W_{p^ν}, $p^\nu \parallel N$, $p \neq 2$. Therefore S_2 induces an isomorphism

$$X_0^*(N) \cong X_0(N)/G,$$

where G is the subgroup of $\operatorname{Aut} X_0(N)$ generated by $\{W_{p^\nu}\}_{p\neq 2} \cup \{V_2\}$. Thus, to obtain the information about the fixed points of V_2 on $X_0^*(N)$, it suffices to consider the fixed points of W_{2^α} $(2^\alpha \parallel N)$ on $X_0(N)/G$.

Example 4. Let $N = 152 = 2^3 \cdot 19$. The genus of $X_0^*(152)$ is three, and V_2 has 4 fixed points on $X_0^*(152)$ (see Table 3). Put $G = \langle W_{19}, V_2 \rangle$, which is abelian of type $(2,2)$. On $X_0(152)$, we can see from [2] that W_8 and W_{152} have 4 and 12 fixed points, respectively. Since G acts fixed-point-freely on the fixed-point set of W_8 (resp. W_{152}), the contribution of the fixed points of W_8 (resp. W_{152}) on $X_0(152)$ to those of W_8 on $X = X_0(152)/G$ is one (resp. three). Further we have

$$h(-4 \cdot 8) = 2, \qquad h(-4 \cdot 152) = 12,$$

where $h(-d)$ is the class number of primitive quadratic forms of discriminant $-d$. This means that the fixed points of W_{152} are all defined over a field of degree exactly 12. Hence W_8 has only one rational fixed point on X, i.e., $X_0^*(152) \cong X$ is not hyperelliptic.

The similar argument can be applied to the cases $N = 216$ and 312. Let $N = 216$. The genus of $X_0^*(216)$ is five, and the involution V_2 has four fixed points on $X_0^*(216)$. On $X_0(216)$, we see from [2] that W_8 (resp. W_{216}) has 4 (resp. 12) fixed points. Also we compute $h(-4 \cdot 8) = 2$, $h(-4 \cdot 216) = 12$. Hence $X_0^*(216)$ is not hyperelliptic.

For $N = 312$, we include the W-splitting of $S_2(312)$:

$$3, 10, 7, 5, 6, 6, 9, 3,$$

which should be read in the same manner as the data in the third column of Table 2. From this, we see that W_{104} has 24 fixed points and W_{312} has 8 fixed points; the corresponding class numbers are $h(-4\cdot 104) = 12, h(-4\cdot 312) = 8$. Therefore W_8 has exactly one rational fixed point on $X_0(312)/G$, where $G = \langle W_3, W_{13}, V_2\rangle$. Hence $X_0^*(312)$ is not hyperelliptic.

Remark 3. Let $9 \parallel N$, and let V_3 be the involution on $X_0(N)$ defined in the previous section. Then V_3 is defined over $\mathbb{Q}(\sqrt{-3})$. Also, it can easily be shown that

$$V_3 W_{p^\nu} V_3 = \begin{cases} W_{p^\nu} & \text{if } p^\nu \equiv 0, 1 \pmod 3, \\ W_9 W_{p^\nu} & \text{if } p^\nu \equiv 2 \pmod 3 \end{cases}$$

for $p^\nu \parallel N$ (cf. Theorem 8 of [1]).

4. Modular involutions on $X_0^*(N)$ (II). In this section, we assume that $4 \parallel N$. Write $N = 4M$. Then

PROPOSITION 7. *We have the isomorphism*

$$X_0^*(N) \cong X_0(N)/G \cong X_0(2M)/\langle\{W_{p^\nu}\}_{p|M}\rangle,$$

where G is a subgroup of $\mathrm{Aut}\, X_0(N)$ *generated by* $\{W_{p^\nu}\}_{p\neq 2} \cup \{S_2\}$.

Proof. Indeed, the first isomorphism is obtained by conjugating $\Gamma_0^*(N)$ by $S_2W_4S_2$, and the second by conjugating $\langle G \cup \Gamma_0(N)\rangle$ by α_2. ∎

Consider the case $N = 300$. Then we have

$$X_0^*(300) \cong X_0(150)/\langle W_3, W_{25}\rangle,$$

the right hand side of which has a covering of degree two to $X_0^*(150)$; since the genus of $X_0^*(300)$ is four and that of $X_0^*(150)$ is one, we find that $X_0^*(300)$ is not hyperelliptic (Proposition 4).

