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On the number of sums and products

by

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Dedicated to the memory of P. Erdős

In what follows $\mathcal{A}$ will always denote a finite subset of the *non-zero* reals, and n the number of its elements. As usual, $\mathcal{A}+\mathcal{A}$ and $\mathcal{A}\cdot\mathcal{A}$ stand for the sets of all pairwise sums $\{a+a' : a, a' \in \mathcal{A}\}$ and products $\{a\cdot a' : a, a' \in \mathcal{A}\}$, respectively. Also, $|S|$ denotes the size of a set S.

The following problem was posed by Erdős and Szemerédi (see [5]):

For a given n, how small can one make $|\mathcal{A}+\mathcal{A}|$ and $|\mathcal{A}\cdot\mathcal{A}|$ simultaneously?

In other words, defining

$$m(\mathcal{A}) := \max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\},$$

a lower estimate should be found for

$$g(n) := \min_{|\mathcal{A}|=n} m(\mathcal{A}).$$

Remark. The philosophy behind the question is that *either* of $|\mathcal{A}+\mathcal{A}|$ or $|\mathcal{A}\cdot\mathcal{A}|$ is easy to minimize—just take an arithmetic or geometric (i.e., exponential) progression for $\mathcal{A}$. However, in both of these examples, the other set becomes very large.

In their above mentioned paper, Erdős and Szemerédi managed to prove the existence of a small but positive constant c_1 such that $g(n) \geq n^{1+c_1}$ for all n. (See also p. 107 of Erdős' paper [3].) Later on, Nathanson and K. Ford found the lower bounds $n^{32/31}$ and $n^{16/15}$, respectively [7].

The goal of this paper is to improve the exponent to 5/4.

Theorem 1. *There is a positive absolute constant c such that, for every n-element set $\mathcal{A}$,*

$$c\cdot n^{5/4} \leq \max\{|\mathcal{A}+\mathcal{A}|, |\mathcal{A}\cdot\mathcal{A}|\}.$$

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1. A tool from geometry. In the proof we shall use a result of Szemerédi and Trotter (see [9]).

PROPOSITION 1 (Szemerédi–Trotter Theorem). *Let t and N be positive integers with $t^2 \leq N$. Let, moreover, $\mathcal{P}$ be a set of N distinct points in the plane and $e_1, \ldots, e_M$ some (also distinct) straight lines. If each of the e_i contains t or more points of $\mathcal{P}$, then*

$$M \leq C \cdot \frac{N^2}{t^3}. \blacksquare$$

(Here C is a huge absolute constant of Szemerédi and Trotter—improved later to the more reasonable value of 3 by Clarkson *et al.* [2]. Recently, a simple and very elegant proof was found by Székely [8].)

Remark. The importance of the above assertion lies in the fact that the exponent of t in the denominator is strictly larger than 2. (A bound of $M \leq \binom{N}{2}/\binom{t}{2}$ would be trivial by just double-counting the pairs of points.) The first result in this direction was that of Beck [1], with an exponent 2.05 of t; this was later improved to t^3 by Szemerédi and Trotter.

2. Proof of the Theorem. Denote the elements of $\mathcal{A}$ by $a_1, \ldots, a_n$, and define the following n^2 functions:

$$f_{j,k}(x) := a_j(x - a_k) \quad \text{for } 1 \leq j, k \leq n.$$

LEMMA 2. *For every $j, k \leq n$, the function $f_{j,k}$ maps at least n elements of $\mathcal{A} + \mathcal{A}$ to some elements of $\mathcal{A} \cdot \mathcal{A}$.* $\blacksquare$

(Indeed, the image of $a_k + a_i$ is $a_j \cdot a_i \in \mathcal{A} \cdot \mathcal{A}$, for every $a_i \in \mathcal{A}$.)

From a geometric point of view, the above lemma asserts that the graph of each of the functions $f_{j,k}$ contains n or more points of $\mathcal{P} := (\mathcal{A} + \mathcal{A}) \times (\mathcal{A} \cdot \mathcal{A})$. Put $N = |\mathcal{P}| = |(\mathcal{A}+\mathcal{A})| \cdot |(\mathcal{A} \cdot \mathcal{A})|$. Then, by applying Proposition 1 to $\mathcal{P}$ and the $f_{j,k}$ (with $M = n^2$ and $t = n$), we get

$$n^2 \leq C \cdot \frac{N^2}{n^3},$$

i.e., $N \geq C^{-1/2} n^{5/2}$—whence the Theorem follows immediately. $\blacksquare$

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