Next we pick up the case $N = 348$. Then

$$X_0^*(348) \cong X := X_0(174)/\langle W_3, W_{29}\rangle.$$

The curve X has an involution induced by W_2. Since (on $X_0(174)$) W_6 and W_{174} have 4 and 12 fixed points each, and since $h(-4\cdot 6) = 2$ and $h(-4\cdot 174) = 12$, it follows that W_2 has only one rational fixed point, as in Example 4. So, by Proposition 6, we conclude that $X_0^*(348)$ is not hyperelliptic.

5. Reduction modulo p. Let p be a prime number and N a positive integer such that $N = pM$, $p \nmid M$. The reduction modulo p of $X_0(pM)$ consists of two copies Z, Z' of $X_0(M)$ in characteristic p, intersecting transversally at the supersingular points [3] (see Figure 1).

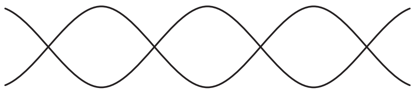

Fig. 1. $X_0(pM) \bmod p$

The Atkin–Lehner involutions $W_{N'}$ $(N' \parallel N)$ still act on $X_0(N) \bmod p$. If $p \nmid N'$, then $W_{N'}$ fixes each component Z, Z', and its action in characteristic p is the same as in characteristic 0. If $p \mid N'$, then $W_{N'}$ interchanges Z and Z'. In particular, if $N' = p$, then W_p fixes each $\mathbb{F}_p$-rational supersingular point, while it exchanges each properly $\mathbb{F}_{p^2}$-rational supersingular point for its conjugate. Let W' be a subgroup of $W(N)$. If W' is generated by some of $W_{N'}$ with $p \nmid N'$, then $X_0(N)/W' \bmod p$ is again of the shape in Figure 1 with $Z = Z' = X_0(M)/W'$. If W' contains some $W_{N'}$ with $p \mid N'$, then $X_0(N)/W' \bmod p$ becomes as in Figure 2:

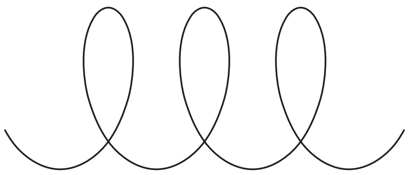

Fig. 2

where $Z'' = X_0(M)/W''$ is some quotient of $X_0(M)$.

Now assume that $X_0(N)/W'$ is hyperelliptic. Assume further for simplicity that the special fibre of the minimal model of $X_0(N)/W'$ at p is as in Figure 2, with $X_0(N)/W''$ being of genus zero. Then, as explained in Appendix C of [7], there must exist an element A of order 2 of $\mathrm{PGL}_2(\mathbb{F}_p)$ such that

$$A\alpha = \overline{\alpha} \tag{1}$$

for all properly $\mathbb{F}_{p^2}$-rational supersingular points α on Z''. We apply this observation to the cases $N = 164$ and 234.

Let $N = 164 = 41 \cdot 4$. We have $X_0^*(164) \cong X_0(82)/\langle W_{41}\rangle =: X$ by Proposition 7. Consider $X_0(82)$ modulo $p = 41$. Then $Z = Z' = X_0(2)$ is of genus zero, and the curve $X_0(2)$ is defined by the equation

$$j = 64\frac{(x+4)^3}{x^2} \tag{2}$$

(see [4]). Since the supersingular j-invariants in characteristic $p = 41$ are given by

$$j(j+38)(j+13)(j+9) = 0,$$

we compute the supersingular points on $X_0(2)$ in characteristic $p = 41$ by solving the equation (2):

$$(x+4)[(x+31)(x^2+29x+10)][(x+23)(x^2+27x+1)][(x+25)(x^2+7x+37)] = 0,$$

where $x = -4$ is the only supersingular point above $j = 0$. Thus the special fibre of the minimal model of X at $p = 41$ is as in Figure 2, with $Z'' = X_0(2)$. But it can easily be checked that there does not exist an element of order 2 of $\mathrm{PGL}_2(\mathbb{F}_{41})$ with the property (1), from which we conclude that $X_0^*(164) \cong X$ is not hyperelliptic.

Let $N = 234 = 13 \cdot 18$, and consider $X_0(234)$ modulo $p = 13$. Then $Z = Z' = X_0(18)$ is of genus zero, and the curve $X_0(18)$ is defined by the equation

$$j = h \circ g \circ f(x), \tag{3}$$

where

$$f(X) = \frac{1}{2}X(X^2 + 6X + 12),$$
$$g(X) = \frac{X(2X+9)^2}{27(X+4)},$$
$$h(X) = 27\frac{(X+1)(9X+1)^3}{X}.$$

The actions of W_2, W_9, W_{18} are given by

$$x|W_2 = \frac{-2(x+3)}{x+2}, \quad x|W_9 = \frac{-3(x+2)}{x+3}, \quad x|W_{18} = \frac{6}{x}$$

(cf. [4]). The only supersingular j-invariant in characteristic $p = 13$ is $j = 5$, and the supersingular points on $X_0(18)$ are obtained by solving the equation (3):

$$\begin{aligned}&(x^2+6)(x^2+7)(x^2+x+5)(x^2+3x+10)(x^2+4x+6)(x^2+4x+9)\\ &\quad\times(x^2+4x+10)(x^2+5x+1)(x^2+5x+5)(x^2+6x+2)(x^2+6x+4)\\ &\quad\times(x^2+7x+1)(x^2+7x+4)(x^2+8x+10)(x^2+9x+2)(x^2+9x+9)\\ &\quad\times(x^2+10x+1)(x^2+10x+6) = 0.\end{aligned}$$

Put $X = X_0^*(234)$ and consider X modulo $p = 13$, which is of the shape in Figure 2 with $Z'' = X_0^*(18)$. The curve $X_0^*(18)$ is parametrized by

$$x' = x + x|W_2 + x|W_9 + x|W_{18} = \frac{(x^2-6)^2}{x(x+2)(x+3)}.$$

There are three conjugate pairs of properly $\mathbb{F}_{13^2}$-rational supersingular points, say, α_i, $i = 1, \ldots, 6$, with $\alpha_{i+3} = \overline{\alpha}_i$ for $i = 1, 2, 3$; they are the roots of the equation

$$(x'^2 - 6x' + 7)(x'^2 - 6x' - 9)(x'^2 - 32) = 0.$$

It can be shown that the number of the points on $Z = Z'$ above each of the three conjugate pairs is equal to the degree of the covering $X_0(238) \to X$. Hence the special fibre of the minimal model of X at $p = 13$ is of the shape in Figure 2, with $Z'' = X_0^*(18)$. But there does not exist an element of order 2 of $\mathrm{PGL}_2(\mathbb{F}_{13})$ with the property (1), from which we conclude that $X_0^*(234)$ is not hyperelliptic.

6. Gap sequences. Let X be an algebraic curve over $\mathbb{C}$ of genus g. The *Weierstrass gap sequence* G_P at a point P of X is defined by

$$G_P = \{n \in \mathbb{Z} \mid n > 0 \text{ and } (f)_\infty \neq n(P) \text{ for all } f \in K(X)\},$$

where $K(X)$ is the function field of X and $(f)_\infty$ is the polar divisor of f. By the Riemann–Roch theorem, we see that the cardinality of G_P is exactly the genus g of X with $\max G_P \leq 2g - 1$. It is easily shown that

$$G_P = \{n \in \mathbb{Z} \mid \exists \omega \in H^0(X, \boldsymbol{\omega}_X) \text{ such that } \mathrm{ord}_P(\omega) = n - 1\}, \tag{4}$$

where $\boldsymbol{\omega}_X$ is the canonical sheaf on X. A point P on X is a *Weierstrass point* if $G_P \neq \{1, 2, \ldots, g\}$. If X is hyperelliptic and P is a Weierstrass point of X, then

$$G_P = \{1, 3, 5, \ldots, 2g - 1\}. \tag{5}$$

Now let Γ be a Fuchsian group of the first kind having $i\infty$, the point at infinity, as its cusp. For simplicity, we assume that the local parameter at the point $i\infty$ is $q = \exp(2\pi i\tau)$. Let $S_2(\Gamma)$ be the space of cuspforms of weight 2 on Γ. If the corresponding algebraic curve $X = X_\Gamma$ is of genus g, then $\dim_{\mathbb{C}} S_2(\Gamma) = g$. Since $S_2(\Gamma)$ can be identified with the space of holomorphic 1-forms on X by sending f to $2\pi i f\, d\tau$, we can interpret (4) as

$$G_P = \{n \in \mathbb{Z} \mid \exists f \in S_2(\Gamma) \text{ such that } f = q^n + \ldots\}.$$

Hence if X is hyperelliptic, then there is a basis $\langle f_1, , \ldots, f_g \rangle$ of $S_2(\Gamma)$ of the form

$$\begin{cases} f_1 = q + a_2^{(1)} q^2 + a_3^{(1)} q^3 + \ldots, \\ f_2 = q^2 + a_3^{(2)} q^3 + \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_g = q^g + a_{g+1}^{(g)} q^{g+1} + \ldots, \end{cases} \tag{6}$$

or

$$\begin{cases} f_1 = q + a_2^{(1)} q^2 + a_3^{(1)} q^3 + \ldots, \\ f_2 = q^3 + a_4^{(2)} q^4 + \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_g = q^{2g-1} + a_{2g}^{(g)} q^{2g} + \ldots \end{cases} \tag{7}$$

according as $\overline{i\infty}$ is an ordinary point or a Weierstrass point. In other words, if $S_2(\Gamma)$ does not have a basis of the form (6) nor (7), then X is not hyperelliptic. In the following, Γ is taken to be some normalizer of $\Gamma_0(N)$, and we use trace formulas of Hecke operators to compute Fourier coefficients of $f_1, \ldots, f_g$ ([8], [13]).

Example 5. Let $N = 160$. Then $X_0^*(160)$ is of genus 4, so $S_2^*(160)$ is of dimension 4. On the other hand, $S_2^*(160)$ contains

$$f_1 - 8f_1^{(8)} = q - 2q^3 - q^5 + 2q^7 + \ldots,$$
$$f_1^{(2)} - 2f_1^{(4)} = q^2 - 2q^4 - 2q^6 - q^{10} + \ldots,$$
$$f_2 = q - 2q^3 - q^5 - 2q^7 + \ldots,$$

where $f_1 \in S_2^0(4\cdot 5)^{(-,+)}$ and $f_2 \in S_2^0(32\cdot 5)^{(+,+)}$. Hence $S_2^*(160)$ cannot have a basis of the form (6) nor (7). Namely, $X_0^*(160)$ is not hyperelliptic.

By the same reason, $X_0^*(175)$ and $X_0^*(270)$ are not hyperelliptic; $X_0^*(175)$ is of genus 3 and $S_2^*(175)$ contains

$$f_1 - 5f_1^{(5)} = q - q^3 + 2q^4 + \ldots, \qquad f_2 = q - q^3 - 2q^4 + \ldots,$$

where $f_1 \in S_2^0(5\cdot 7)^{(-,+)}$ and $f_2 \in S_2^0(25\cdot 7)^{(+,+)}$; $X_0^*(270)$ is of genus 3 and $S_2^*(270)$ is spanned by

$$g_1 := f_1 - 9f_1^{(9)} = q - q^2 + q^3 + q^4 + \ldots,$$
$$g_2 := f_2 + 2f_2^{(2)} - 3f_2^{(3)} - 6f_2^{(6)} = q + 3q^2 - 3q^3 + q^4 + \ldots,$$
$$g_3 := f_3 + 2f_3^{(2)} = q - 2q^4 + \ldots,$$

where $f_1 \in S_2^0(2\cdot 3\cdot 5)^{(+,-,+)}$, $f_2 \in S_2^0(9\cdot 5)^{(-,+)}$ and $f_3 \in S_2^0(27\cdot 5)^{(+,+)}$, so

$$3g_1 + g_2 - 4g_3 = 12q^4 + \ldots \in S_2^*(270).$$

Example 6. Let $N = 396$. Then $X_0^*(396)/\langle V_3\rangle$ is of genus 3, and the space $S_2^*(396)^{V_3}$ is spanned by $f_1 + 4f_1^{(4)}$, $f_1^{(2)}$ and $f_2 - 2f_2^{(2)}$, where $f_1 \in S_2^0(9\cdot 11)^{(+,+)}$ and $f_2 \in S_2^0(2\cdot 9\cdot 11)^{(-,+,+)}$ (see [2] and Proposition 3). But since their levels are divisible by 3^2, f_1 and f_2 have zero as their third Fourier coefficients, implying that $S_2^*(396)^{V_3}$ cannot have a basis of the form (6) nor (7). This shows that $X_0^*(396)/\langle V_3\rangle$ is not hyperelliptic. Hence $X_0^*(396)$ is also non-hyperelliptic.

By the same reason, $X_0^*(450)$ is not hyperelliptic (the space $S_2^*(450)^{V_3}$ is spanned by $f_1 - 5f_1^{(5)}$, $f_2 + 2f_2^{(2)}$ and f_3, where $f_1 \in S_2^0(2\cdot 9\cdot 5)^{(+,+,-)}$, $f_2 \in S_2^0(9\cdot 25)^{(+,+)}$ and $f_3 \in S_2^0(2\cdot 9\cdot 25)^{(+,+,+)}$).

7. Conclusion: Remaining cases and the defining equations of hyperelliptic curves $X_0^*(N)$. So far, we have determined the hyperellipticity of $X_0^*(N)$ except for the following values of N:

(8) $$N = 144, 162, 196, 240, 294, 420, 476, 630.$$

We are now going to treat these remaining cases. Let Γ be as in the previous section, and $\langle f_1, \ldots, f_g\rangle$ a basis of $S_2(\Gamma)$. Assume for simplicity that $\overline{i\infty}$ is an ordinary point of X_Γ, and that $\langle f_1, \ldots, f_g\rangle$ is of the form (6). Put

$$z = \frac{f_{g-1}}{f_g}, \qquad w = \frac{dz}{2\pi i f_g d\tau} = \left(\frac{f_g}{q}\right)^{-1}\frac{dz}{dq}$$

and define

$$G(T) = T^{2g+2} + v_{2g+1}T^{2g+1} + \ldots + v_0 \in \mathbb{C}[T]$$

by the condition $\mathrm{ord}_q(w^2 - G(z)) \geq 1$, i.e., the Laurent series $w^2 - G(z)$ consists only of positive q-power terms. Thus we can write

$$w^2 - G(z) = \sum_{j\geq 1} d_j q^j. \tag{9}$$

Proposition 8. *Notation being as above, the curve $X = X_\Gamma$ is hyperelliptic if and only if the following two conditions hold*:

(i) $G(T)$ *is separable*,
(ii) $d_1 = \ldots = d_h = 0$ *where* $h = 4g^2 + 8g - 20$.

Moreover, if X is hyperelliptic, then it is defined by the equation $w^2 = G(z)$.

Proof. See [7]. ∎

Example 7. Let $N = 144$. Then a basis of $S_2^*(144)$ is given by

$$f_1 = q - 4q^4 - 4q^7 + 2q^{13} + O(q^{17}),$$
$$f_2 = q^2 - 4q^4 + 2q^5 + 2q^6 - 4q^7 + 2q^9 - 2q^{10} + 4q^{13} + O(q^{17}),$$
$$f_3 = q^3 - 2q^4 + q^5 - 2q^7 + q^9 + 2q^{11} + 2q^{13} - 2q^{15} + O(q^{17}).$$

Put $x = f_1/f_3$ and $y = f_2/f_3$. Then we can see that they satisfy no quadratic equations. In fact, by obtaining much more precise expressions for $f_1,\ f_2,\ f_3$, we can verify that x and y satisfy a quartic equation

$$x^3 - x^2(y^2+4) + 2x(y^3 - 2y^2 + 2y + 3) - (y^4 - 4y^3 + 8y^2 - 8y + 7) = 0,$$

which is the defining equation of $X_0^*(144)$. Hence $X_0^*(144)$ is not hyperelliptic. We can give an alternative proof which uses Proposition 8. In the present case, we have

$$G(T) = T^8 - 12T^7 + 76T^6 - 272T^5 + 626T^4 - 820T^3 + 720T^2 - 184T + 1$$

and $w^2 - G(z) = -864\,q + \ldots$, hence again we conclude that $X_0^*(144)$ is not hyperelliptic.

Example 8. Let $N = 207$. By Table 3 and Proposition 5, we know that $X_0^*(207)$ is hyperelliptic. Let us compute the defining equation of $X_0^*(207)$. A basis of $S_2^*(207)$ is given by

$$\begin{aligned} f_1 = q + q^2 - q^3 - 2q^4 - 2q^5 - q^6 - q^7 \\ - 2q^8 - q^9 - q^{10} + q^{12} - 3q^{13} + O(q^{14}), \end{aligned}$$

$$f_2 = q^2 - 2q^4 - q^5 + q^7 + q^8 - q^{10} - 2q^{11} + O(q^{14}),$$

$$f_3 = q^3 - q^4 - 2q^5 + q^6 + q^7 + q^8 + q^9 - q^{10} - 4q^{11} - q^{12} + 3q^{13} + O(q^{14}).$$

Then we compute $f_2^2 - f_1 f_3 = O(q^{15})$, or equivalently, $x^2 - y = O(q^9)$ if we write $x = f_2/f_3$ and $y = f_1/f_3$. Since the degree of the divisors of poles of x and y are bounded by

$$2 \cdot 3 - 2 = 4,$$

and since x^2, y have a pole of order 2 at $\overline{i\infty}$, we see that the f_i's in fact satisfy a quadratic equation

$$f_2^2 - f_1 f_3 = 0.$$

Hence again we find that $X_0^*(207)$ is hyperelliptic. Its defining equation is given by

$$w^2 = z^8 - 6z^7 + 11z^6 - 12z^5 + 9z^4 - 12z^3 + 11z^2 - 6z + 1,$$

where we put

$$z = \frac{f_2}{f_3} \quad \text{and} \quad w = \left(\frac{f_3}{q}\right)^{-1} \frac{dz}{dq}.$$

Using Proposition 8, we can show that $X_0^*(N)$ is not hyperelliptic for all N in (8) (see Table 6 for data of Fourier coefficients). Hence we have

THEOREM. *Assume that $X_0^*(N)$ is of genus ≥ 3. Then $X_0^*(N)$ is hyperelliptic if and only if $N = 136, 171, 207, 252, 315, 176$ or 279. For $N \neq 207$ and $N \neq 315$, the hyperelliptic involution of $X_0^*(N)$ is of type V in the notation of Section 2; namely, $V = S_2 W_8 S_2$ for $N = 136$, $V = S_2 W_{16} S_2$ for $N = 176$, and $V = S_3 W_9 S_3^2$ for $N = 171,\ 252$ and 279. Their defining equations are given in Table 4 below.*

Table 4

N	Defining equation $w^2 = f(z)$ of $X_0^*(N)$	Discriminant of $f(z)$
136	$w^2 = z(z+1)(z^2+3z-2)(z^4+4z^3+5z^2+2z-4)$	$-2^{28}17^3$
171	$w^2 = (z^2-z+1)(z^6+z^5+2z^4-7z^3-2z^2-3z+9)$	$2^{16}3^6 19^4$
207	$w^2 = z^8-6z^7+11z^6-12z^5+9z^4-12z^3+11z^2-6z+1$	$-2^{16}3^6 23^3$
252	$w^2 = (z^2+3)(z^2-z+1)(z^4-5z^3+8z^2-7z+7)$	$2^{28}3^4 7^4$
315	$w^2 = (z^4+z^3+3z^2+z+1)(z^4+5z^3+3z^2+5z+1)$	$-2^{16}3^6 5^2 7^3$
176	$w^2 = z(z^3-4z+4)(z^3-2z^2+2)(z^3+2z^2-2)$	$-2^{40}11^5$
279	$w^2 = (z^6-z^5+z^4+2z^3-z^2+1)$ $\times (z^6+3z^5+5z^4+6z^3+7z^2+12z+9)$	$2^{24}3^8 31^6$

R e m a r k 4. Let N be an integer which is in Table 3. Then $X_0^*(N)/\langle V \rangle$ is of genus 2 for $N = 207,\ 315,\ 160,\ 216,\ 630,\ 336$ and 360. We also give their defining equations in Table 5.

Table 5

N	Defining equation $w^2 = f(z)$ of $X_0^*(N)$	Discriminant of $f(z)$
207	$w^2 = (z-1)(z+3)(z^4 - 2z^3 - 5z^2 + 6z - 3)$	$-2^{12}3^6 23^3$
315	$w^2 = (z-3)(z+1)(z^2 - z + 1)(z^2 + 3z - 3)$	$-2^{12}3^6 5^4 7^3$
160	$w^2 = (z^2+4)(z^2 - 2z + 2)(z^2 + 2z + 2)$	$-2^{26}5^4$
216	$w^2 = (z^2 - 3z + 3)(z^3 - 3z^2 + 3z + 3)$	$2^4 3^8$
630	$w^2 = (z^2 + z - 1)(z^4 - z^3 + 2z^2 + 7z + 7)$	$2^{12}3^6 5^3 7^2$
336	$w^2 = (z^2 - 3)(z^4 - 11z^2 + 32)$	$2^{23}3 \cdot 7^2$
360	$w^2 = (z^2 + 3)(z^2 - z + 4)(z^2 + z + 4)$	$-2^{18}3^3 5^2$

Table 6

N	A basis of $S_2^*(N)$	d_1
162	$(1,0,0,0,-4,-3,-4,-2,0,4,1,3)$	10
	$(0,1,0,-1,-1,-3,0,-1,0,1,1,3)$	
	$(0,0,1,0,-2,-1,-1,0,0,2,1,1)$	
196	$(1,0,0,-1,-2,-2,-3,0,0,2,-1,0)$	-760
	$(0,1,0,0,-2,-2,-3,1,1,2,1,0)$	
	$(0,0,1,0,-2,-1,0,0,0,2,0,-1)$	
240	$(1,0,0,-2,-1,0,-1,0,-2,0,0,-2)$	49600
	$(0,1,0,0,-2,-1,-2,0,0,3,-2,0)$	
	$(0,0,1,-2,0,0,3,0,-3,0,0,-2)$	
294	$(1,0,0,0,-1,-1,-3,-1,-2,-2,-1,1)$	-522
	$(0,1,0,0,-1,-1,0,-3,0,-3,0,0)$	
	$(0,0,1,1,0,-1,-3,-3,-3,-1,1,0)$	
420	$(1,0,0,-1,0,-1,-1,0,0,-1,-2,0)$	8
	$(0,1,0,0,-2,-1,0,1,0,1,0,0)$	
	$(0,0,1,0,-1,-1,0,0,-1,1,2,-1)$	
476	$(1,0,0,-1,-1,0,0,0,-1,-1,0,0)$	-48
	$(0,1,0,0,-1,0,-1,1,-2,-1,-2,0)$	
	$(0,0,1,0,0,-1,-1,0,0,0,-2,-1)$	
630	$(1,0,0,0,0,-1,0,-2,-2,1,-2,2,0,-1,-2,3,-5,1)$	24354
	$(0,1,0,0,0,0,0,-3,0,-1,-2,0,2,-1,0,2,-4,0)$	
	$(0,0,1,0,0,-1,-1,0,0,0,-1,1,-1,1,-1,0,-3,0)$	
	$(0,0,0,1,0,0,1,-3,-1,0,-2,0,3,-1,0,4,-3,1)$	
	$(0,0,0,0,1,-1,0,-1,-1,1,0,2,1,0,-2,4,-4,0)$	

In Table 6, we give a basis of $S_2^*(N)$ explicitly for all $N \neq 144$ in (8) (for $N = 144$, see Example 7).

If $f_i(\tau) \in S_2^*(N)$ $(1 \leq i \leq g)$ has the Fourier expansion $f_i(\tau) = \sum_{n\geq 1} a_n^{(i)} q^n$, we give its Fourier coefficients by $(a_1^{(i)}, a_2^{(i)}, \ldots, a_r^{(i)})$ with $r = 3g+3$. Note that for all N in Table 6, the f_i's are of the form (6), and $r = 3g+3$ is the lowest bound to be able to calculate d_1 (see (9) and Proposition 8). We have $d_1 \neq 0$ for all N in (8).

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